On the Changes in the Spectral Distribution of Kinetic Energy for Twodimensional, Nondivergent Flow

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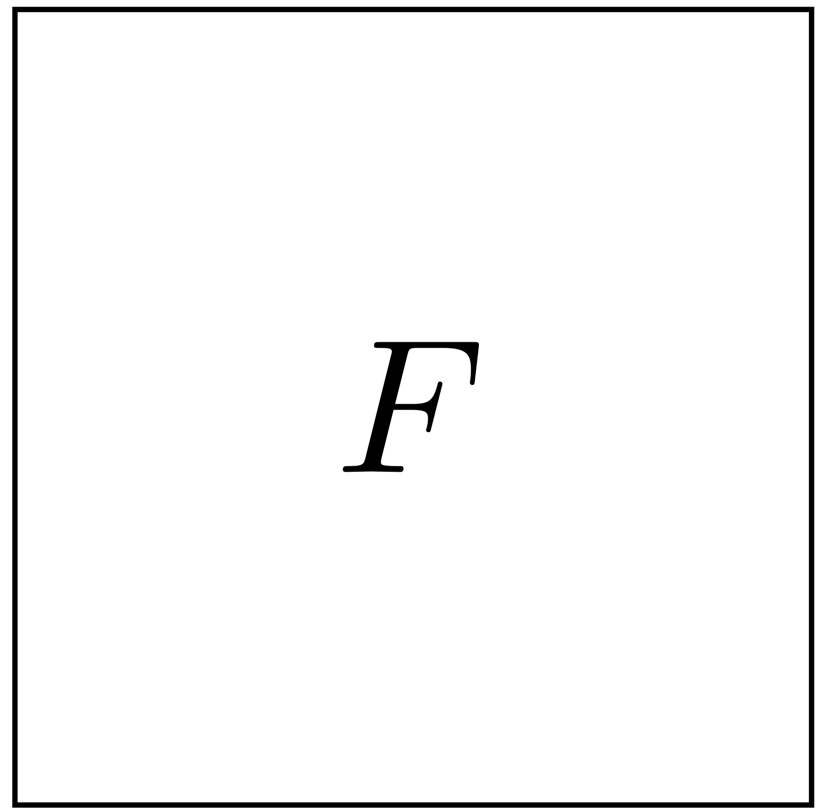
Important Questions

- Does energy flow from small to large scales, from large to small scales, or both?

Navier-Stokes ~~on a sphere~~ (no rotation)

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\left(\frac{p}{\rho}\right) - (\mathbf{v}\cdot\nabla\mathbf{v})$$

$$\mathbf{v} = -\nabla\psi \times \mathbf{k}$$



Doubly-periodic

Conservation of energy

$$\int_F (\nabla\psi)^2 \, dF = \mathrm{const}$$

Eliminating $\frac{p}{\rho}$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\left(\frac{p}{\rho}\right) - (\mathbf{v}\cdot\nabla\mathbf{v})$$

$$\mathbf{v} = -\nabla\psi \times \mathbf{k}$$

$$\frac{\partial \nabla^2\psi}{\partial t} = -\mathbf{v}\cdot\nabla\nabla^2\psi$$

An equation for the rate of change of vorticity

$$\times\frac{1}{2}\nabla^2\psi$$

$$\frac{\partial(\nabla^2\psi)^2}{\partial t} = -\frac{1}{2}\nabla^2\psi\mathbf{v}\cdot\nabla\nabla^2\psi$$

$$\int_F$$

Integrate over the surface of the sphere F

$$\frac{\partial}{dt}\int_F(\nabla^2\psi)^2 dF = 0$$

enstrophy is the square of vorticity

$$\int_F(\nabla^2\psi)^2\, dF = \mathrm{const}$$

Conservation of enstrophy

2 conserved quantities

$$\int_F (\nabla \psi)^2 \, dF = \mathrm{const}$$

$$\int_F (\nabla^2 \psi)^2 \, dF = \mathrm{const}$$

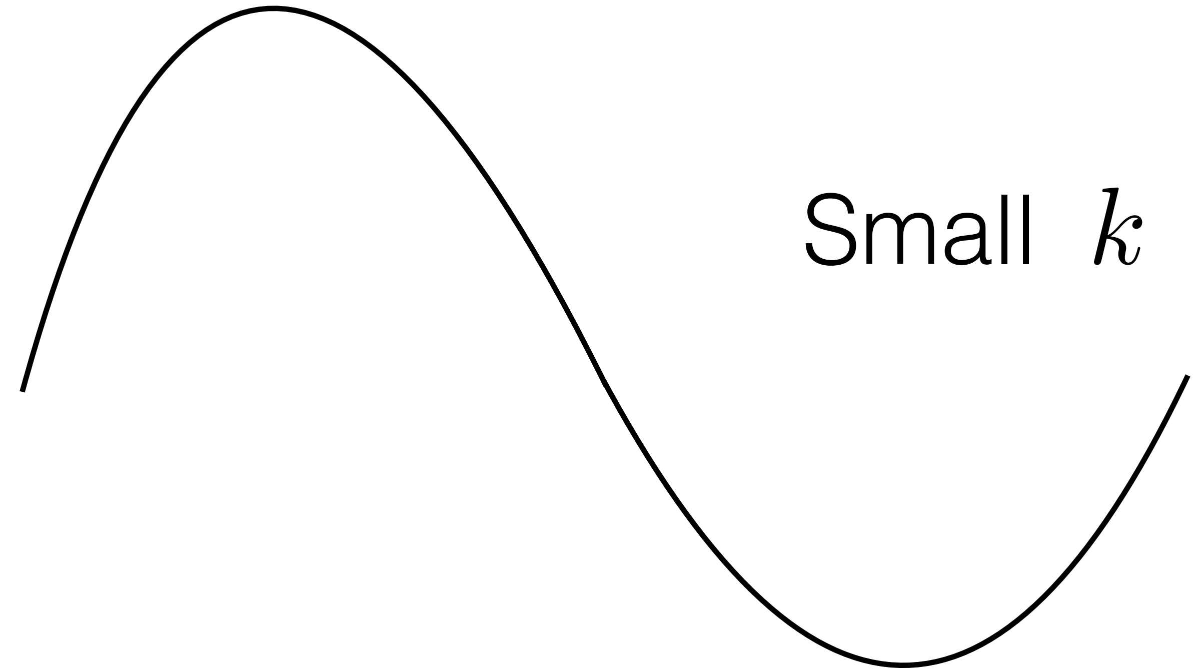
Spectral decomposition into modes

$$\psi = \sum_{k=1}^{\infty} \psi_k$$

$$\psi_k = \Psi_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

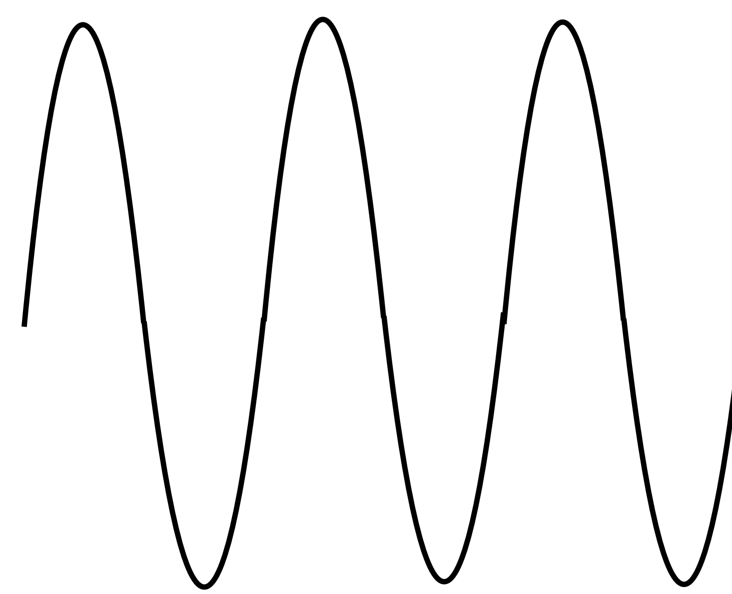
$$\nabla^2 \psi_k + k^2 \psi_k = 0$$

$$\nabla^2 \psi = -\sum_{k=1}^{\infty} k^2 \psi_k$$



Small k

Long spatial scales



Large k

Short spatial scales

Conservation of energy

$$\int_F (\nabla\psi)^2\, dF = \text{const}$$

$$\int_F (\nabla\psi)^2\, dF = \sum_{k=1}^{\infty} \int k^2 \psi_k^2\, dF$$

$$H_k \equiv \int_F k^2 \psi_k^2\, dF$$

$$\sum_{k=1}^{\infty} H_k = \text{const}$$

Define change in energy at wavenumber k

$$\Delta H_k = H_k|_t - H_k|_{t=0}$$

$$\sum_{k=1}^{\infty} H_k|_t = \sum_{k=1}^{\infty} H_k|_{t=0}$$

$$\sum_{k=1}^{\infty} \Delta H_k = 0$$

Conservation of enstrophy

$$\int_F (\nabla^2\psi)^2\, dF = \text{const}$$

$$\int_F (\nabla^2\psi)^2\, dF = \sum_{k=1}^{\infty} \int k^4 \psi_k^2\, dF$$

$$H_k \equiv \int_F k^2 \psi_k^2\, dF$$

$$\sum_{k=1}^{\infty} k^2 H_k = \text{const}$$

Define change in energy at wavenumber k

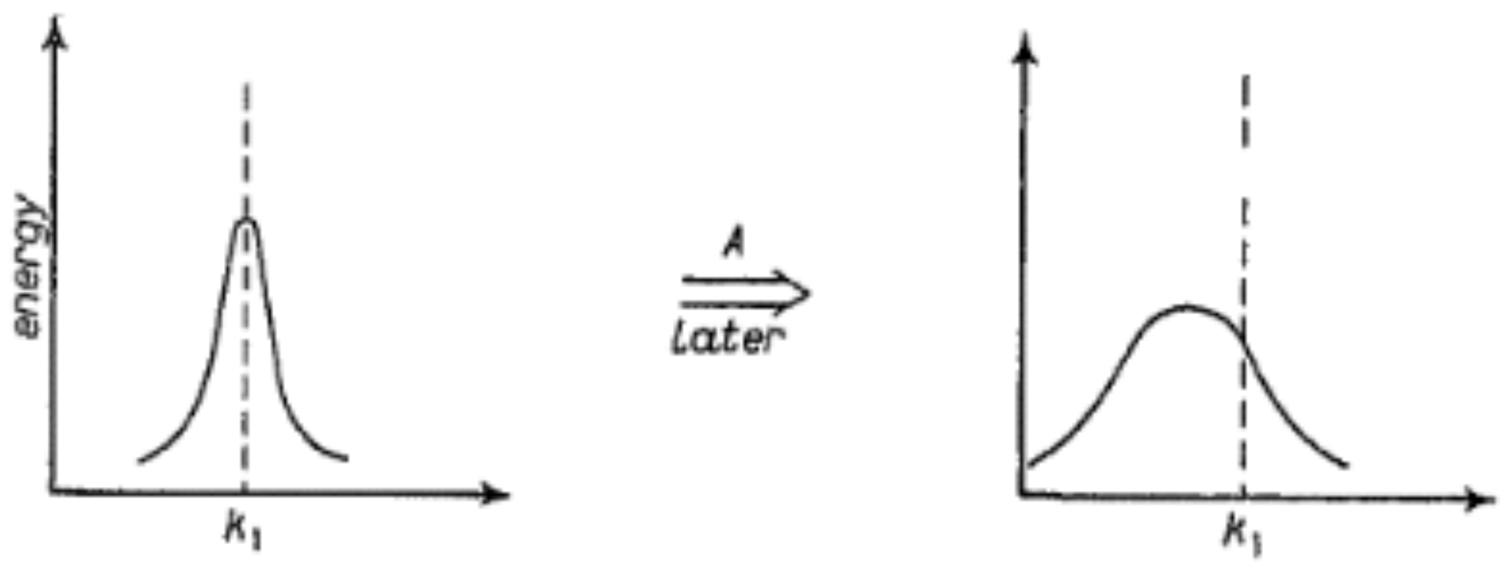
$$\Delta H_k = H_k|_t - H_k|_{t=0}$$

$$\sum_{k=1}^{\infty} k^2 H_k|_t = \sum_{k=1}^{\infty} k^2 H_k|_{t=0}$$

$$\sum_{k=1}^{\infty} k^2 \Delta H_k = 0$$

Energy spreads out

- Energy begins at wavelength k_1
- We then assume that it spreads out



(Salmon 1982)

- Which direction does it go?

