Non-geostropic Baroclinic Stability

Stone, PH. On Non-Geostrophic Baroclinic Stability, JAS 1966

Stone, PH. On Non-Geostrophic Baroclinic Stability: Part II, JAS 1966

Haine, TWN. & Marshall, J. Gravitational, Symmetric and Baroclinic Instability of the Ocean Mixed Layer, JPO 1998


Ruth Musgrave
The background state is in thermal wind balance

\[ N^2 = -\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z} = \bar{b}_z \]

\[ M^2 = -\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial y} = \bar{b}_y \]

\[ f \frac{dU}{dz} = \frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial y} = -M^2 \]

\[ U = -\frac{M^2 z}{f} \]
Primitive equations (not QG!), linearized around the background state

Background + perturbations:

\[ u' = u + U \quad v' = v \quad w' = w \]
\[ b' = b + \bar{b} \quad p' = p + \bar{p} \]

Linearized set of equations:

\[ u_t + Uu_x + wU' - fv = -p_x \]
\[ v_t + Uv_x + fu = -p_y \]
\[ 0 = -p_z + b \]
\[ b_t + Ub_x + v\bar{b}_y + w\bar{b}_z = 0 \]
\[ u_x + v_y + w_z = 0 \]
Primitive equations (not QG!), linearized around the background state

![Diagram showing a box with arrows indicating the directions of x, y, and z, and a vector U(z) pointing along the z-axis.]

Non-dimensionalize:

\[
\begin{align*}
(x^*, y^*) &= \frac{U}{f} (x, y) \quad z^* = H_0 z \\
(u^*, v^*) &= U (u, v) \quad w^* = H_0 f w \\
b^* &= N^2 H_0 b \quad p^* = N^2 H_0^2 p \\
t^* &= \frac{1}{f} t \quad \sigma^* = f \sigma \\
H^* &= \frac{1}{H_0} H \\
\end{align*}
\]

Assume a solution like:

\[e^{i(kx+ly+\sigma t)}\]

Combine to form eigenvalue equation:

\[
[1 - (\sigma + kU(z))^2] w'' - 2 \left[ \frac{k}{\sigma + kU(z)} - il \right] w' - \left[ Ri(k^2 + l^2) - \frac{2ikl}{\sigma + kU(z)} \right] w = 0
\]

Boundary conditions \( w(0) = w(1) = 0 \)

\[
Ri = \frac{N^2 H_0^2}{U^2} = \frac{N^2 f^2}{M^4}
\]
Instabilities of the non-geostropic eigenvalue equation

\[
[1 - (\sigma + kU(z))^2]w'' - 2 \left[ \frac{k}{\sigma + kU(z)} - il \right] w' - \left[ Ri(k^2 + l^2) - \frac{2ikl}{\sigma + kU(z)} \right] w = 0
\]

Boundary conditions \( w(0) = w(1) = 0 \)

- Gravitational (Ri < 0)
- Kelvin-helmholtz (0 < Ri < 0.25)
- Symmetric (k=0, Ri < 1)
- Baroclinic (l=0, Ri > 1)
- Also allows inertia-gravity waves!
No zonal derivatives: symmetric instability

Let $k = 0$

Eigenvalue equation: $[1 - \sigma^2]w'' + 2ilw' - \text{Ri} l^2 w = 0$

Assume solution: $w = w_0 e^{imz}$

$$m^2 + \frac{2l}{1 - \sigma^2} m + \frac{\text{Ri} l^2}{1 - \sigma^2} = 0$$

$$m = \frac{l}{\sigma^2 - 1} \pm \sqrt{\frac{l^2}{(\sigma^2 - 1)^2} + \frac{\text{Ri} l^2}{\sigma^2 - 1}}$$

$$w = w_0 (ae^{im+z} + be^{im-z})$$

Boundary conditions: $w = 0$ at $z = 0, 1$

$\implies m_+ - m_- = 2n\pi$ and $a = -b$
Symmetric instability: growth rates

Boundary conditions impose

\[ \eta \pi = \sqrt{\frac{l^2}{(\sigma^2 - 1)^2} + \frac{Ril^2}{\sigma^2 - 1}} \]

