The transition from geostrophic to stratified turbulence

By MICHAEL L. WAITE† AND PETER BARTELLO
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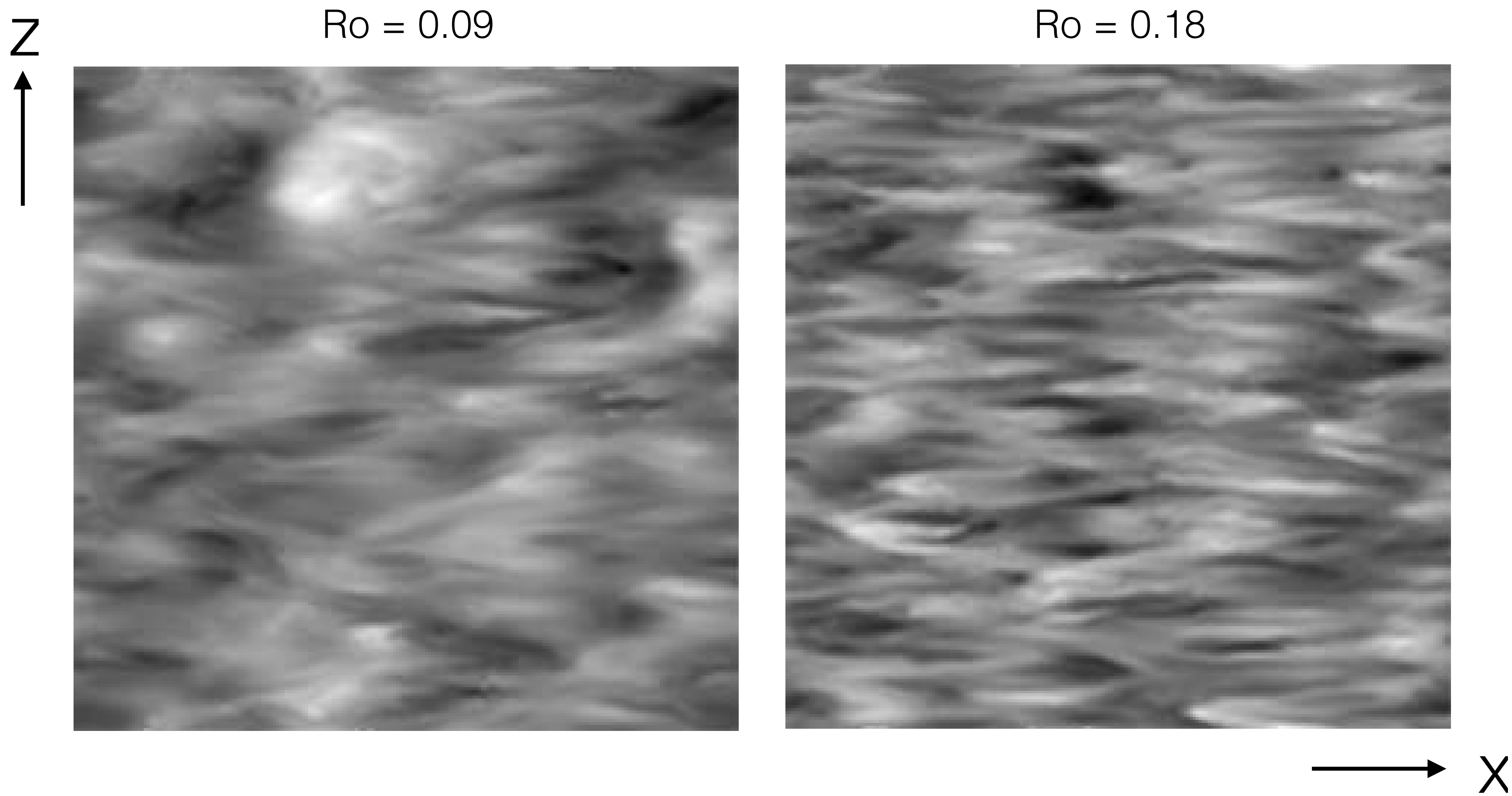

interpretation by Gregory Wagner

Waite and Bartello's parameter study

1. Force the triply-periodic Boussinesq equations at large-ish scales $(k_f = 5)$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f \hat{\boldsymbol{z}} \times \boldsymbol{u} = -\nabla p + b' \hat{\boldsymbol{z}} + \boldsymbol{F}_u + D_u(\boldsymbol{u}),$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{\partial b'}{\partial t} + \boldsymbol{u} \cdot \nabla b' + N^2 w = F_{b'} + D_{b'}(b'),$$

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2. Achieve an approximate steady-state by removing energy at small *and* large scales

$$\hat{D}_{\boldsymbol{k}}(q) = -\left(\nu\left(k_h^8 + k_z^8\right) + r(\boldsymbol{k})\right)\hat{q}_{\boldsymbol{k}},$$

hyperviscosity

large-scale drag on k=1

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hyperviscosity

large-scale drag on k,l =1

3. Hold N, forcing constant and vary f → N/f not constant

The simulations

f	N/f	Ro_ω	Fh_ω	Fz_ω	Ro_u	Fh_u	Fz_u
0	∞	∞	0.19	0.53	∞	0.026	0.056
1/16	128	25.0	0.19	0.52	3.4	0.026	0.056
1/8	64	12.0	0.19	0.52	1.7	0.026	0.055
1/4	32	6.3	0.19	0.52	0.83	0.026	0.054
1/2	16	3.2	0.19	0.51	0.39	0.025	0.051
3/4	32/3	2.2	0.21	0.49	0.25	0.024	0.051
1	8	1.7	0.21	0.47	0.18	0.022	0.048
3/2	16/3	1.2	0.22	0.42	0.12	0.022	0.040
2	4	0.89	0.22	0.37	0.090	0.022	0.033
4	2	0.47	0.23	0.21	0.048	0.024	0.018
8	1	0.24	0.23	0.10	0.024	0.024	0.0091

TABLE 1. The Coriolis parameters f used in our primary simulations ($N = 8$) along with N/f and the microscale (vorticity-based) and macroscale (velocity-based) Rossby and Froude numbers. The Rossby and Froude numbers are from the simulations with large-scale damping, and are averaged over $70 \leqslant t \leqslant 100$.

$$Ro_\omega = \frac{\sqrt{[\omega_z^2]}}{f},$$

$$U = \sqrt{[u^2 + v^2 + w^2]}, \quad L = 2\pi \left(\frac{E^{(0)}}{\int k_h^{1/2} E_h^{(0)}(k_h)\, dk_h} \right)^2$$

"micro-scales"

"large-scales"

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$$Fz_\omega = \frac{\sqrt{[\omega_x^2 + \omega_y^2]/2}}{N}.$$

vertical Froude number

Brillant and Chomaz 2001:

$$H = \frac{U}{N} \quad \text{and} \quad Fz = O(1)$$

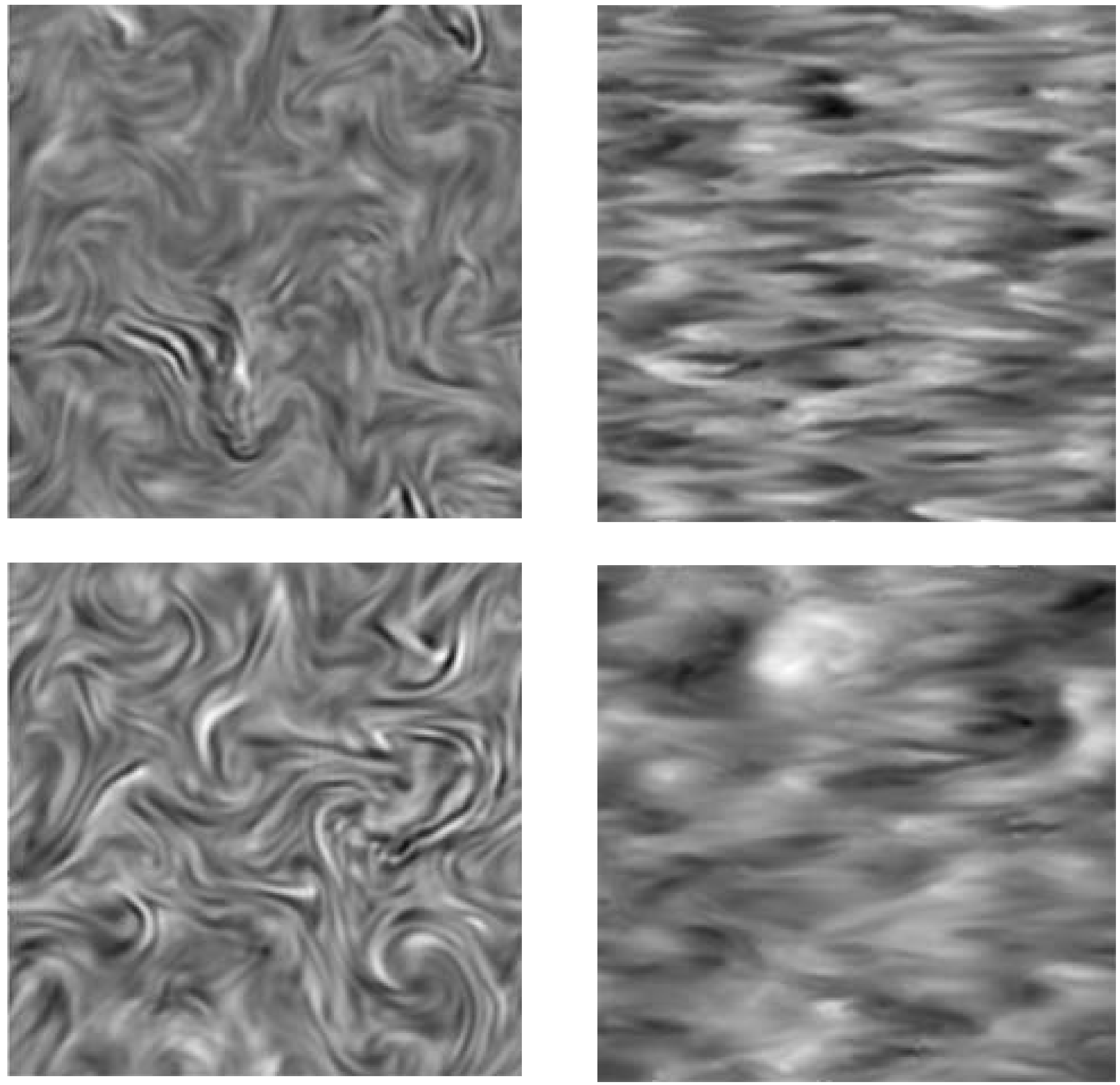
"stratified turbulence"

