GENERAL QUESTIONS. With regard to the little homeworks asked last time, note that \((i^i)^i\) is not the same as \(i^{(i^i)}\) so that when U were asked to evaluate \(i^i\) you could justifiably have claimed that the question was ambiguous.

Several questions Monday emphasized the importance of understanding expressions such as \(\int ((1/x)dx)\) as a single symbol and not trying to deconstruct them too deeply into their individual components, a process that could erroneously lead to the expression \(\int (d)\), which is itself not a standard expression but might in the heat of battle mistakenly be interpreted as \(\int (dx)\). In instances of doubt (which do arise when the notion of derivative is extended to e.g. functional derivative) I find it helpful to always remember the elementary calculus definitions of \(df/dx\) and \(\int (f dx)\).

Several people effectively commented "I took this stuff several years ago, I couldn’t remember it then and I can’t remember it now. What to do?" For me, the answer has been to break the material into problems/topics short enough that I can explain each in one or two pages, and then in idle moments try to reproduce the argument without looking at the pages. That is a lot easier than starting with the course outline and trying to reproduce any item at once from memory. Remember, the vignettes I’ve chosen are THE most useful and common ones, and there are not too many of them.

PARTICULAR PROBLEMS.

(1) Simple Harmonic Oscillator with Damping. Newton’s law for the motion of the particle says \(ma = -kx - Rv\) where \(v\) is the velocity of the particle and \(R\) is a damping coefficient. For simplicity suppose the particle starts at \(x(0) = 1\) with zero initial speed \(dx/dt|_{t=0} = 0\).

Go to the notation \(x_t = dx(t)/dt,\) etc. We have to solve the ode \(x_{tt} + rx + \omega^2x = 0\) subject to the initial conditions \(x(0) = 1, x_t(0) = 0\) in which \(\omega^2 = k/m\) and \(r = R/m\). The equation (i) is linear (neither \(r\) nor \(\omega\) depend on \(x\)), (ii) it has constant coefficients (neither \(r\) nor \(\omega\) depend on \(t\)), (iii) it is homogeneous (no forcing on the right hand side), (iv) it is second order and has two initial conditions. Because the eq is second order it has two independent ("mathematically different") solutions, because it has constant coefficients they are of the form \(Ae^{at}\), \(Be^{bt}\) in which \(A\), \(B\) are constants and \(a\), \(b\) must be two DIFFERENT constants. Because there is no forcing, the
solution has the form $x(t) = Ae^{at} + Be^{bt}$. If $Ae^{at}$ is to solve the ode, then $a^2 + ra + \omega^2 = 0$. Similarly, $b^2 + rb + \omega^2 = 0$. Thus (quadratic formula) $a = (1/2)(-r + \sqrt{(r^2 - 4\omega^2)})$ while $b = (1/2)(-r - \sqrt{(r^2 - 4\omega^2)})$. Note that $a, b$ differ by the choice of sign, if the same sign were chosen for both then $e^{at}$ and $e^{bt}$ would be the same functions, not independent.

The solution is easiest if we suppose that $r << \omega$. In that case the quadratic formula results reduce to $a = -r/2 + i\omega$, $b = -r/2 - i\omega$. Somone asked effectively "how do you know you can keep the $−$ quadratic formula results reduce to $a$ and $b$ differ by the choice of sign, if the same sign were chosen for both then $e^{at}$ and $e^{bt}$ would be the same functions, not independent.

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Thus $x(t) = (Ae^{i\omega t} + Be^{-i\omega t})e^{-rt/2}$. This nearly solves the ode (not-perfectly. because of the approximation $r << \omega$) but doesn’t yet satisfy the initial conditions. To satisfy them we need to choose $A, B$ such that $A + B = 1$, $i\omega(A - B) - (r/2)(A + B) = 0$. I’ll spare us all the algebra (but you’d better check it if this is part of an airplane design), we get, after Euler’s theorem, $x(t) = (cos(\omega t) - (r/\omega)sin(\omega t))e^{-rt/2}$.

Clearly the solution oscillates with period $2\pi/\omega$ and the oscillations decay with time scale $2/r$. This solution only works for small $r$, in fact for $r << \omega$. From the expressions for $a, b$ above we see that if $r > \omega$ then $a, b$ are entirely real and there are no oscillations at all.

Finally notice that you can solve a general problem of the form $b_n d^n x/dx^n + b_{n-1} d^{n-1} x/dx^{n-1} ... b_1 x = 0$ with the Ansatz $x(t) = A_1 e^{a_1 t} + A_2 e^{a_2 t} ... + A_n e^{a_n t}$. Each of the $a...$ satisfy the same $n^{th}$ order algebraic equation $b_n a_n^n + b_{n-1} a_{n-1}^{n-1} ... + b_1 = 0$. Pay close attention to the meaning of the subscripts here, the important thing is to not use $n$ both to index the coefficients $b...$ and the roots $a...$.

(2) Falling Body With Friction. The location of the body is $z(t)$. $a = F/m$, with friction, says $z_{tt} + rz_t = -g$. This is a second order ode, the initial conditions are $z(0) = 0$ (body starts at vertical location $z = 0$ and $Z(0) = 0$ (body starts with zero initial vertical velocity.

With no friction ($r = 0$) the solution was $z(t) = -(1/2)gt^2$. With or without friction, this is an inhomogeneous or forced problem (the forcing is $−g$ on the right hand side, if $g = 0$ the equation would be homogeneous. Because the equation is inhomogeneous the solution is of the form $z(t) = Ae^{at} + Be^{bt} + P$ in which the first two are solutions of the homogeneous equation (sometimes called the free solution) and $P$ is ANY solution of the forced equation we can manage to find (often called the particular solution). $P$ is not unique, that is, you can find many different $P$, but all you need
is one. To find it in this case think about what happens after very long
times; the particle falls at a constant terminal velocity so that \( z_{tt} = 0 \) and
\( Z_t \) is constant. In that case, one solution of the full forced equation is clearly
\( P(t) = -gt/r \); falling at uniform speed. The free solutions satisfy
\( z_{tt} + rz_t = 0 \) and you can find them with the Ansatz \( z(t) = Ae^{rt} + Be^{-rt} \),
but it is easier to guess them. Just look at the free equation \( z_{tt} + rz_t = 0 \).
One solution is \( Z = A \), a constant. Another is found by rewriting the
equation as \( z_{tt} = -rz_t \). This is the decay equation, so we know the solution
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formula using parts integration (as opposed to reviewing all your old notes!).