Lecture 12

Ekman layers

12.1 The role of friction

The linearized time-dependent momentum equations, in the hydrostatic limit are:

$$\rho \left[\frac{\partial u}{\partial t} - fv \right] = -\frac{\partial p}{\partial x} + \rho A_v \frac{\partial^2 u}{\partial z^2} \tag{12.1}$$

$$\rho \left[\frac{\partial v}{\partial t} + fu \right] = -\frac{\partial p}{\partial y} + \rho A_v \frac{\partial^2 v}{\partial z^2} \tag{12.2}$$

$$0 = -\frac{\partial p}{\partial z} - \rho g \tag{12.3}$$

where A_v is the vertical eddy diffusivity (the turbulent equivalent to the kinematic viscosity), and the lateral friction terms have been ignored on the grounds of the thinness of the ocean.

12.2 Ekman layers driven by wind-stress at the surface

In this case we set the pressure gradient terms to zero:

For boundary conditions we require that the vertical gradient of the horizontal velocity be equal to the ration of the stress to the eddy viscosity

12.2.1 An infinitely deep ocean

$$
\begin{aligned}
-fv^\tau &= A_v \frac{\partial^2 u^\tau}{\partial z^2} \\
fu^\tau &= A_v \frac{\partial^2 v^\tau}{\partial z^2}
\end{aligned}
\tag{12.4}
$$

where the velocities are functions of z. The boundary conditions are:

$$
\frac{\partial u^\tau}{\partial z} = \tau_w^x/\rho A_v \quad \frac{\partial v^\tau}{\partial z} = \tau_w^y/\rho A_v \quad \text{at} \quad z = 0 \tag{12.5}
$$

$$
u^\tau = v^\tau = 0 \quad \text{at} \quad z = -\infty \tag{12.6}
$$

Equations 12.4 are a coupled set of second order ordinary differential equations for u and v as a function of z. If we define a length scale

$$
\delta_E = \sqrt{\frac{2A_v}{f}} \tag{12.7}
$$

and if the wind is blowing in the x-direction (to the east) alone, then it can be verified that the general solution is of the form:

$$
u^\tau(z) = \frac{\sqrt{2}\tau_w^x \delta_E}{2\rho A_v} e^{z/\delta} \cos\left(z/\delta - \pi/4\right) \tag{12.8}
$$

$$
v^\tau(z) = \frac{\sqrt{2}\tau_w^x \delta_E}{2\rho A_v} e^{z/\delta} \sin\left(z/\delta - \pi/4\right) \tag{12.9}
$$

(eg. Kundu & Cohen, 4th ed. pg 619) These solutions suggest that δ_E is the vertical distance over which the effect of friction is confined.

Talk about vertically integrated balance. For wind blowing in the x-direction only:

$$
[u] = \int_{-h}^{0} u \, dz = 0; \tag{12.10}
$$

Talk about vertically integrated balance. For wind blowing in the x-direction only:

$$
[v] = \int_{-h}^{0} u \, dz = -\frac{\tau_w^x}{\rho f}. \tag{12.11}
$$

so long as $h/\delta_E \gg 1$. Very different from no rotation case! With positive τ_w^y ,we have negative transport (offshore), leading to sea level change, and pressure gradients, so:

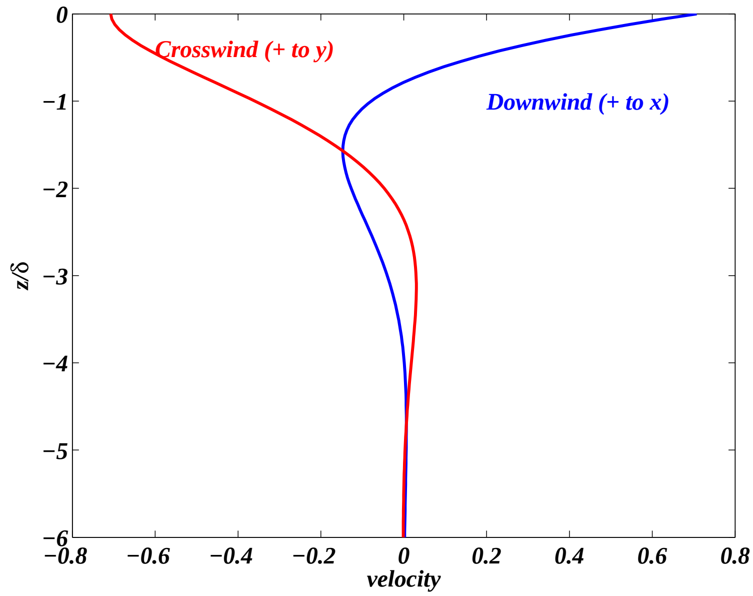

Figure 12.1: Vertical profiles of downwind (u) and crosswind (v) current driven by a wid stress directed along the x (toward the east) axis.

12.3 Ekman layers driven by pressure gradients

Consider a pressure gradient in the y direction only (ie the flow aloft is along the x axis, over a solid boundary:

$$\begin{aligned} \rho\left[\frac{\partial u}{\partial t} - fv\right] &= \rho A_v \frac{\partial^2 u}{\partial z^2} \qquad (12.12) \\ \rho\left[\frac{\partial v}{\partial t} + fu\right] &= -\frac{\partial p}{\partial y} + \rho A_v \frac{\partial^2 v}{\partial z^2} \end{aligned}$$

The boundary conditions are:

$$\frac{\partial u^p}{\partial z} = \frac{\partial v^p}{\partial z} = 0 \quad \text{at} \quad z = 0 \qquad (12.13)$$

$$u^p = v^p = 0 \quad \text{at} \quad z = -h \qquad (12.14)$$

In the case when h/δ_E is very large, it help to recast the problem in terms of $z' = z + h$, so that $z' = 0$ at the bottom, and increases to $z' = h$ at the surface.