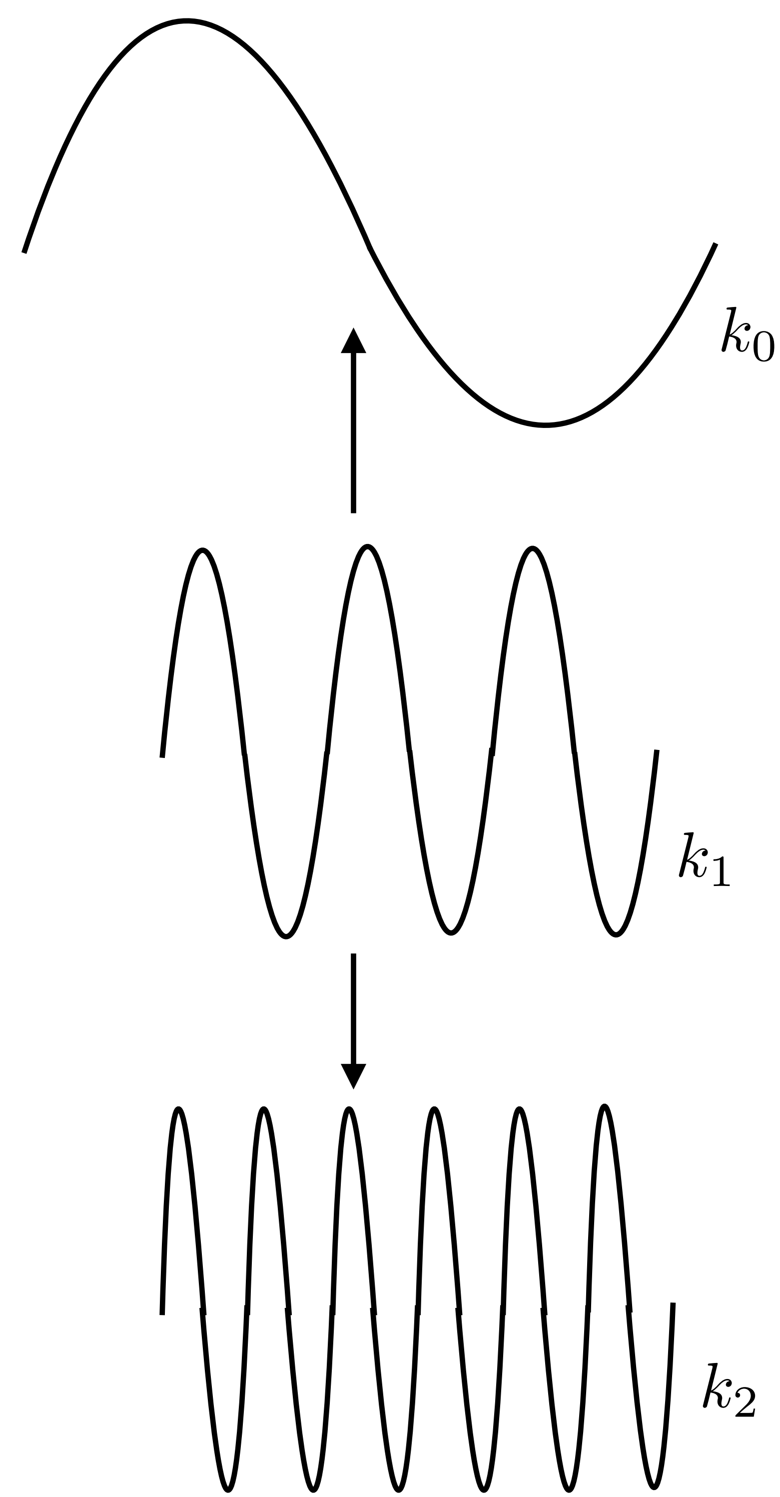
Using 3 scales as an example

$$\Delta H_k = H_k|_t - H_k|_{t=0}$$

$$\Delta H_{k_0} + \Delta H_{k_1} + \Delta H_{k_2} = 0$$ energy conservation

$$k_0^2 \Delta H_{k_0} + k_1^2 \Delta H_{k_1} + k_2^2 \Delta H_{k_2} = 0$$ enstrophy conservation

$$k_0 < k_1 < k_2$$



$$\Delta H_{k_0} = - \frac{k_2^2 - k_1^2}{k_2^2 - k_0^2} \Delta H_{k_1}$$