Snapshots

$w(x,y)$ $v(x,z)$

Ro = 0.18

Ro = 0.09



Why large-scale damping?

without damping

with damping

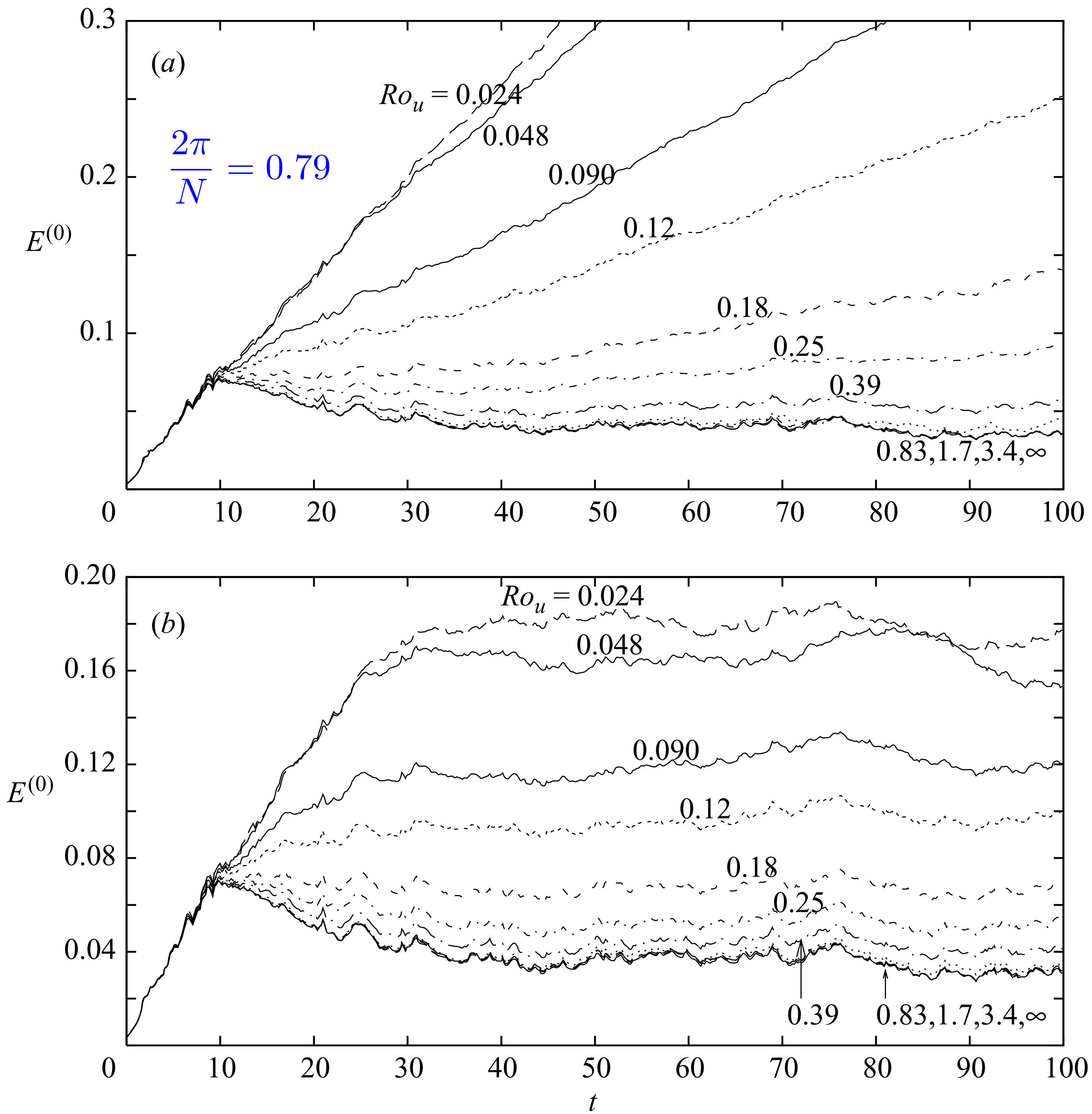


FIGURE 1. Time series of vortical energy for different Ro_u, (*a*) without and (*b*) with large-scale damping. At each time, the energy increases with decreasing Ro_u.

Main result (in my opinion)

small-scale dissipation

$$\epsilon_s = 2\nu \sum_{\boldsymbol{k}} \left(k_h^8 + k_z^8\right)(|\hat{u}_{\boldsymbol{k}}|^2 + |\hat{v}_{\boldsymbol{k}}|^2 + |\hat{w}_{\boldsymbol{k}}|^2 + |\hat{b}'_{\boldsymbol{k}}|^2/N^2),$$

large-scale dissipation

$$\epsilon_l = 2r_0 \sum_{1\leqslant k_h \leqslant \sqrt{2}} (|\hat{u}_{\boldsymbol{k}}|^2 + |\hat{v}_{\boldsymbol{k}}|^2 + |\hat{w}_{\boldsymbol{k}}|^2 + |\hat{b}'_{\boldsymbol{k}}|^2/N^2),$$

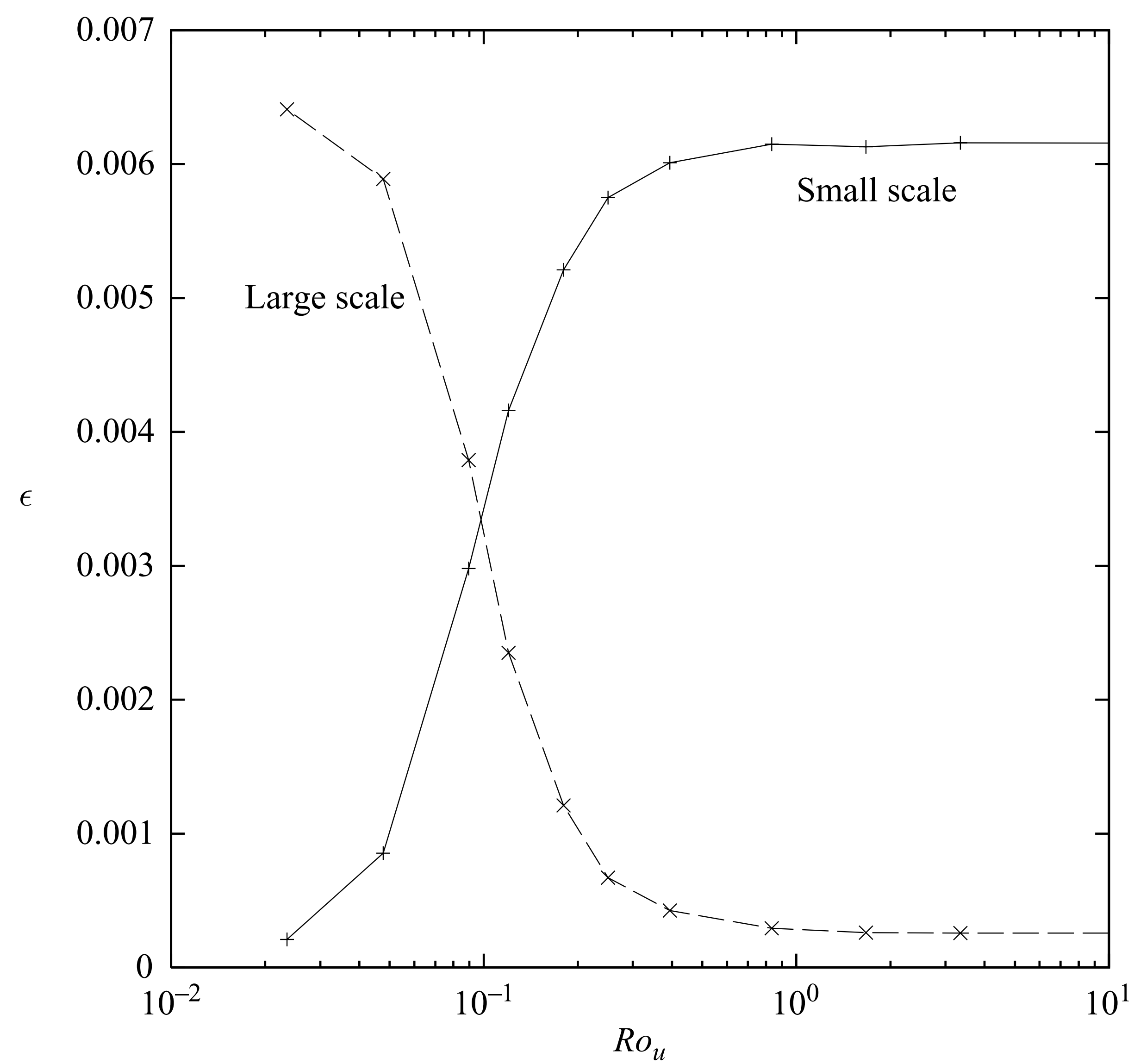


FIGURE 2. The small- and large-scale energy dissipation rates ϵ_s and ϵ_l.