Growth rate:

\[ \sigma^2 = 1 + \frac{Ril^2}{2n^2\pi^2} \pm \sqrt{\frac{Ri^2l^4}{4n^4\pi^4} + \frac{l^2}{n^2\pi^2}} \]

With our assumed solution:

\[ e^{i(kx+ly+\sigma t)} \]

\[ \sigma_+^2 \text{ is always stable} \]

\[ \sigma_-^2 \text{ is sometimes stable} \]

Instability condition:

\[ Ri < 1 - \frac{n^2\pi^2}{l^2} \]
There are stable and unstable solutions

\[
\sigma^2 = 1 + \frac{Ril^2}{2n^2\pi^2} \pm \sqrt{\frac{Ri^2l^4}{4n^4\pi^4} + \frac{l^2}{n^2\pi^2}}
\]

Growth rates are fastest for lowest modes
Symmetric instability: growth rates

\[ \sigma^2_- = 1 + \frac{Ri l^2}{2n^2 \pi^2} - \sqrt{\frac{Ri^2 l^4}{4n^4 \pi^4} + \frac{l^2}{n^2 \pi^2}} \]
Symmetric instability: eigenmodes

\[ \psi = \text{Re}[ (e^{im+z} - e^{im-z})e^{il0y} ] \]
\[ \psi, n = 1, Ri = 0.5 \]

Instability forms as rolls aligned with the background flow
Symmetric instability: a mixed gravitational-centrifugal instability

1. Gravitational instability: source is PE

\[ \frac{Dw}{Dt} = \frac{D^2}{Dt^2}(\delta z) = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} \]

\[ = g \left( \frac{\rho_0 - \rho}{\rho} \right) \]

\[ = -N^2 \delta z \]

\[ \frac{\partial p_0}{\partial z} = -\rho_0 g \]

\( N^2 > 0 \) stable

\( N^2 = 0 \) neutral

\( N^2 < 0 \) unstable

Haine & Marshall, JPO 1998 following Holton
Symmetric instability:
a mixed gravitational-centrifugal instability

2. Centrifugal instability: source is KE

Lines of constant M for \( u_g = u_g(y) \)

\[
\frac{Dv}{Dt} = \frac{D^2}{Dt^2}(\delta y) = f(u_g - u)
\]

\[
= -f \left( f - \frac{\partial u_g}{\partial y} \right) \delta y
\]

\[
= -f \frac{\partial M}{\partial y} \delta y
\]

Absolute momentum: \( M = fy - u_g \)

\[
f \frac{\partial M}{\partial y} > 0 \text{ stable}
\]

\[
f \frac{\partial M}{\partial y} = 0 \text{ neutral}
\]

\[
f \frac{\partial M}{\partial y} < 0 \text{ unstable}
\]

following Holton
Symmetric instability:
a mixed gravitational-centrifugal instability

3. Symmetric instability

\[ s = \frac{z_2 - z_1}{y_2 - y_1} \]

\[ s_b = -\frac{M^2}{N^2} \]
Symmetric instability: a mixed gravitational-centrifugal instability

3. Symmetric instability

\[ \Delta PE = \rho_0 (z_2 - z_1)(b_2 - b_1) \]
\[ = \rho_0 (z_2 - z_1)[M^2(y_2 - y_1) + N^2(z_2 - z_1)] \]
\[ = \rho_0 N^2 \Delta y^2 s(s - s_b) \]

\[ s = \frac{z_2 - z_1}{y_2 - y_1} \]
\[ s_b = -\frac{M^2}{N^2} \]
\[ \delta b = \bar{b}_y \delta y + \bar{b}_z \delta z \]
Symmetric instability: a mixed gravitational-centrifugal instability

\[ \Delta PE = \rho_0 (z_2 - z_1)(b_2 - b_1) = \rho_0 (z_2 - z_1)[M^2(y_2 - y_1) + N^2(z_2 - z_1)] = \rho_0 N^2 \Delta y^2 s(s - s_b) \]

\[ \Delta KE = (1/2)\rho_0 \left[ \left\{ u_1 + f(y_2 - y_1) \right\}^2 + \left\{ u_2 - f(y_2 - y_1) \right\}^2 - u_1^2 - u_2^2 \right] = \rho_0 (y_2 - y_1)^2 f \left( f - s \frac{\partial u}{\partial z} \right) = \rho_0 \Delta y^2 [f^2 - N^2 ss_b] \]
Symmetric instability:
a mixed gravitational-centrifugal instability