$$\Delta H_{k_2} = - \frac{k_1^2 - k_0^2}{k_2^2 - k_0^2} \Delta H_{k_1}$$

$$\frac{k_1^2 - k_0^2}{k_2^2 - k_0^2}, \frac{k_2^2 - k_1^2}{k_2^2 - k_0^2} > 0$$

If the kinetic energy flows from the intermediate scale to both short and long scales

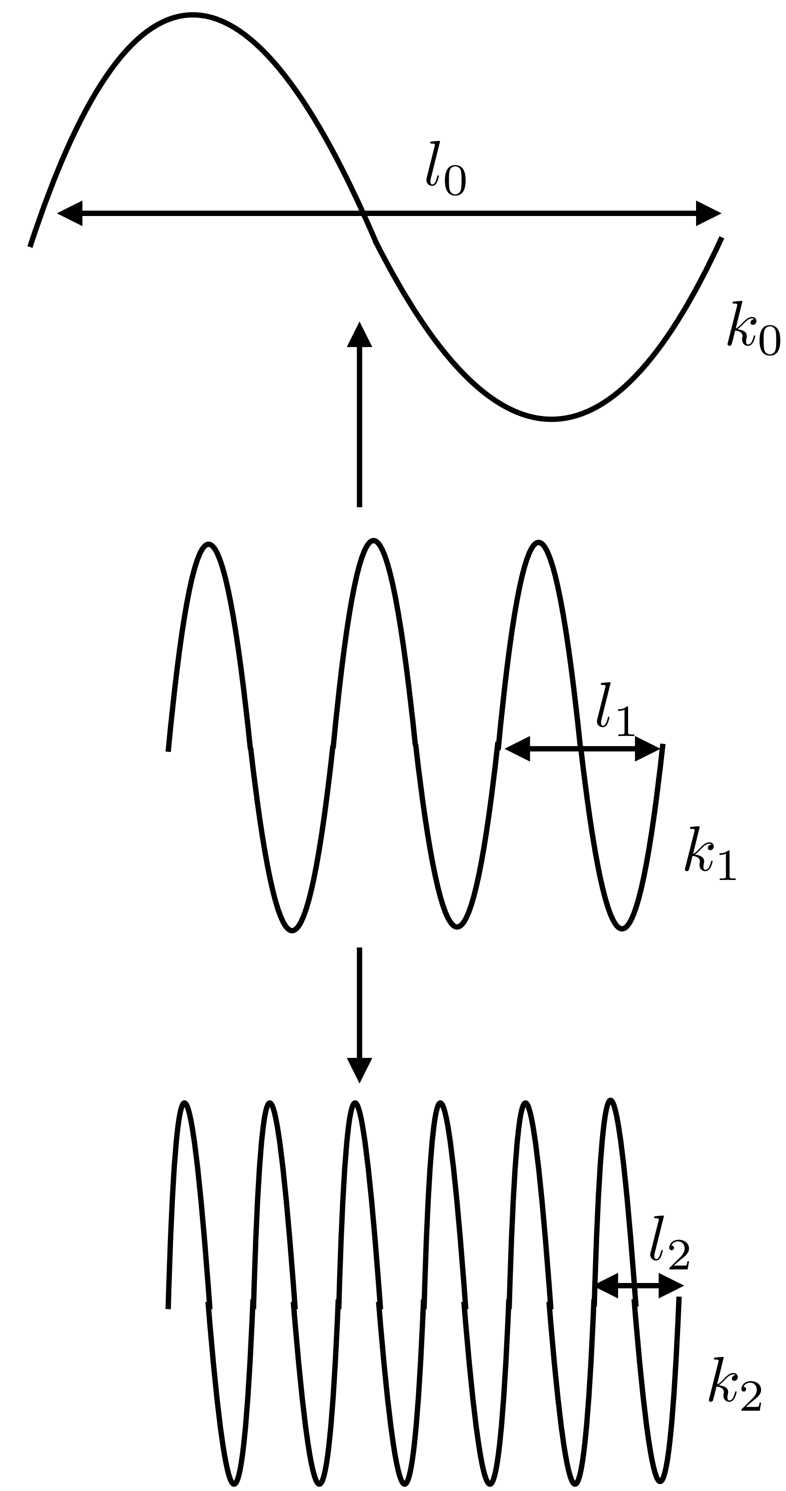
$$\frac{\Delta H_{k_0}}{\Delta H_{k_2}} = \frac{k_2^2 - k_1^2}{k_2^2 - k_0^2} > 1$$