The general solution to equation 12.12 can be written in terms of

$$U = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \tag{12.15}$$

representing the geostrophic velocity, as

$$u^p(z') = U\left(1 - e^{-z'/\delta_E}\cos z'/\delta_E\right) \tag{12.16}$$

$$v^p(z') = Ue^{-z'/\delta_E}\sin z'/\delta_E \tag{12.17}$$

In this case the bottom stress is related to the geostrophic velocity by:

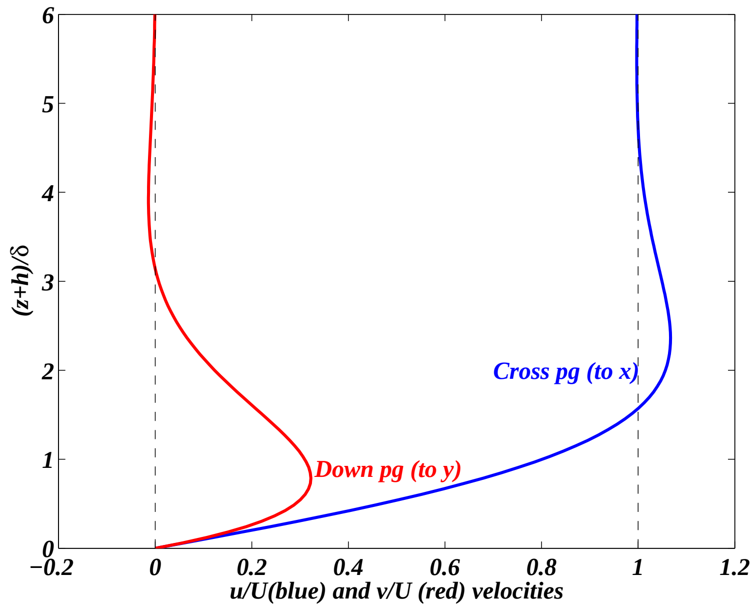

Figure 12.2: Vertical profiles of downwind (u) and crosswind (v) current driven by a wid stress directed along the x (toward the east) axis.

Describe Ekman pumping/suction

12.4 Ekman layers driven by combined stress and pressure gradients

In the most general case, when a fluid layer of depth h is driven by a combination of wind stress at the surface and pressure gradients, we can write the

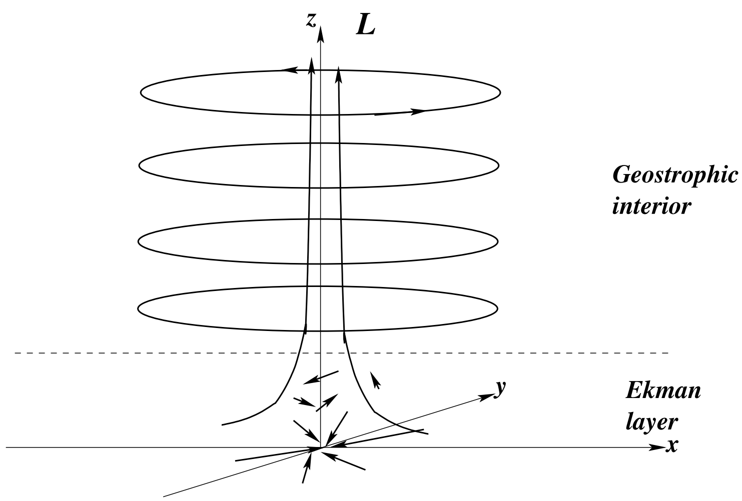

Figure 12.3: Above the bottom boundary the flow is in geostrophic balance: counterclockwise in the northern hemisphere around a low pressure center. Very close to the bottom, where the velocities and the Coriolis force are weak, the pressure gradient is balance by the bottom stress, parallel to the flow, so the flow is toward the center of the low pressure. This creates a convergence toward the center, with a net upward flow throughout the field (not confined to the Ekman layer)

general formula:

$$u = q_r^N \frac{\partial \eta}{\partial x} - q_i^N \frac{\partial \eta}{\partial y} + q_r^\tau \tau^x - q_i^\tau \tau^y \tag{12.18}$$

$$v = q_r^N \frac{\partial \eta}{\partial y} + q_i^N \frac{\partial \eta}{\partial x} + q_r^\tau \tau^y - q_i^\tau \tau^x \tag{12.19}$$

where q_r^N and q_i^N arre the real and imaginary parts of:

$$q^N = \frac{ig}{f}\left[1 - \frac{cos((1-i)z/\delta_E)}{cos((1-i)h/\delta_E)}\right] \tag{12.20}$$

and q_r^τ and q_i^τ arre the real and imaginary parts of:

$$q^\tau = -\frac{\sqrt{2}(1+i)\delta_E}{2\rho A_v}\frac{sin((1-i)(z+h)/\delta_E)}{cos((1-i)h/\delta_E)} \tag{12.21}$$

To find the transports, these expressions have to be integrated vertically: