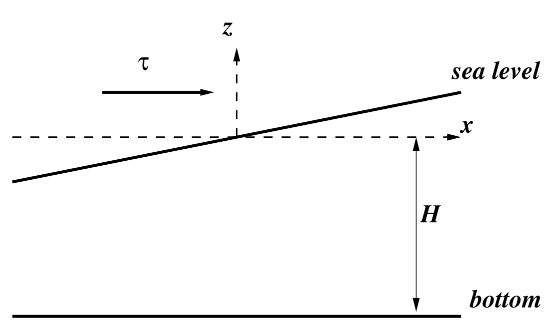
SIO 210 - Exercises - week 2 - due 9 Oct.

The following is a question I gave on the PO Departmental Exam in 2007. Obviously it is at the outer limit of what you can do, but there is so much to be gained from it, and you have done well enough in the first homework, that I think that with the added clues, you can do this with a little time. Don't hesitate to ask if you need help. Organize a little session where we can all talk about this together.

Think about an elongated lake over which the wind has been blowing for a long time (the flow is steady, any derivatives with respect to time can be taken to be zero). The direction of the wind is aligned with the long axis of the lake, which we take to be the x-axis, and to simplify things, we assume that the geometry doesn't change in the y-direction, what we call a two-dimensional flow, so any derivatives with respect to time can be taken to be zero as well. Finally assume (without much justification at this point) that the velocity is not a function of x (this is obviously not true near the ends of the lake)

The lake is large, but not so large that we need to include the Coriolis force (we will redo this problem in a few week including the Coriolis force, so you can understand how upwelling and downwelling come about). So we take a fixed (non-rotating) coordinate system. The length of the lake is $2L$ and the (constant) depth, H is much, much less than L. The origin of the coordinate system is at the center of the surface of the lake. The x coordinate runs downwind (i.e. the wind is aligned with the long axis of the basin), and z is measured positive up from the undisturbed surface. The (disturbed) surface is located at $z = \eta(x, y)$ and the bottom is located at $z = -H$.

To help you, here is a brief explanation of what happens. As soon as the wind is applied (before the steady state that you are asked to quantify is reached), the stress imposed by the wind forces the surface to flow downwind (this is the definition of a fluid: stress causes continuous deformation). In the very early moment, this causes water to pile up at the downwind end of the basin, while the water level is slightly lower at the upwind end. As soon as this happens a positive sea level gradient (sea level increases downwind) is set up. That sea level forces a return flow upwind, and steady state is reached when the upwind flow directly driven by the wind balances the downwind flow due to the pressure gradient, as in this picture:



1. Write the full momentum and mass conservation, and simplify to show that when the vertical momentum equation is hydrostatic and lateral friction terms (e.g. $\partial^2 u/\partial x^2$ very small) the x momentum equation reduces to:

$$0 = -g\frac{\partial \eta}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} = -g\eta_x + \nu u_{zz} \tag{1}$$

   The terms after the second equal sign are shorthand symbols. This is a differential equation and so requires boundary conditions, which we haven't said much about yet. For this problem, take the bottom boundary condition to be $u(z = -H) = 0$ (called the no-slip boundary condition), whereas at the surface, we say that the stress in the water is equal to the stress imposed by the wind:

$$\rho\nu\frac{\partial u}{\partial z} = \tau_w \quad \text{at} \quad z = 0 \tag{2}$$

   Integrate the differential equation to find a relation between u, z and the alongshore sea level gradient η_x.

2. Write an expression relating the flux $[u]$ (defined below) to sea level gradients and the imposed wind stress. This expressions is of the form:

$$[u] = \int_{-h}^{0} u dz = A(H)\tau_w^x + B(H)\eta_x \tag{3}$$

3. Think now in term of mass conservation. What is the flux of mass $[u]$ at the (closed) end of the basin? What do you deduce about the flux at any location. Use this result to relate η_x to τ_s. This is a *key* point: the sea level gradient sets itself up in response to the wind stress, in such a way to force the flux to be zero.

4. Finally, knowing η_x as a function of τ_s, show that

$$u(z) = \frac{\tau_w}{\mu}\left[\frac{z^2}{H} + z\right] \tag{4}$$

   Plot $\mu u/(\tau_w H)$ as a function of z/H. Do you see how near the surface, the flow is *downwind* while at depth the flow is *upwind*?