The change in kinetic energy change will be smallest for the short scales

So what does this mean?

For example if: $\frac{k_2^2}{k_1^2} = 4$ $\frac{k_1^2}{k_0^2} = 4$

$\frac{l_1}{l_2} = 2$ $\frac{l_0}{l_1} = 2$



Then $\frac{\Delta H_{k_0}}{\Delta H_{k_2}} = 4$

i.e. 4 times as much energy flows to large scales than small scales

These results can be generalized to include all scales (rather than just 3).

ε is the rate of energy transfer past k and has units $\left[L^2/T^3\right]$

η is the rate of enstrophy transfer past k and has units $\left[1/T^3\right]$ $\quad Z = \int_0^\infty dk\, k^2\, E(k),$

$E(k)$ is the energy spectrum and has units $\left[L^3/T^2\right]$

Energy is injected at k_1

k_D is the scale at which viscosity becomes important and k has units $[1/L]$

If all energy with wave numbers higher than k_1 is removed at higher wave numbers than k_D

$$\eta > k_D^{\;2}\varepsilon\,.$$

However, if viscosity $\nu \to 0$

$$k_D \to \infty, \qquad \varepsilon \to 0$$

Therefore at high wavenumbers ε is not important for determining the shape of the spectrum.
The only important variables are η and k

$$E(k) = C_1\, \eta^{2/3} k^{-3}, \qquad\qquad k_D \sim \left(\frac{\eta}{\nu^3}\right)^{1/6}$$