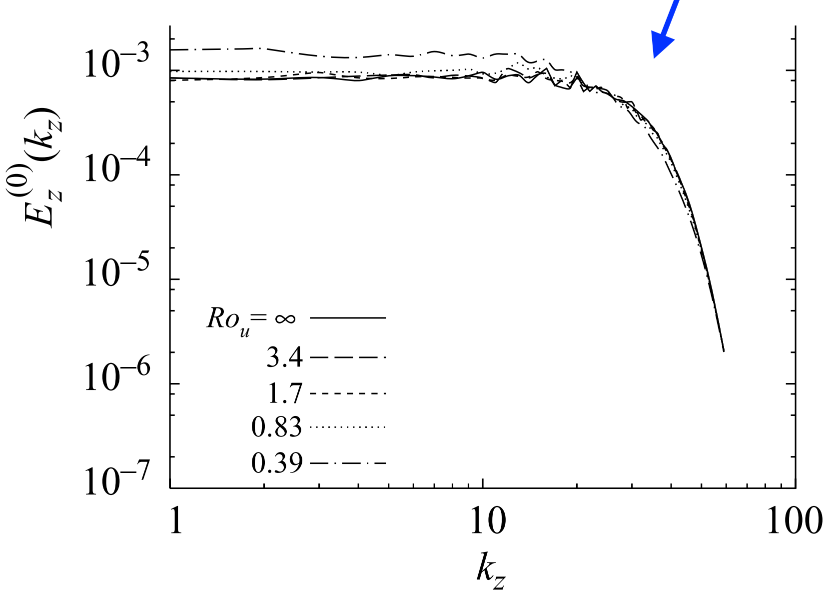
A transition in vertical scale

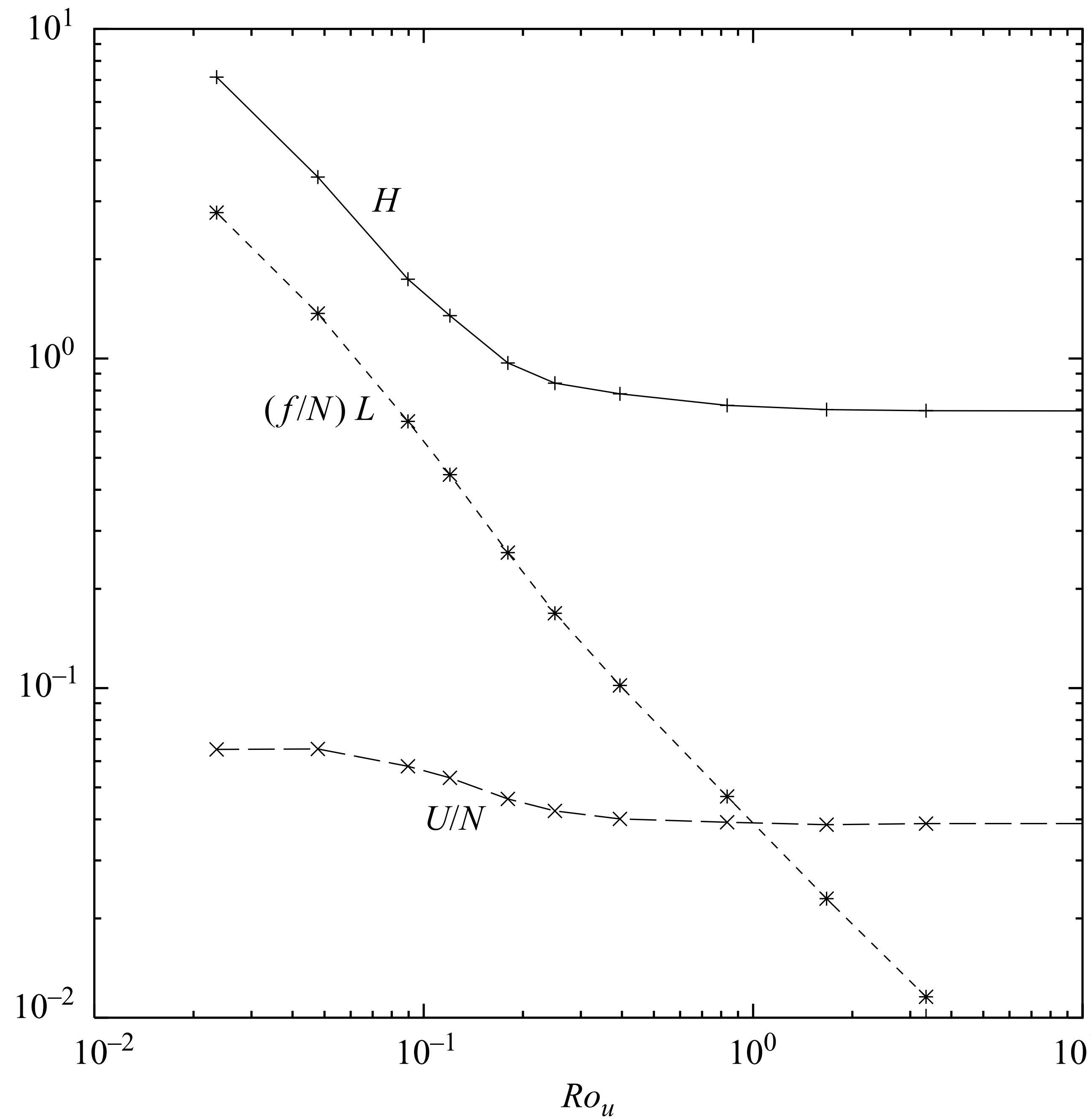
$$U = \sqrt{[u^2 + v^2 + w^2]},$$

$$L = 2\pi \left(\frac{E^{(0)}}{\int k_h^{1/2} E_h^{(0)}(k_h)\, dk_h} \right)^2$$

$$H = 2\pi \left(\frac{E^{(0)}}{\int k_z^{1/2} E_z^{(0)}(k_z)\, dk_z} \right)^2$$

H (apparently)





Vertical scales as a function of Ro: the measured vertical scale H, the QG scale $(f/N)L$, and U/N.

Bartello's key: the modal decomposition

$$
\begin{aligned}
u_t - fv + p_x &= -\boldsymbol{u} \cdot \boldsymbol{\nabla} u \\
v_t + fu + p_y &= -\boldsymbol{u} \cdot \boldsymbol{\nabla} v \\
w_t - b + p_z &= -\boldsymbol{u} \cdot \boldsymbol{\nabla} w \\
b_t + wN^2 &= -\boldsymbol{u} \cdot \boldsymbol{\nabla} b \\
u_x + v_y + w_z &= 0
\end{aligned}
$$

Bartello’s key: the modal decomposition

$$\hat{u}_t - f\hat{v} + \mathrm{i}k\hat{p} = 0$$
$$\hat{v}_t + f\hat{u} + \mathrm{i}\ell\hat{p} = 0$$
$$\hat{w}_t - \hat{b} + \mathrm{i}m\hat{p} = 0$$
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$$\omega \stackrel{\mathrm{def}}{=} v_x - u_y$$
$$\delta \stackrel{\mathrm{def}}{=} u_x + v_y$$

$\longrightarrow$

$$\frac{\partial}{\partial t}\begin{bmatrix}\hat{\omega}\\ \hat{\delta}\\ \hat{b}\end{bmatrix} = \mathrm{i}\begin{bmatrix} 0 & -\mathrm{i}f & 0 \\ -\frac{\mathrm{i}fm^2}{|\boldsymbol{k}|^2} & 0 & -m\frac{k^2+\ell^2}{|\boldsymbol{k}|^2} \\ 0 & -\frac{N^2}{m} & 0 \end{bmatrix}\begin{bmatrix}\hat{\omega}\\ \hat{\delta}\\ \hat{b}\end{bmatrix}$$

$\boldsymbol{W}_{\boldsymbol{k}}$ $\quad$ $\boldsymbol{L}_{\boldsymbol{k}}$ $\quad$ $\boldsymbol{W}_{\boldsymbol{k}}$

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$\boldsymbol{W}_{\boldsymbol{k}}$ $\quad L_{\boldsymbol{k}}$ $\quad \boldsymbol{W}_{\boldsymbol{k}}$

$$\frac{\partial}{\partial t}\boldsymbol{W}_{\boldsymbol{k}} = \mathrm{i}L_{\boldsymbol{k}}\boldsymbol{W}_{\boldsymbol{k}} \quad \xrightarrow{L_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}}^{(j)} = \lambda_{\boldsymbol{k}}^{(j)}\boldsymbol{X}_{\boldsymbol{k}}^{(j)}} \quad \frac{\mathrm{d}B_{\boldsymbol{k}}^{(j)}}{\mathrm{d}t} + \mathrm{i}\lambda_{\boldsymbol{k}}^{(j)}B_{\boldsymbol{k}}^{(j)} = 0$$

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$$\frac{\partial}{\partial t}\boldsymbol{W}_{\boldsymbol{k}} = \mathrm{i}\boldsymbol{L}_{\boldsymbol{k}}\boldsymbol{W}_{\boldsymbol{k}} \quad \xrightarrow{L_{\boldsymbol{k}}\boldsymbol{X}_{\boldsymbol{k}}^{(j)} = \lambda_{\boldsymbol{k}}^{(j)}\boldsymbol{X}_{\boldsymbol{k}}^{(j)}} \quad \frac{\mathrm{d}B_{\boldsymbol{k}}^{(j)}}{\mathrm{d}t} + \mathrm{i}\lambda_{\boldsymbol{k}}^{(j)}B_{\boldsymbol{k}}^{(j)} = 0$$

frequencies $\quad \lambda_{\boldsymbol{k}}^{(0)} = 0, \quad \lambda_{\boldsymbol{k}}^{(\pm)} = \left(N^2k_h^2 + f^2k_z^2\right)^{1/2}/k,$

$B_{\boldsymbol{k}}^{(0)}$: vortical mode $\qquad$ $B_{\boldsymbol{k}}^{(\pm)}$: wave modes

Mode by mode

Boussinesq equations

$$\frac{\mathrm{d}B_{\boldsymbol{k}}^{(j)}}{\mathrm{d}t} + \mathrm{i}\lambda_{\boldsymbol{k}}^{(j)} B_{\boldsymbol{k}}^{(j)} = \sum_{\boldsymbol{k}=\boldsymbol{p}+\boldsymbol{q}} \Gamma_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}^{jrs} B_{\boldsymbol{p}}^{(r)} B_{\boldsymbol{q}}^{(s)} + \hat{F}_{k}^{(j)} + \hat{D}_{k}^{(j)}$$

$$\lambda_{\boldsymbol{k}}^{(0)} = 0, \quad \lambda_{\boldsymbol{k}}^{(\pm)} = \left(N^2 k_h^2 + f^2 k_z^2\right)^{1/2} / k,$$

*a multiple time-scale expansion to O(ε) yields the QG equation for the (0) modes

vortical modes satisfy

$$-fv + p_x = 0$$
$$fu + p_y = 0$$
$$-b + p_z = 0$$

waves have no linear PV.

$$N^2\omega + f b_z = 0$$

Mode by mode

Boussinesq equations

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Energy

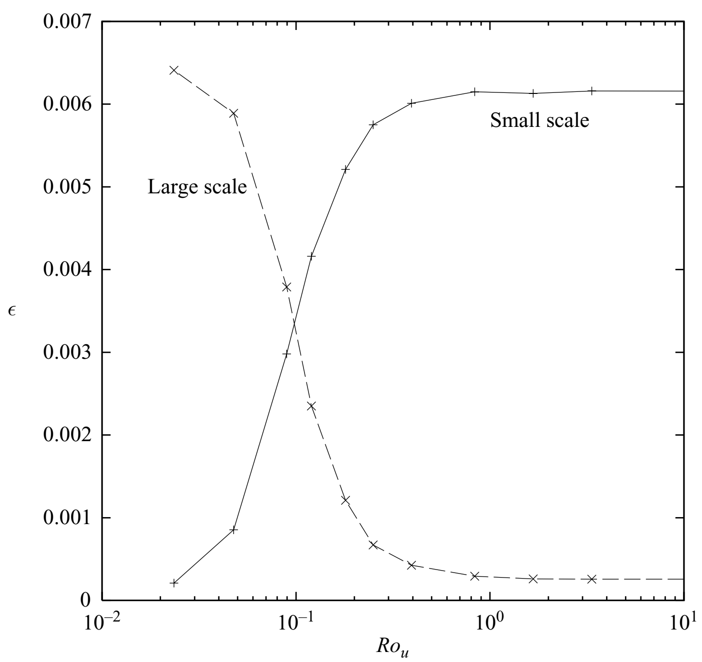
Note: only the vortical mode is forced.

vortical $$E^{(0)} = \frac{1}{2} \sum_{k_h \neq 0} \left|B_{\boldsymbol{k}}^{(0)}\right|^2,$$

wave $$E^{(\pm)} = \frac{1}{2} \sum_{k_h \neq 0} \left|B_{\boldsymbol{k}}^{(+)}\right|^2 + \left|B_{\boldsymbol{k}}^{(-)}\right|^2,$$

"shear" $$E^{(S)} = \frac{1}{2} \sum_{k_h = 0} |\hat{u}_{\boldsymbol{k}}|^2 + |\hat{v}_{\boldsymbol{k}}|^2 + |\hat{b}'_{\boldsymbol{k}}|^2 / N^2$$

Energy partition vs. Ro



Notes:

- Rotation suppresses energy transfer to wave modes.
- "spontaneous emission"

$$e^{-1/Ro}$$

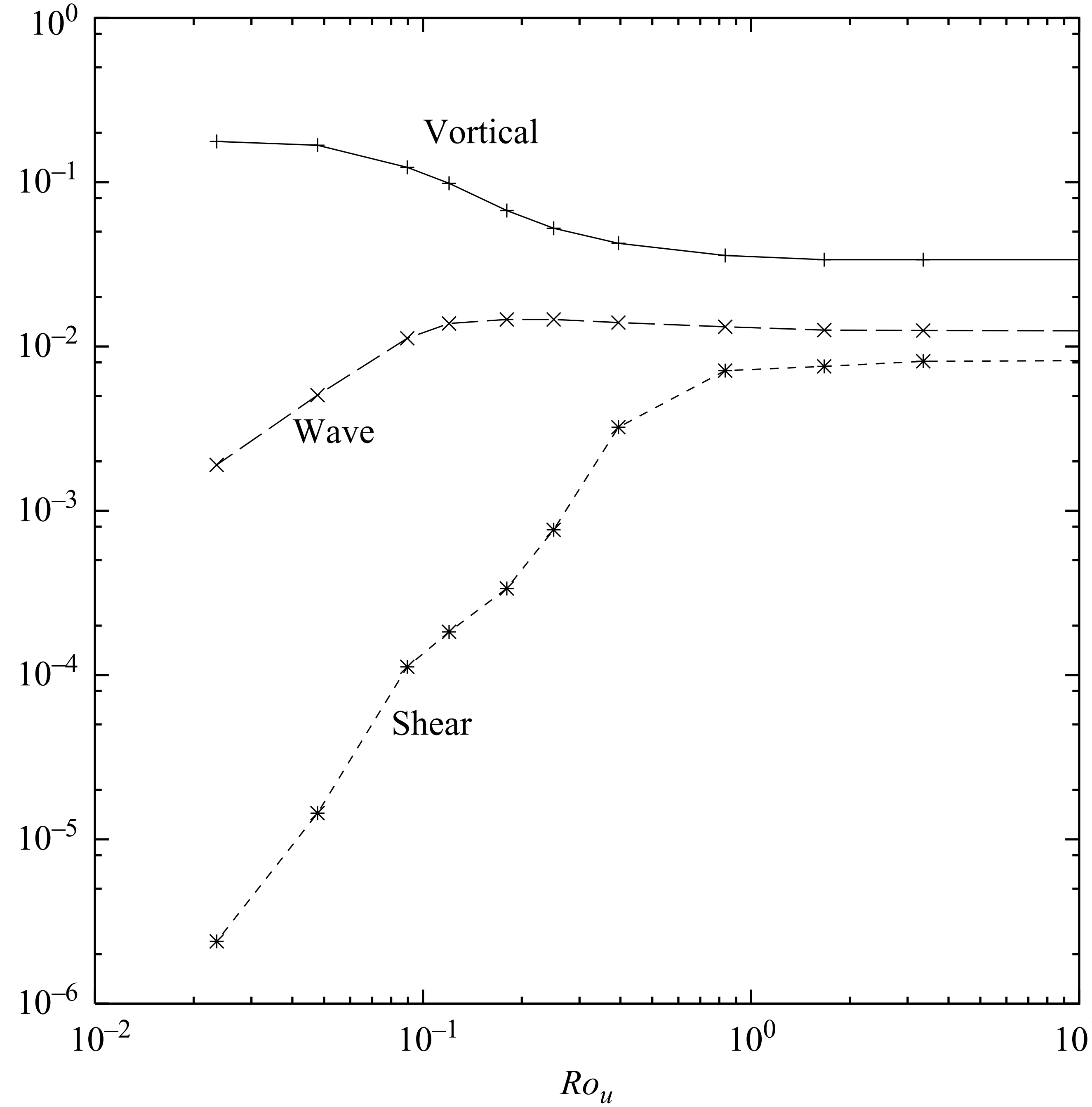


FIGURE 4. The time-averaged vortical, wave and shear mode kinetic energy as defined in § 2 for the simulations with large-scale damping.

Energy partition vs. Ro and N

Low Ro Notes:

- Energy in vortical and shear modes is independent of N/f
- Energy in wave modes is not?

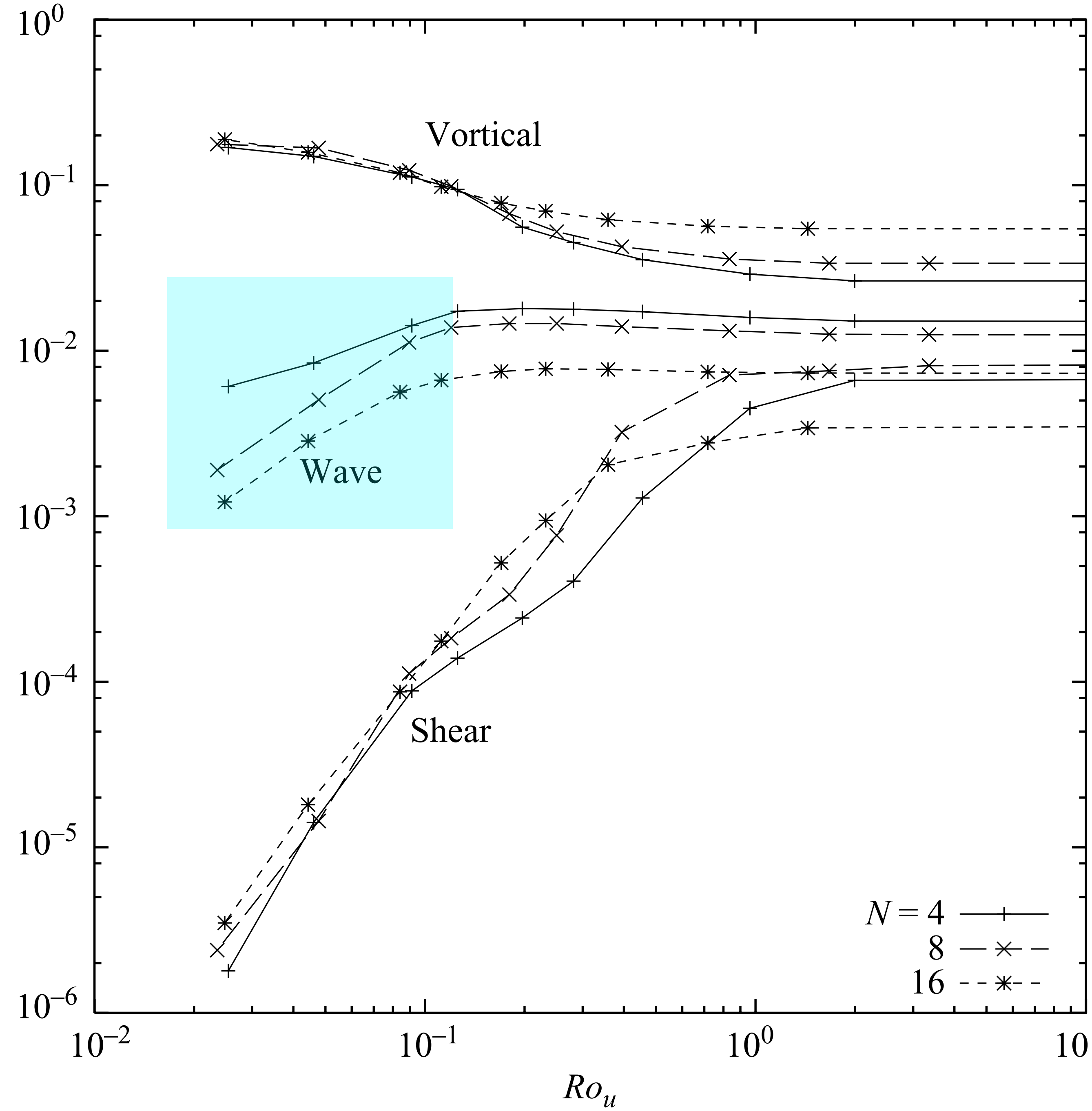
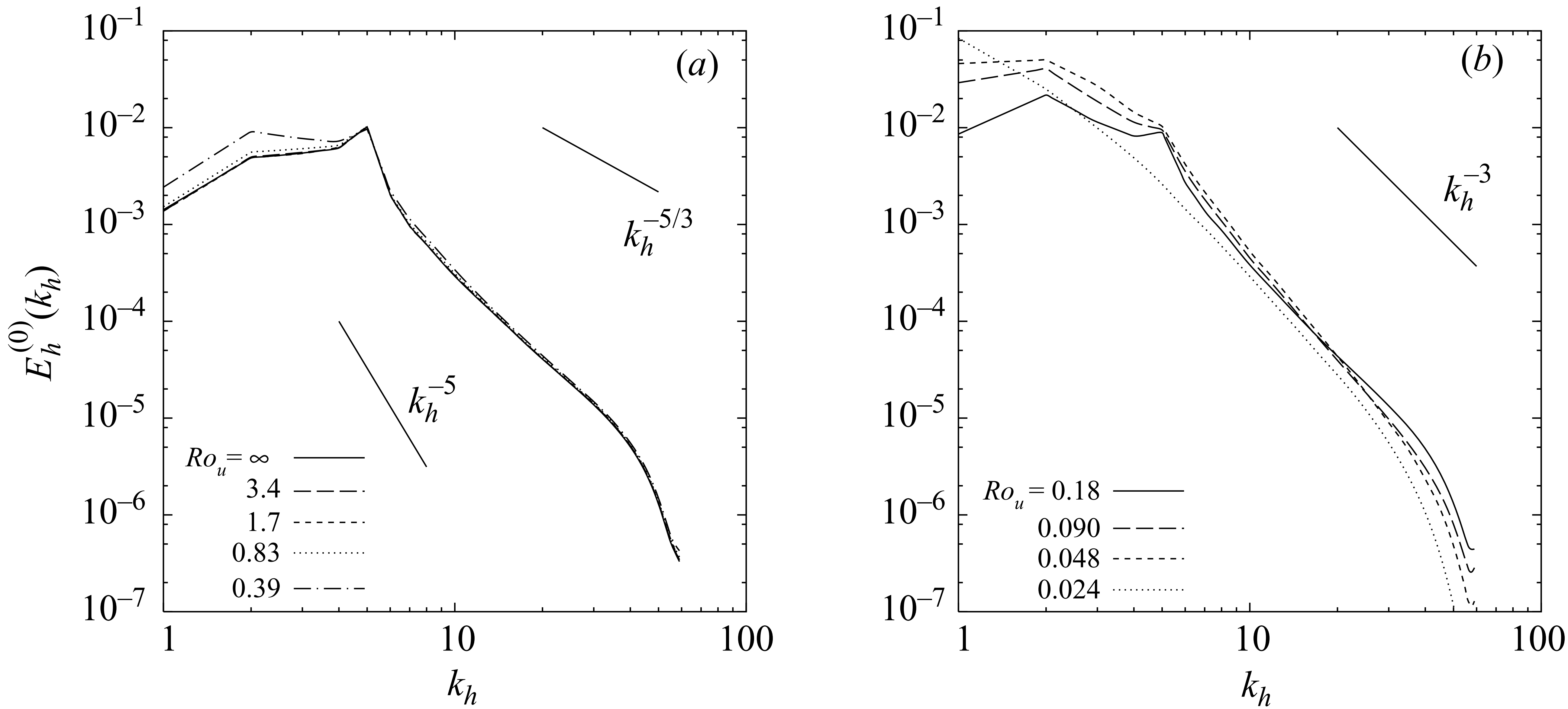


FIGURE 11. The time-averaged vortical, wave and shear mode kinetic energy as defined in § 2 for the sets of simulations with $N = 4$, $N = 8$ (as in figure 4) and $N = 16$.

But what about the *spectra?!?*



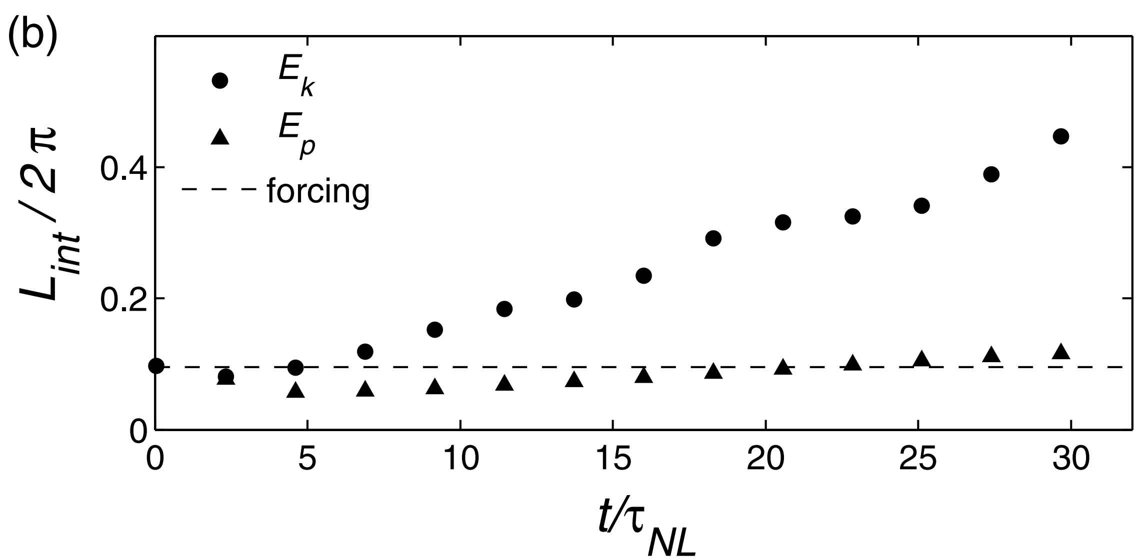
Comments:

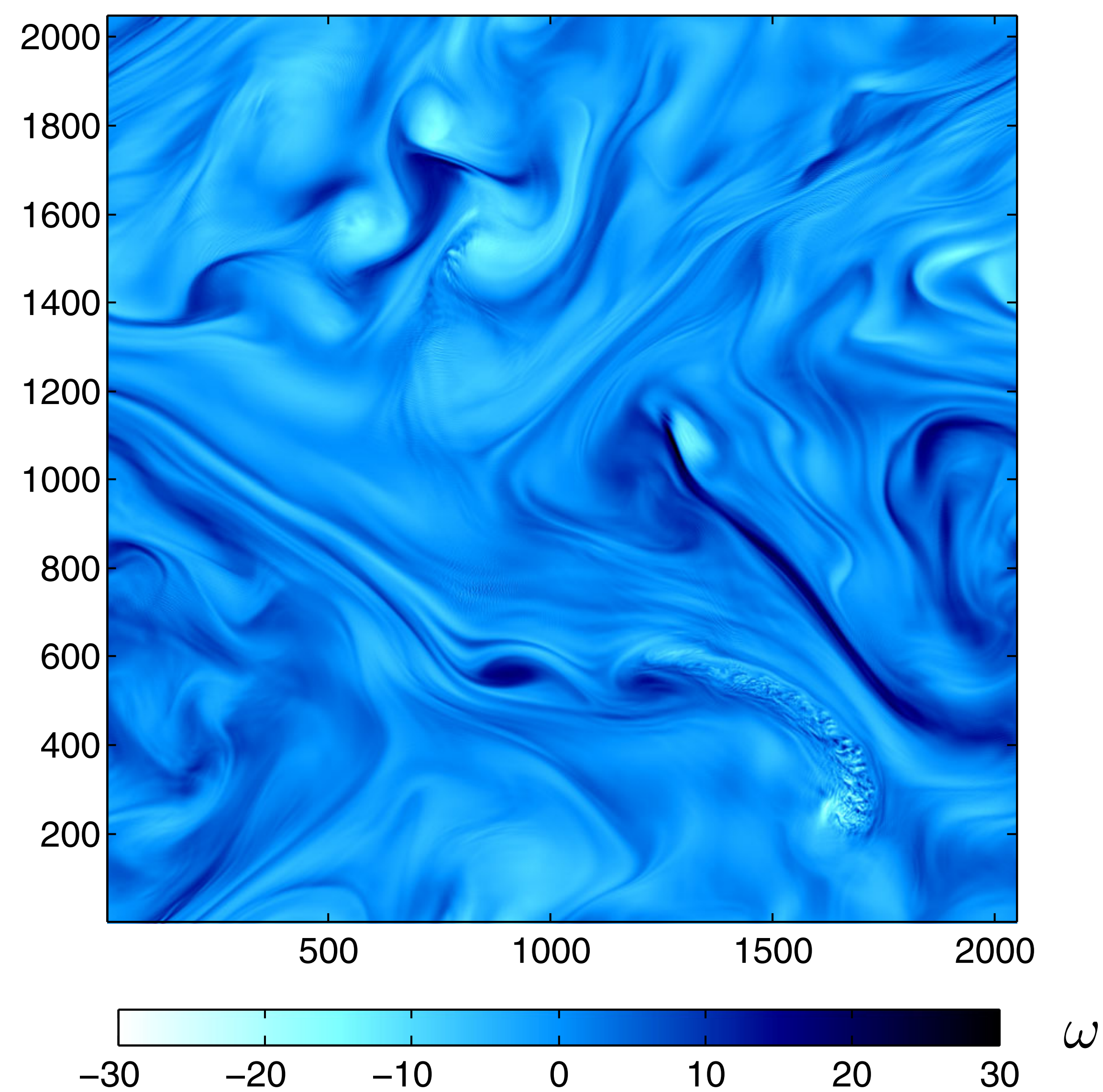
1. Spectra steepen at low Ro, but the effect is *very* subtle on a log-log plot.
2. Resolution is too low for a proper inertial range.

Resolving the Paradox of Oceanic Large-Scale Balance and Small-Scale Mixing

R. Marino,[1,2,3] A. Pouquet,[4,1] and D. Rosenberg[5]

- Boussinesq simulations at 1024^3 and 2048^3 resolution
- Forced at k = 10,11
- "DNS": no hyperviscosity or large-scale drag

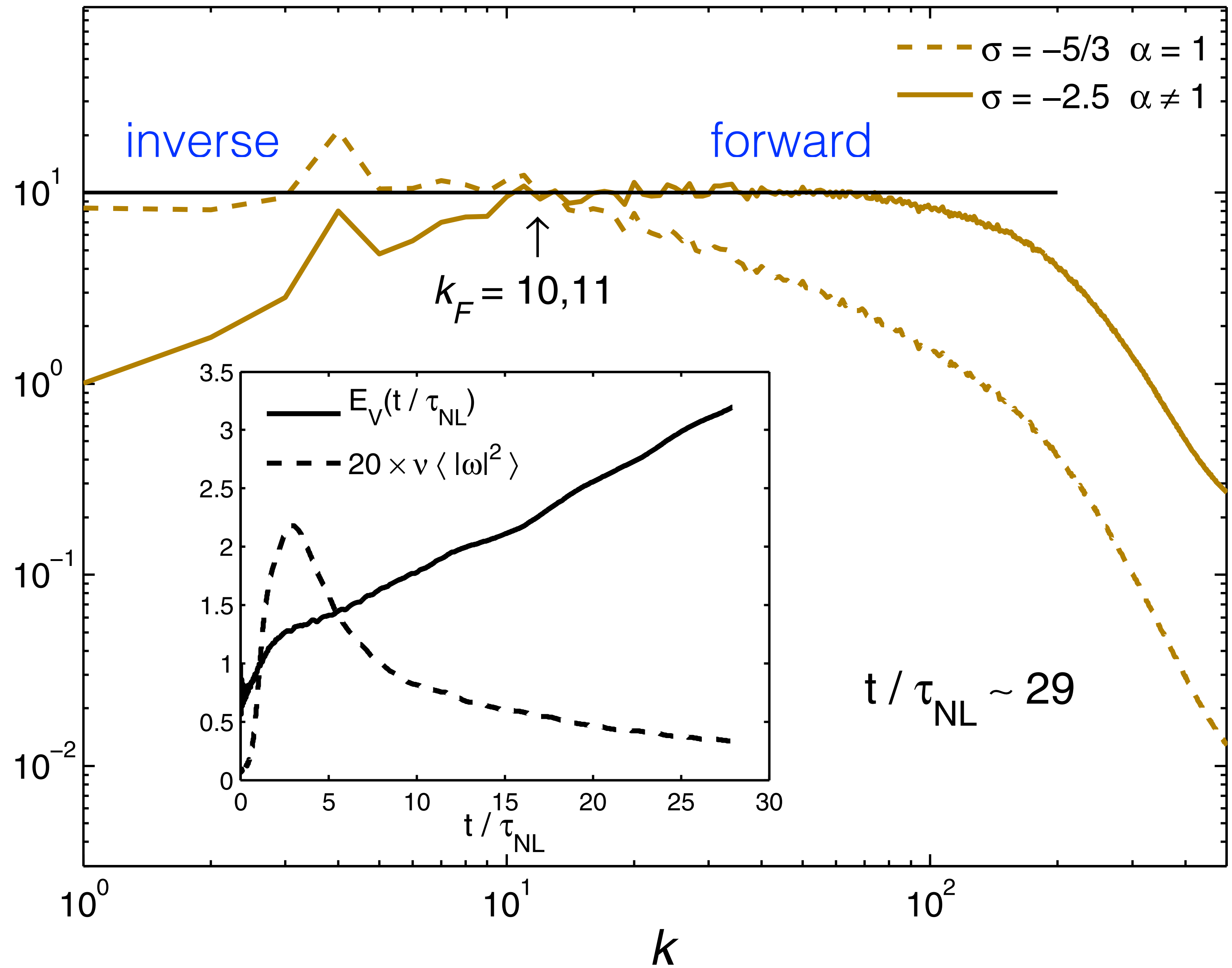




The simplest observation stemming from this extensive high-resolution study is that the flux ratio varies substantially.

Dual cascades

$$\frac{E(k)}{\alpha\epsilon^{2/3}k^{\sigma}}$$



A speculative scaling

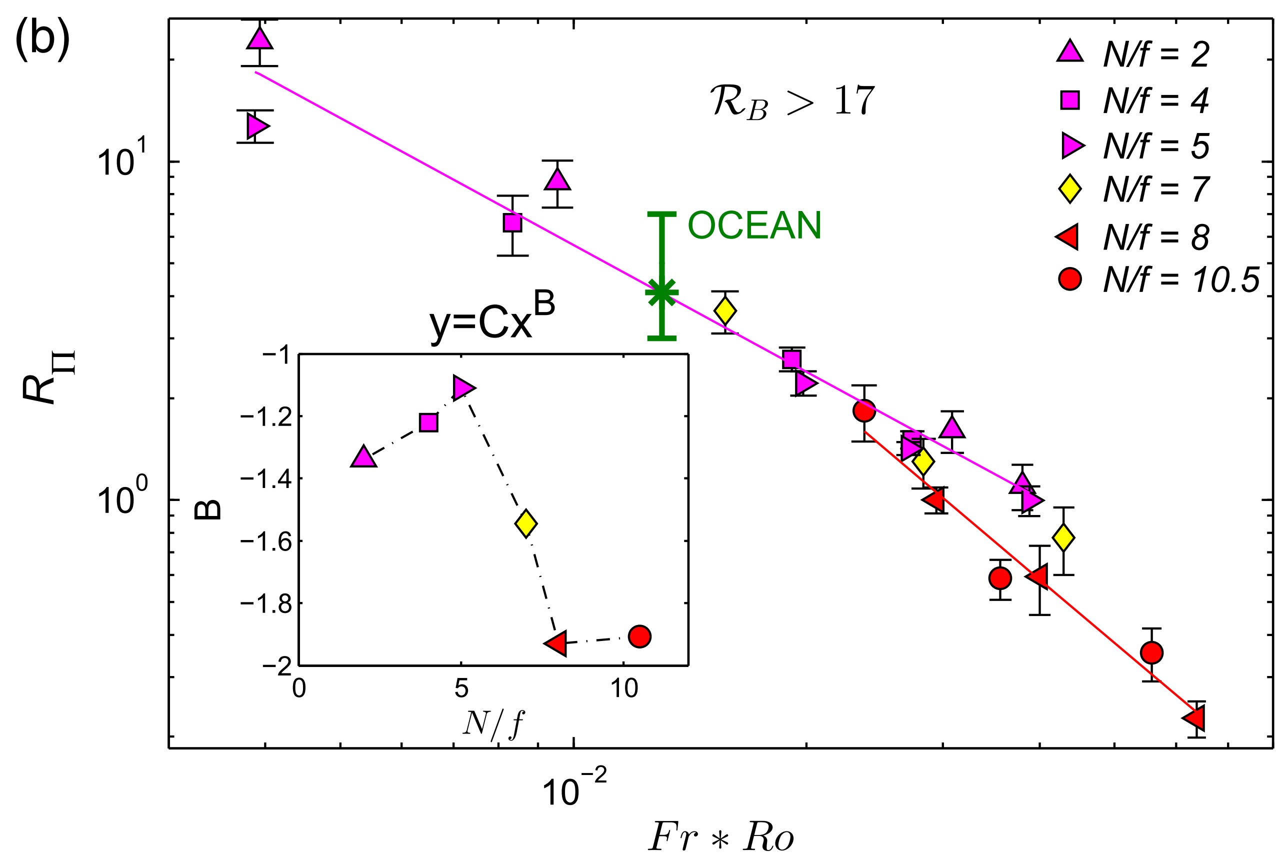
$$R_\Pi = \frac{\text{flux to large scales}}{\text{flux to small scales}} = \left|\frac{\epsilon_L}{\epsilon_S}\right|$$

$\epsilon_S \sim Fh$ and $\epsilon_L \sim \frac{1}{Ro}$ $\longrightarrow$ $R_\Pi \sim \dfrac{1}{Fh \cdot Ro}$

stratification controls forward cascade

rotation controls inverse cascade

sort of works for small N/f.



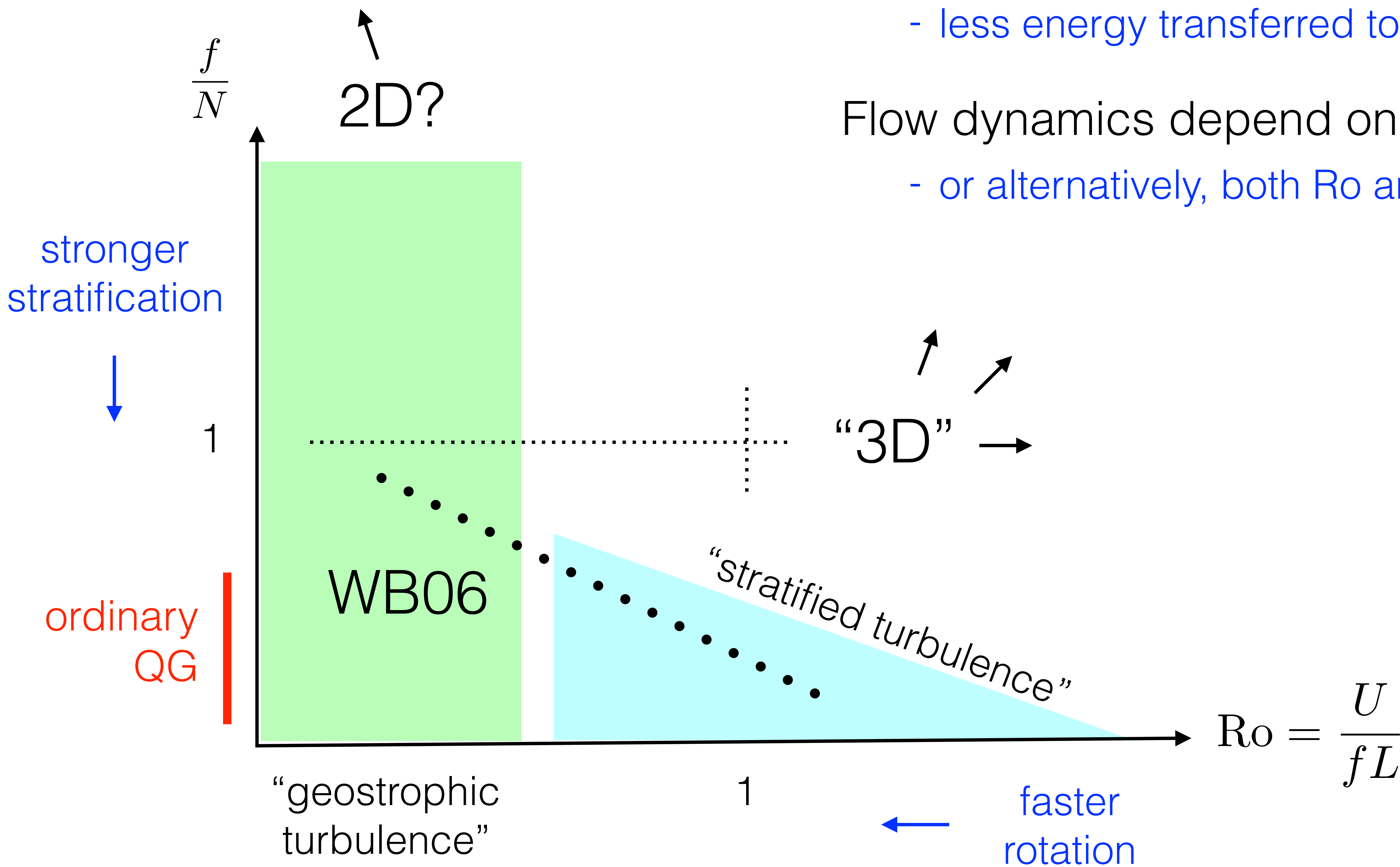
Rotating, stratified turbulence

Fixed N, increasing f implies

- more energy goes upscale
- less energy transferred to waves

Flow dynamics depend on Ro and f/N

- or alternatively, both Ro and Fr



Questions

- Waves are energetic and directly forced.

  Does direct forcing of waves change geostrophic turbulent dynamics?

- Boundary processes can produce small-scales.

  "Intrinsic cascade" versus direct transfer to small-scales

- Non-uniform stratification, rotation?

  Relationship between inhomogeneous and homogeneous dynamics?

