# GFD Homework 2

DUE 11 Feb in class

1. Balanced flows:

   (a) Show that a purely zonal flow [u=u(y), v=0] in geostrophic balance with a meridional pressure field [$\eta = \eta(y)$] is an exact solution of the steady inviscid *fully nonlinear* shallow water equations.

   **The nonlinear shallow water equations are**

$$\frac{du}{dt} + \vec{u} \cdot \nabla u = -g\frac{d\eta}{dx} + fv \quad (1)$$

$$\frac{dv}{dt} + \vec{u} \cdot \nabla v = -g\frac{d\eta}{dy} - fu \quad (2)$$

$$\nabla \cdot \vec{u} = 0' \quad (3)$$

   **If we plug in purely zonal flow, every term is identically zero except the last two in the second equation, namely geostrophic balance:** $g\frac{d\eta}{dy} = -fu$

   (b) Consider a circular pressure field $\eta = \eta(r)$, where $r = \sqrt{x^2 + y^2}$. Show that one can find an azimuthal velocity field $u_a(r)$ such that $\eta$ and $u_a$ satisfy steady, inviscid nonlinear shallow water equations under rotation. Discuss the balance of forces in this case. Such a balance is called cyclostrophic. [hint - it's easiest if you switch to cylindrical coordinates]

   **The easiest thing is to convert to cylindrical coordinates,**

$$x = r\cos\theta; \quad y = r\sin\theta; \quad r = \sqrt{x^2 + y^2}; \quad \tan\theta = y/x; \quad u_r = \frac{dr}{dt}; \quad u_\theta = r\frac{d\theta}{dt}$$

   **Plugging in we can figure out both $u = dx/dt$ and $du/dx$ etc in the new coordinates, which gives (assuming $u_r = 0$)**

$$u = -\sin(\theta)u_\theta \quad (4)$$

$$v = \cos(\theta)u_\theta \quad (5)$$

$$\frac{d}{dx} = \cos(\theta)\frac{d}{dr} - \frac{1}{r}\sin(\theta)\frac{d}{d\theta} \quad (6)$$

$$\frac{d}{dy} = \sin(\theta)\frac{d}{dr} + \frac{1}{r}\cos(\theta)\frac{d}{d\theta} \quad (7)$$

   **Plugging into the shallow water equation for <u>either</u> u or v gives, in steady state;**

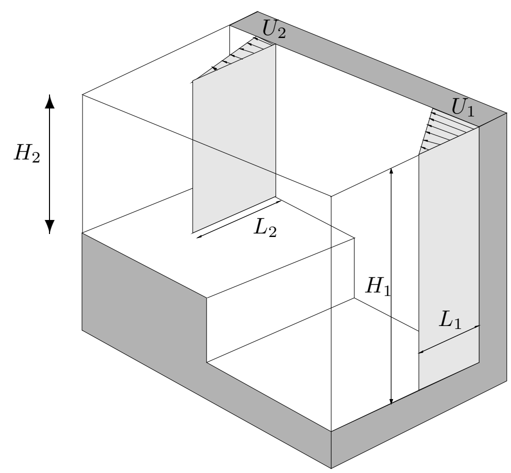
$$\frac{u_\theta^2}{r} + fu_\theta = g\frac{d\eta}{dr}$$

   **If we just consider the second two terms, we have a simple geostrophic balance. We could also imagine (for f=0) a balance between the first and third term, which is cyclostrophic balance. Physically this is a balance between centrifugal acceleration and the pressure force.**

(c) Does cyclostrophic balance cause the flow to go around faster or slower than geostrophic balance for the same pressure distribution? Consider both positive and negative pressure cases

**The centrifugal force is always pointed outward.**

- **Consider a positive sea-surface height anomaly ($d\eta/dr < 0$), in which case the pressure force is outward. In geostrophic balance (in the NH) the flow goes around CW ($u_\theta < 0$) and the coriolis force is directed inward. In this case the centrifugal force <u>adds</u> to the pressure force, so in order for the force balance to add up the coriolis force has to be bigger - in other words the flow goes <u>faster</u> than it would in pure geostrophic balance.**
- **Now consider a case with a negative sea-surface height anomaly ($d\eta/dr > 0$), in which case the pressure force is inward. In geostrophic balance (in the NH) the flow goes around CWW ($u_\theta > 0$) and the coriolis force is directed outward. In this case the centrifugal force <u>opposes</u> the pressure force, so a smaller coriolis force is needed to balance the sum of the other two forces and the flow goes <u>slower</u> than it would in pure geostrophic balance.**




2. As depicted in the figure above (from C-R Chapter 7), a vertically uniform but laterally sheared northern hemisphere coastal current must climb a bottom escarpment. Assuming that the jet velocity still vanishes offshore, use conservation of the *nonlinear shallow water PV* (e.g. DQ/Dt=0 following a water parcel) to determine the velocity prole and the width of the jet downstream of the escarpment. Use $H_1 = 200$ m, $H_2 = 160$ m, $U_1 = 0.5$ m/s, $L_1 = 10$ km and f $= 10^{-4}$ s$^{-1}$. What would happen if the downstream depth were only 100 m?

**We have two unknowns ($U_2, L_2$) and need two equations. The first is convservation of potential vorticity. Assuming $H + \eta \approx H$ we can write PV conservation as**

$$\frac{D}{Dt}\left[\frac{\zeta + f}{H}\right]$$

**As always, the substantial derivative can be viewed as time change <u>following</u> the flow, so the flow must have the same value before and after is crosses the escarpment. Setting those equal gives us**

$$\frac{1}{H_1}(\frac{U_1}{L_1} + f) = \frac{1}{H_2}(\frac{U_2}{L_2} + f)$$

**Our second equation comes from the need to have equal volume transport on both sides (steady state means that no water is piling up anywhere)**

$$U_1 L_1 H_1 = U_2 L_2 H_2$$

**Combining and going a bit of algebra gives us an equation for $L_2$**

$$L_2 = \sqrt{\frac{U_1 L_1^2 H_1^2}{H_2[(U_1 + f L_1) - f L_1 H_1]}}$$

**For $H_2$=160 meters, plugging in gives us**

$$L_2 = 17.6\text{km} \qquad U_2 = 0.35\text{ms}^{-1}$$

**Physically, since the height decreases, the flow compensates by losing some relative vorticity by becoming a bit slower and more spread out horizontally. On the other hand, for $H_2$=100 meters, the denominator in the square root above is negative, meaning there is no solution! Physically, the flow has lost so much height it would need to have a net <u>negative</u> relative vorticity, meaning it would have to increase in the offshore direction. This cannot conserve volume transport, so there is no solution. What actually happens in this (physically reasonable) case is that the flow turns offshore and follows the escarpment as a triangular jet. On the left side of that jet the flow is deeper and the vorticity is cyclonic, while on the right side the water is shallower and the vorticity is anticyclonic.**

3. Two-layer geostrophic adjustment. Consider a two-layer fluid with resting depths $H_1, H_2$ and densities $\rho_1, \rho_2$. We will discuss the setup for this problem in class on Tuesday, or you can start by yourself. Let's consider the baroclinic version of the "dam break problem"- in particular assume that at time=0 both layers are at rest and the surface height ($\eta$) and the interface height ($h$) are given by:

$$\eta(t=0) = 0 \quad (8)$$
$$h(t=0) = h_0 \quad x <= 0 \quad (9)$$
$$h(t=0) = -h_0 \quad x > 0 \quad (10)$$

(a) Making a rigid lid assumption ($\eta << h$), and assuming a small density difference ($\rho_1/\rho_2 \approx 1$) and assume that PV is conserved within each layer (we'll go over this in class Tuesday, but it's also straightforward to just extrapolate what you have in your notes from 1 to 2 layers), derive a single differential equation that governs the interface height once the system has reached a steady-state solution. It's easier to do this for $x > 0$ and $x < 0$ separately.

