1 Shallow water equations PV

There were some questions about the mathematical steps that I left out in class when deriving the full shallow water potential vorticity (PV) conservation equation. So here it is in full(?) glory: we start with our Boussinesq, hydrostatic equations

$$\begin{aligned}
\frac{du}{dt} + \vec{u}\cdot\nabla u &= -\frac{1}{\rho_0}\frac{dp}{dx} + fv + \nu\frac{d^2u}{dz^2} && (1)\\
\frac{dv}{dt} + \vec{u}\cdot\nabla v &= -\frac{1}{\rho_0}\frac{dp}{dy} - fu + \nu\frac{d^2v}{dz^2} && (2)\\
\frac{dp}{dz} &= -g\rho' && (3)\\
\nabla\cdot\vec{u} &= 0 && (4)
\end{aligned}$$

We're still going to assume a small Ekman number and constant density because we're considering motion of the entire fluid depth, and assume that pressure is related to the sea surface height. But we're not going to assume small amplitude flow. The result is:

$$\begin{aligned}
\frac{du}{dt} + \vec{u}\cdot\nabla u &= -g\frac{d\eta}{dx} + fv && (5)\\
\frac{dv}{dt} + \vec{u}\cdot\nabla v &= -g\frac{d\eta}{dy} - fu && (6)\\
\frac{dp}{dz} &= -g\rho' = 0 && (7)\\
\nabla\cdot\vec{u} &= 0' && (8)
\end{aligned}$$

As with the last time, note that nothing on the rhs of the momentum equations has any vertical dependence, so if there's no vertical structure to u or v to start with, there never will be. Which means two things. First, that we can still integrate continuity to get

$$\frac{D\eta}{Dt} = -(H+\eta)\nabla\cdot\vec{u_H} \qquad (9)$$

And second that the vertical derivatives in the advective term in the momentum equations go away, so we can write them as

$$\begin{aligned}
\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} &= -g\frac{d\eta}{dx} + fv && (10)\\
\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} &= -g\frac{d\eta}{dy} - fu && (11)\\
& && (12)
\end{aligned}$$

To get the general form of potential vorticity conservation we're going to do a bit of algebra. Let u_H but the vector of the horizontal component of velocity. We can write

$$\begin{aligned}
\frac{Du_H}{Dt} + f\hat{z}\times u_H &= -g\nabla\eta && (13)\\
\frac{du_H}{dt} + u_H\cdot\nabla u_H + f\hat{z}\times u_H &= -g\nabla\eta && (14)\\
& && (15)
\end{aligned}$$

Note that from vector identities we know that

$$\nabla(u \cdot u) = 2(u \cdot \nabla u) + 2u \times (\nabla \times u)$$

Dividing by two and plugging in we get

$$\begin{aligned}
\frac{du_H}{dt} + \frac{1}{2}\nabla(u_H \cdot u_H) - u_H \times (\nabla \times u_H) + f\hat{z} \times u &= -g\nabla\eta && (16)\\
\frac{du_H}{dt} + \frac{1}{2}\nabla(u_H \cdot u_H) - u_H \times \zeta + f\hat{z} \times u &= -g\nabla\eta && (17)\\
\frac{du_H}{dt} + \frac{1}{2}\nabla(u_H \cdot u_H) + \zeta \times u_H + f\hat{z} \times u &= -g\nabla\eta && (18)\\
\frac{du_H}{dt} + ([f+\zeta]\hat{z}) \times u_H &= -\nabla(g\eta + \frac{1}{2}|u_H|^2) && (19)\\
& && (20)
\end{aligned}$$

Now take the curl of the whole thing to get an equation for rate of change of zeta and note for the right hand side that the curl of the gradient of anything is zero.

$$\frac{d\zeta}{dt} + \nabla \times ([f+\zeta]\hat{z}) \times u_H = 0$$

Apply basic vector identities to this to get

$$\frac{d\zeta}{dt} + (f+\zeta)(\nabla \cdot u_H) - u_H(\nabla \cdot [f+\zeta]\hat{z}) + u_H \cdot \nabla[f+\zeta] - [f+\zeta]\hat{z} \cdot \nabla u_H = 0$$

We can simplify this a bit by realizing that 1) ζ is the curl of u_H and the divergence of the curl of anything is zero, and 2) the last term has vertical vector dotted into a horizontal one, so is also zero. So now we have

$$\frac{d\zeta}{dt} + (f+\zeta)(\nabla \cdot u_H) + u_H \cdot \nabla[f+\zeta] = 0$$

Combining the first and third terms into a material derivative gives us

$$\frac{D\zeta}{Dt} = -(f+\zeta)\nabla \cdot u_H \qquad (21)$$

Now look at our continuity equation from the last page and notice that is also has a divergence of horizontal velocity terms in it. So let's take the continuity equation, (9) and multiply both sides by $(f+\zeta)$ and then divide by $(H+\eta)$ to get

$$(f+\zeta)\frac{D\eta}{Dt} = -(f+\zeta)(H+\eta)\nabla \cdot u_H$$

$$\frac{(f+\zeta)}{(H+\eta)}\frac{D\eta}{Dt} = -(f+\zeta)\nabla \cdot u_H$$

which now has a rhs just like Equation (21) above. Combining them gives us

$$\begin{aligned}
\frac{D\zeta}{Dt} &= \frac{(\zeta+f)}{(H+\eta)}\frac{D\eta}{Dt} && (22)\\
\frac{D\zeta}{Dt} - \frac{(\zeta+f)}{(H+\eta)}\frac{D\eta}{Dt} &= 0 && (23)\\
& && (24)
\end{aligned}$$

Finally, notice that this looks like the chain rule expansion of the following

$$\frac{D}{Dt}\left[\frac{\zeta+f}{H+\eta}\right] = 0 \qquad (25)$$