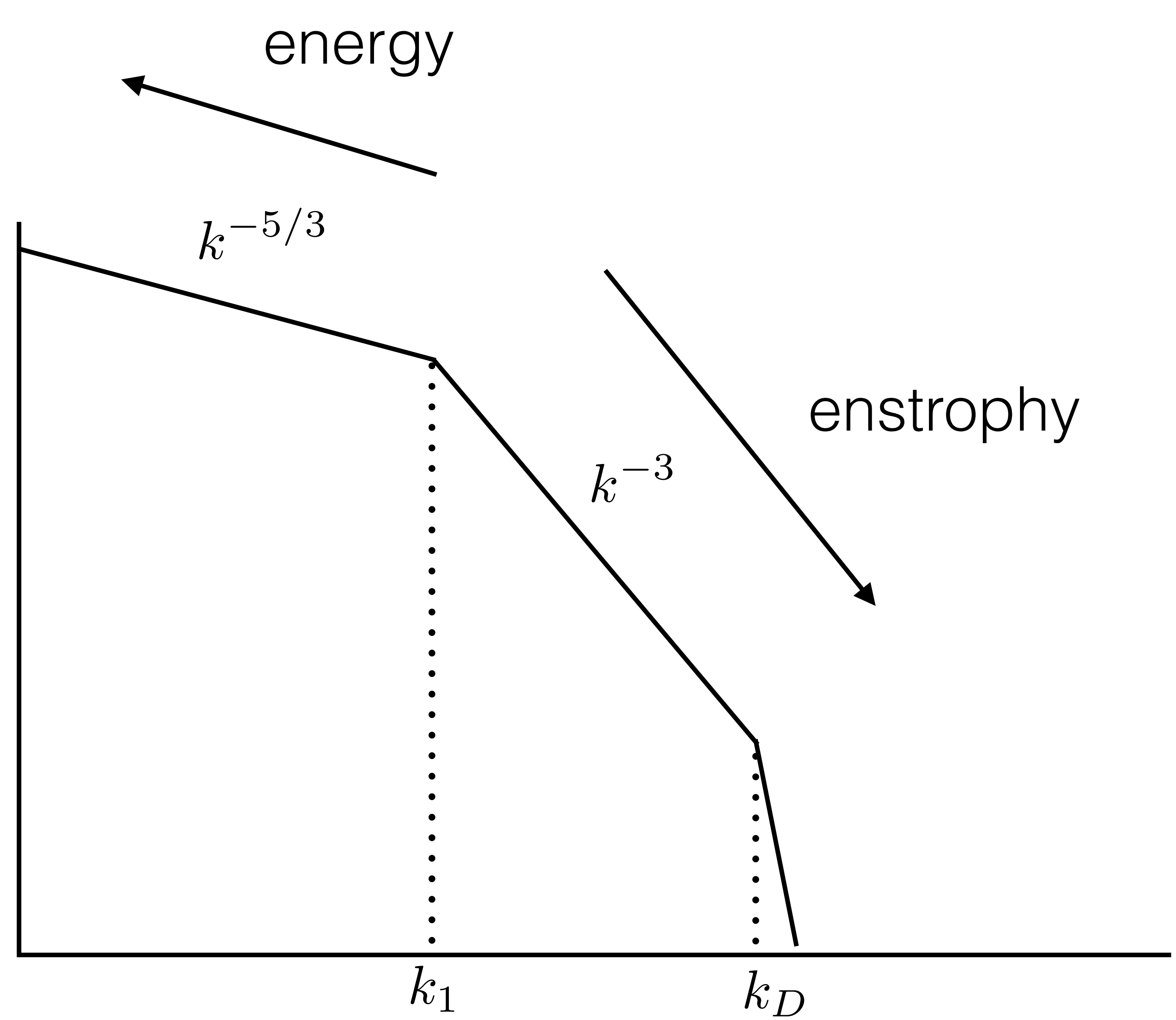
At low wavenumbers energy must be removed somewhere if we are in a steady state

k_R is the wavenumber at which energy is removed

As $k_R/k_1 \to 0$ $\qquad \eta \to 0$

Therefore, η is not important for determining the shape of the spectrum

$$E(k) = C_2 \varepsilon^{2/3} k^{-5/3},$$



Conclusions

In 2D turbulence, energy flows from small scales to large scales, and enstrophy flows for large scales to small scales.

The energy spectrum for 2D turbulence can be found from dimensional analysis.