**We can set the initial PV equal to the final PV for each layer. Initially the relative vorticity is zero for each layer because there is no flow. Since are initially only gradients in the x-direciton, we can assume that is always the case, which implies $\frac{d=}{dy}0$ and $u_1 = u_2 = 0$. The PV conservation for each layer becomes**

$$\frac{f}{H_1 - h_{(t=0)}} = \frac{\zeta_1 + f}{H_1 - h} = \frac{\frac{dv_1}{dx} + f}{H_1 - h} \quad (11)$$
$$\frac{f}{H_2 + h_{(t=0)}} = \frac{\zeta_2 + f}{H_2 + h} = \frac{\frac{dv_2}{dx} + f}{H_2 + h} \quad (12)$$

**The conservation statement above is true at all times, even while the flow is in the process of adjusting. However, we're interested in the steady-state solution, in which case we can also assume the layer velocities and interface/surface heights are related through geostrophy:**

$$v_1 = \frac{g}{f}\frac{d\eta}{dx} \tag{13}$$
$$v_2 = \frac{\rho_1}{\rho_2}\frac{g}{f}\frac{d\eta}{dx} + \frac{g'}{f}\frac{dh}{dx} \approx v_1 + \frac{g'}{f}\frac{dh}{dx} \tag{14}$$

**where for the last approximation we have assumed $\rho_1/\rho_2 \approx 1$. If we plug these into the two equations above (4 and 5), and start by working with the $x < 0$ part of the domain (so $h_{t=0} = h_0$) we get two equations and two unknowns:**

$$f(H_1 - h) = (\frac{dv_1}{dx} + f)(H_1 - h_0) \tag{15}$$
$$f(H_2 + h) = (\frac{dv_2}{dx} + f)(H_2 + h_0) = (\frac{dv_1}{dx} + \frac{g'}{f}\frac{d^2h}{dx^2} + f)(H_2 + h_0) \tag{16}$$

**where for the last step we've used the geostrophic equations to express $v_2$ in terms of $v_1$. Now we have two equations and two unknowns $(h, v_1)$. Eliminating $v_1$ from these equations gives the following equation for h(x):**

$$\frac{d^2h}{dx^2} = \frac{f^2}{g'}\left[\frac{(h-h_0)(H_1+H_2)}{(H_1-h_0)(H_2+h_0)}\right]$$

(b) Assuming the initial displacement is small $h_0 << H_1, H_2$, rewrite this equation in terms of the baroclinic Rossby radius: $a'^2 = g'H_{\text{eff}}/f^2$, $H_{\text{eff}} = (H_1H_2)/(H_1 + H_2)$
**Plugging in this assumption simplifies our differential equation to**

$$\frac{d^2h}{dx^2} = \frac{f^2}{g'}\left[\frac{(h-h_0)(H_1+H_2)}{H_1H_2}\right] = \frac{h-h_0}{a'^2}$$

(c) Solve for $h(x)$ in the entire domain and sketch the solution. You'll need to apply reasonable matching conditions at x=0.
**To solve let's temporarily change variables to $\tilde{h} = h - h_0$, so the equation becomes**

$$\frac{d^2\tilde{h}}{dx^2} = \frac{1}{a'^2}\tilde{h}$$

**the solution to which is**

$$\tilde{h} = Ae^{x/a'} + Be^{-x/a'} \qquad h = Ae^{x/a'} + Be^{-x/a'} + h_0$$

**Recall that we're still working in the domain $(x < 0)$ so we'll choose B=0. For the right side of our domain the analysis works exactly the same with all instances of $h_0$ replaced with $-h_0$, so the solution look like**

$$h = Ae^{x/a'} + h_0 \quad (x < 0)$$
$$h = Ce^{-x/a'} - h_0 \quad (x > 0)$$

**Requiring both h and its derivative (velocity) to be continuous across x=0 allows us to determine the integration constants and the final solution is**

$$h = h_0(1 - e^{x/a'}) \quad (x < 0)$$
$$h = h_0(e^{-x/a'} - 1) \quad (x > 0)$$

**Just as with the single-layer equation considered in class, the system as smoothed the initial discontinuity, in this case over the width of the baroclinic Rossby radius.**
**** Note that in class there was confusion about whether the nonlinear form of PV conservation (with a material instead of local derivative) was appropriate for a case like this, and I'm not sure if I explained it clearly. In a case like this we essentially ARE using the material derivative, by assuming that the 'chunk' of fluid to the right and left of the initial discontinuity conserves it's own PV as the system evolves. We are also assuming that the steady-state solution is in geostrophic balance, which we showed in the first problem above is an exact solution to the full shallow water equations. So we can solve the equations as done above and get the general result.**

(d) What are typical values of the barotropic and baroclinic rossby radii of deformation? (use $H_1 = 100m, H_2 = 1000m, f = 10^{-4}s^{-1}$)