3. Symmetric instability

$$\Delta PE = \rho_0 (z_2 - z_1)(b_2 - b_1)$$
$$= \rho_0 (z_2 - z_1)[M^2(y_2 - y_1) + N^2(z_2 - z_1)]$$
$$= \rho_0 N^2 \Delta y^2 s(s - s_b)$$

$$\Delta KE = (1/2)\rho_0 \left[\{u_1 + f(y_2 - y_1)\}^2 + \{u_2 - f(y_2 - y_1)\}^2 - u_1^2 - u_2^2\right]$$
$$= \rho_0 (y_2 - y_1)^2 f \left(f - s \frac{\partial u}{\partial z}\right)$$
$$= \rho_0 \Delta y^2 [f^2 - N^2 ss_b]$$

$$\Delta (KE + PE) = \rho_0 \Delta y^2 [f^2 (1 - 1/Ri) + N^2 (s - s_b)^2]$$

For instability, energy change must be -ve, so $$Ri < 1$$

Haine & Marshall, JPO 1998
Symmetric instability: a mixed gravitational-centrifugal instability

3. Symmetric instability

- Slopes of absolute momentum can be less than isopycnal slopes in regions of weak vertical stratification and strong horizontal stratification.
- Also known as “isentropic inertial instability”.

*Holton*
Non-geostrophic baroclinic instability is qualitatively similar to Eady instability: \( l=0 \)

\[
[1 - (\sigma + kU)^2] \partial_{zz} w - \frac{2k}{\sigma + kU} \partial_z w - R_i k^2 w = 0
\]

Unstable for \( \text{Ri} > 1 \)

Fig. 4. A snapshot of the temperature perturbation for the fastest-growing mode.

Unstable for \( \text{Ri} > 1 \)
Stone: Non-geostrophic stability with perturbations in 2 directions

\[ [1 - (\sigma + kU(z))^2] w'' - 2 \left[ \frac{k}{\sigma + kU(z)} - il \right] w' - \left[ Ri(k^2 + l^2) - \frac{2ikl}{\sigma + kU(z)} \right] w = 0 \]

Boundary conditions \[ w(0) = w(1) = 0 \]

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**Fig. 5.** Schematic diagram of the \( k - \lambda \) plane, showing the various regions where the analyses of Sections 2 through 5 apply. The locations of the three local maxima in the growth rate are indicated schematically by x's, and the range of values of Ri for which these maxima exist are also given.

Stone, JAS 1966
Stone: Non-geostrophic stability with perturbations in 2 directions

Symmetric, $k=0$: $\mathcal{R}_i < 1$

Baroclinic, $l=0$: $\mathcal{R}_i > 1$

Stone, JAS 1966
Non-geostrophic instabilities arising from 2D perturbations

**Ri = 2:**

~Baroclinic. Max growth rate at $l=0$.

Single unstable mode.

~ Eady case ($l=0$)

**Fig. 1.** Growth rates vs $k$ when **Ri** = 2.

Stone, JAS 1966
Non-geostrophic instabilities arising from 2D perturbations

**Ri = 0.92:**
More than one vertical mode unstable. Symmetric instability appears.

- Symmetric instability
- Baroclinic instability

Secondary unstable modes have smaller growth rates and smaller scales

**Fig. 3.** Growth rates for the most unstable mode when Ri = 0.92.
Non-geostrophic instabilities arising from 2D perturbations

**Ri = 0.5:**
Symmetric has largest growth rate

![Graph showing growth rates for the most unstable mode when Ri=0.5.](image)

**Fig. 4.** Growth rates for the most unstable mode when Ri=0.5.

Stone, JAS 1966
Summary

• A flow in thermal wind balance is subject to a range of instabilities when the full primitive equations are considered
  • $0 < Ri < 0.25$ -> Kelvin Helmholtz
  • $0.25 < Ri < 0.95$ -> Symmetric
  • $0.95 < Ri$ -> Baroclinic
• Baroclinic instability is not significantly modified in full equations compared to QG for $Ri >> 1$
• Symmetric instability can be thought of as a combined centrifugal and gravitational instability