Summary and questions

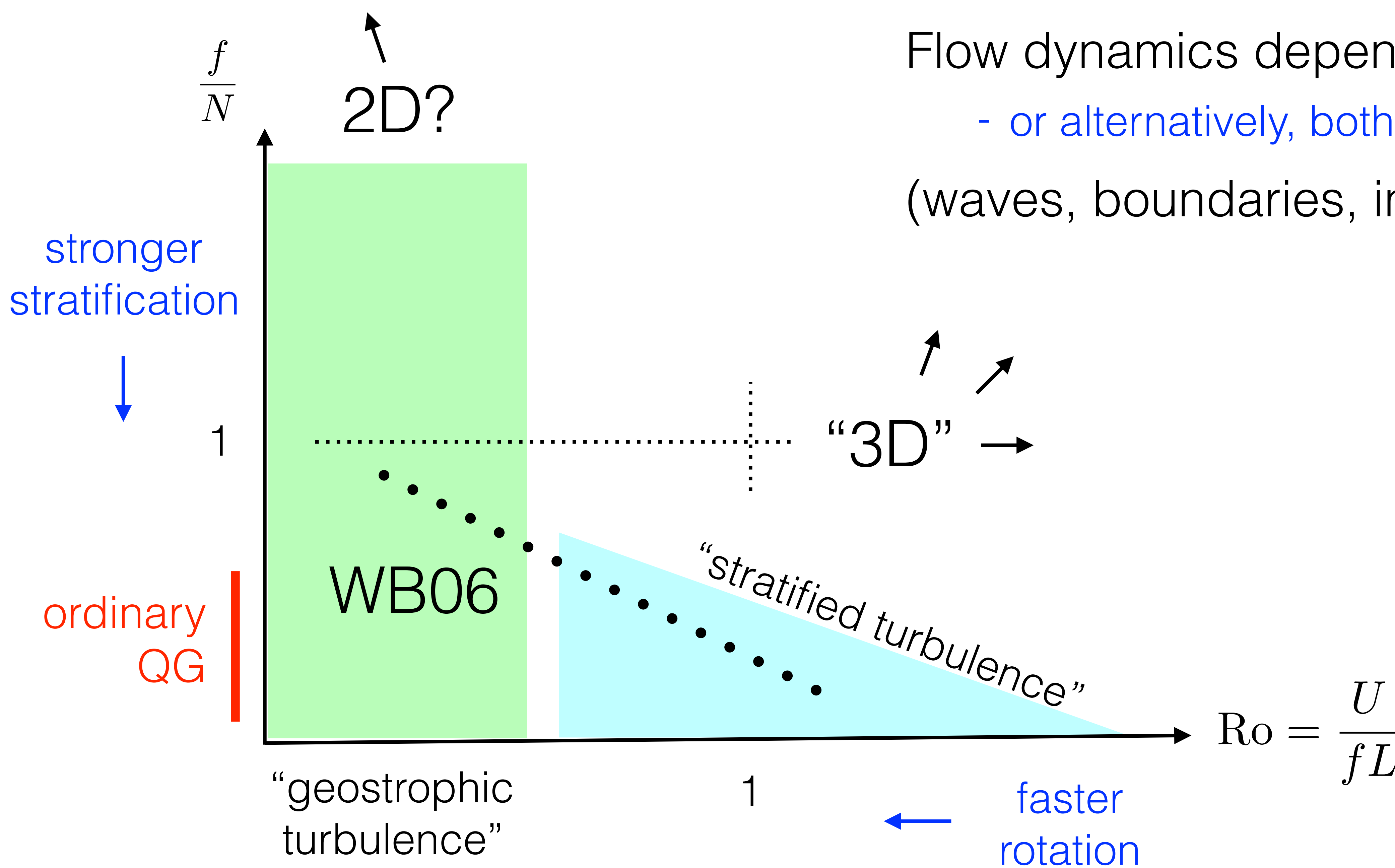
Fixed N, increasing f implies

- more energy goes upscale
- less energy transferred to waves

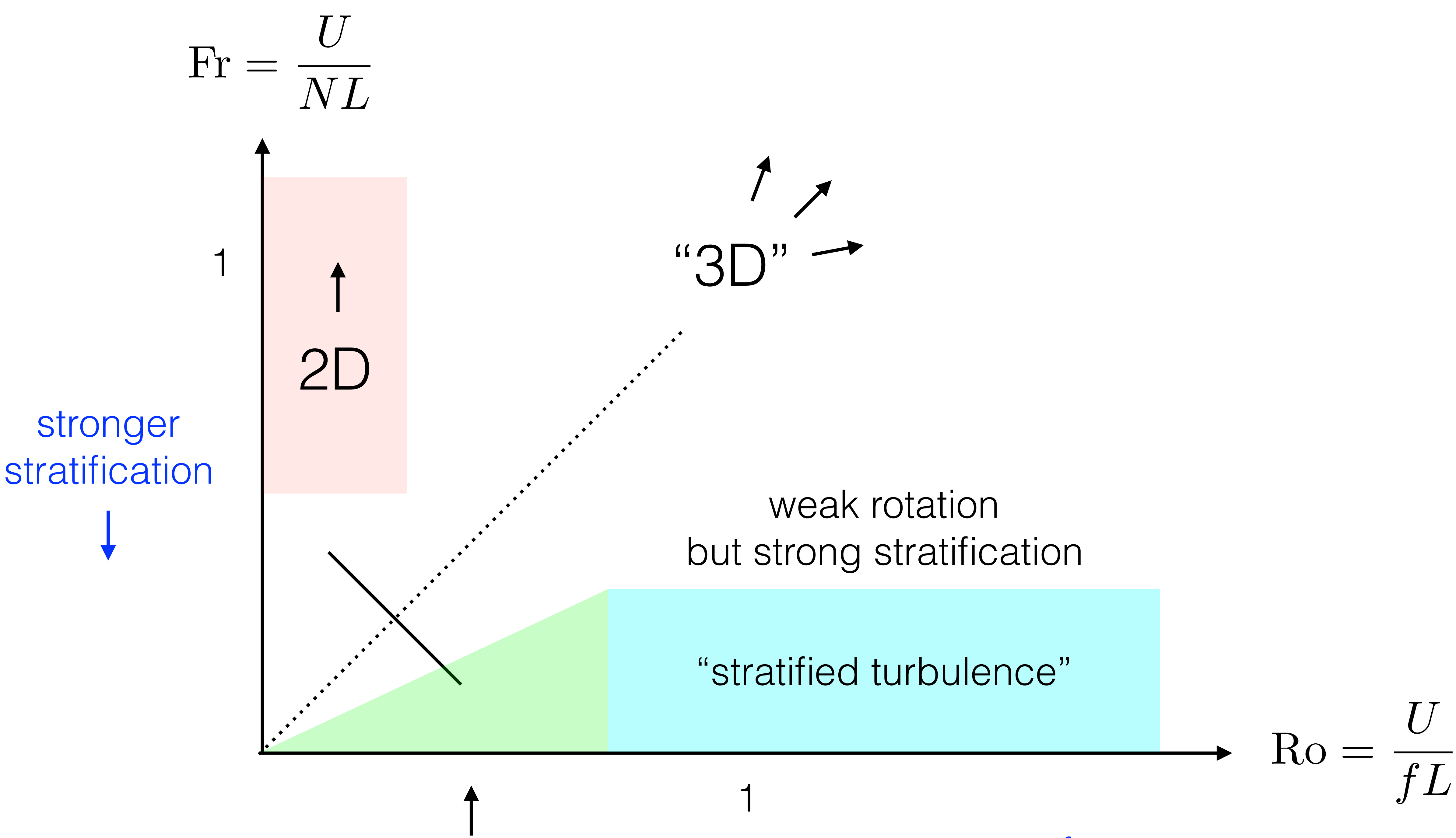
Flow dynamics depend on Ro and f/N

- or alternatively, both Ro and Fr

(waves, boundaries, inhomogeneity?)



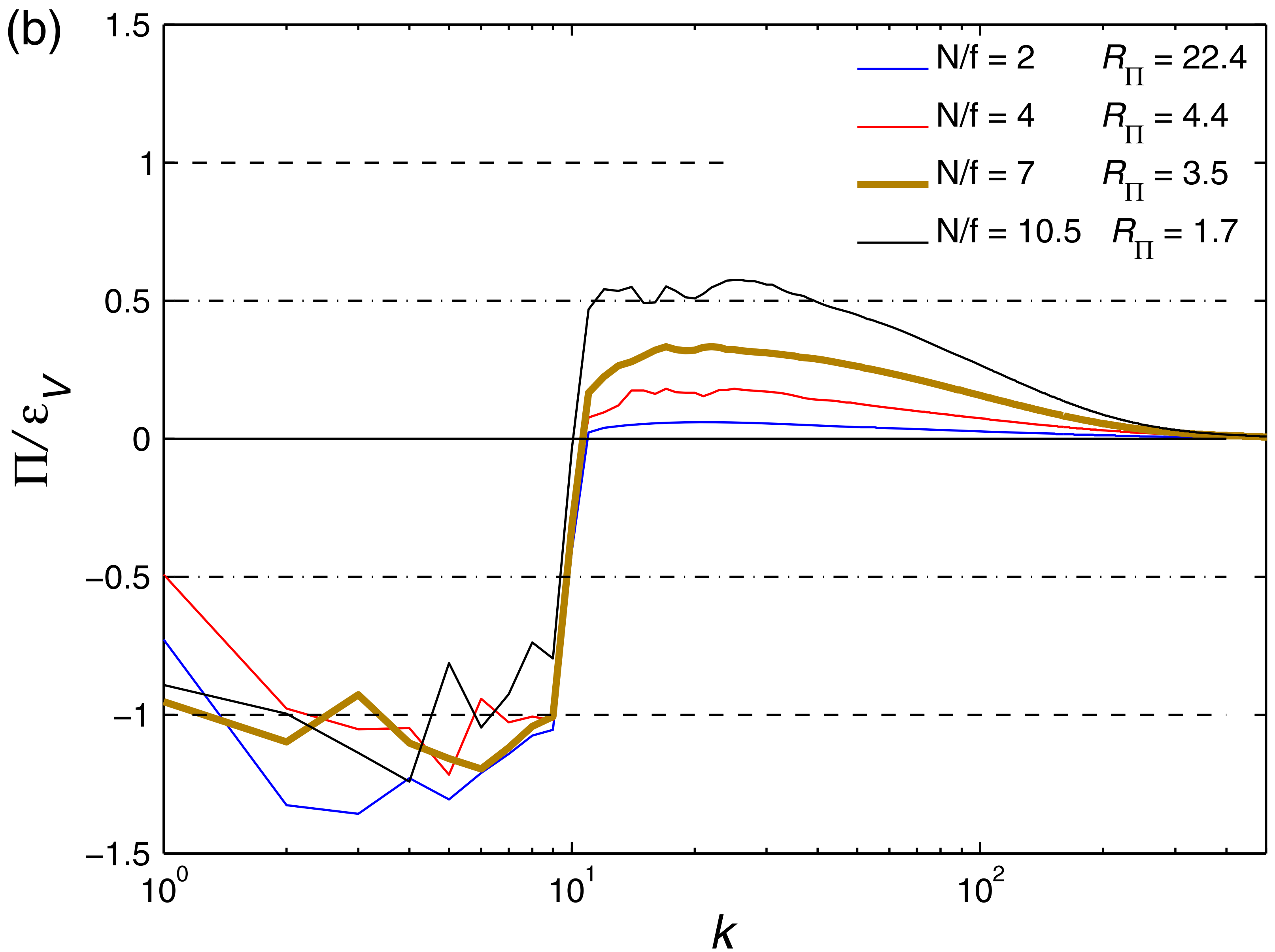
Rotating, stratified turbulence



"geostrophic turbulence"

Strong rotation, stronger stratification.

The dual cascade

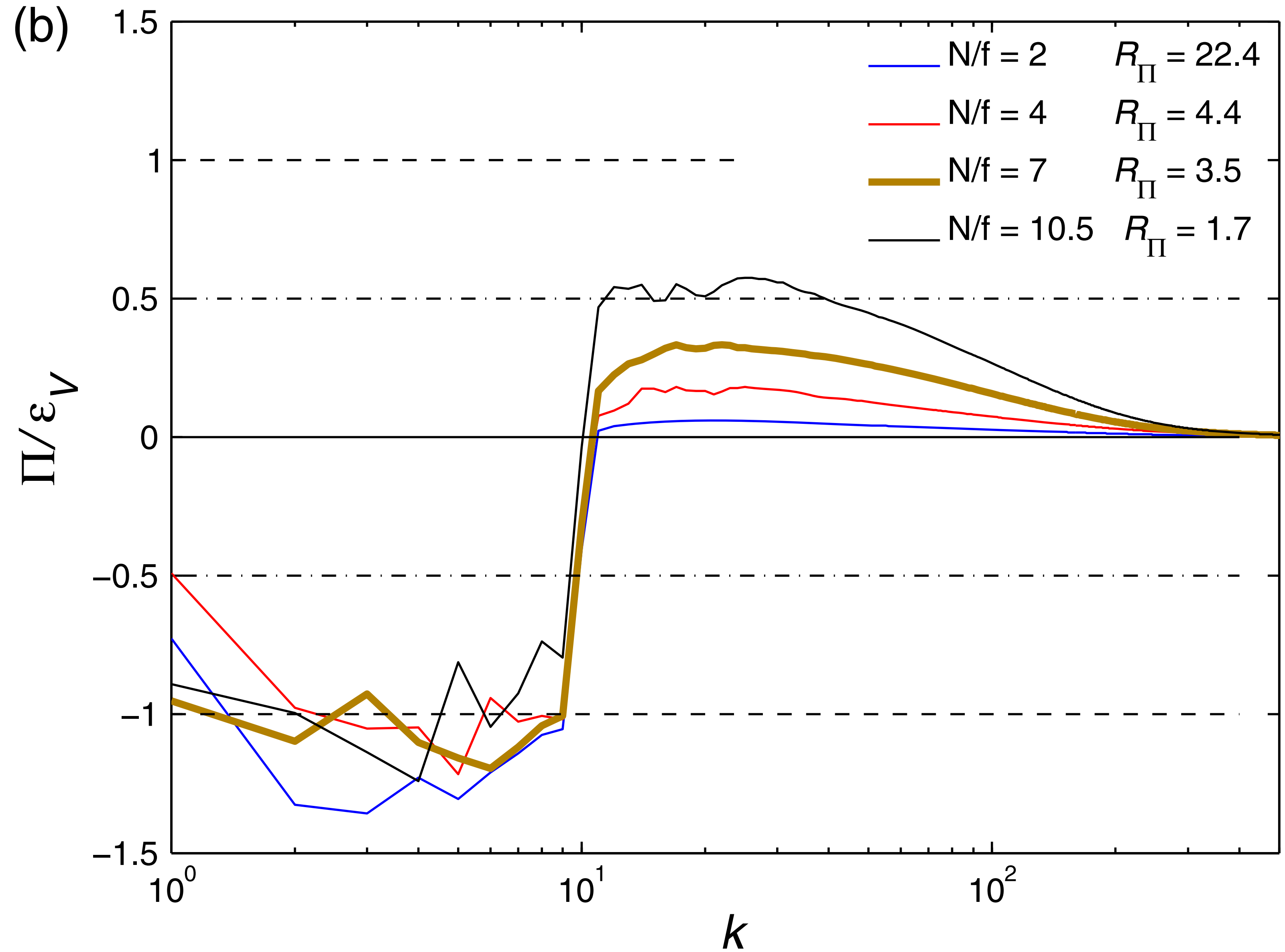


$$\Pi_V(k) = \int_{k_{\min}}^{k} T_V(q)dq, \qquad T_V(q) = -\sum_{\mathcal{C}_q} \hat{\mathbf{u}}^\star_{\mathbf{q}} \cdot (\widehat{\mathbf{u} \cdot \nabla \mathbf{u}})_{\mathbf{q}}$$

“Dual cascades”

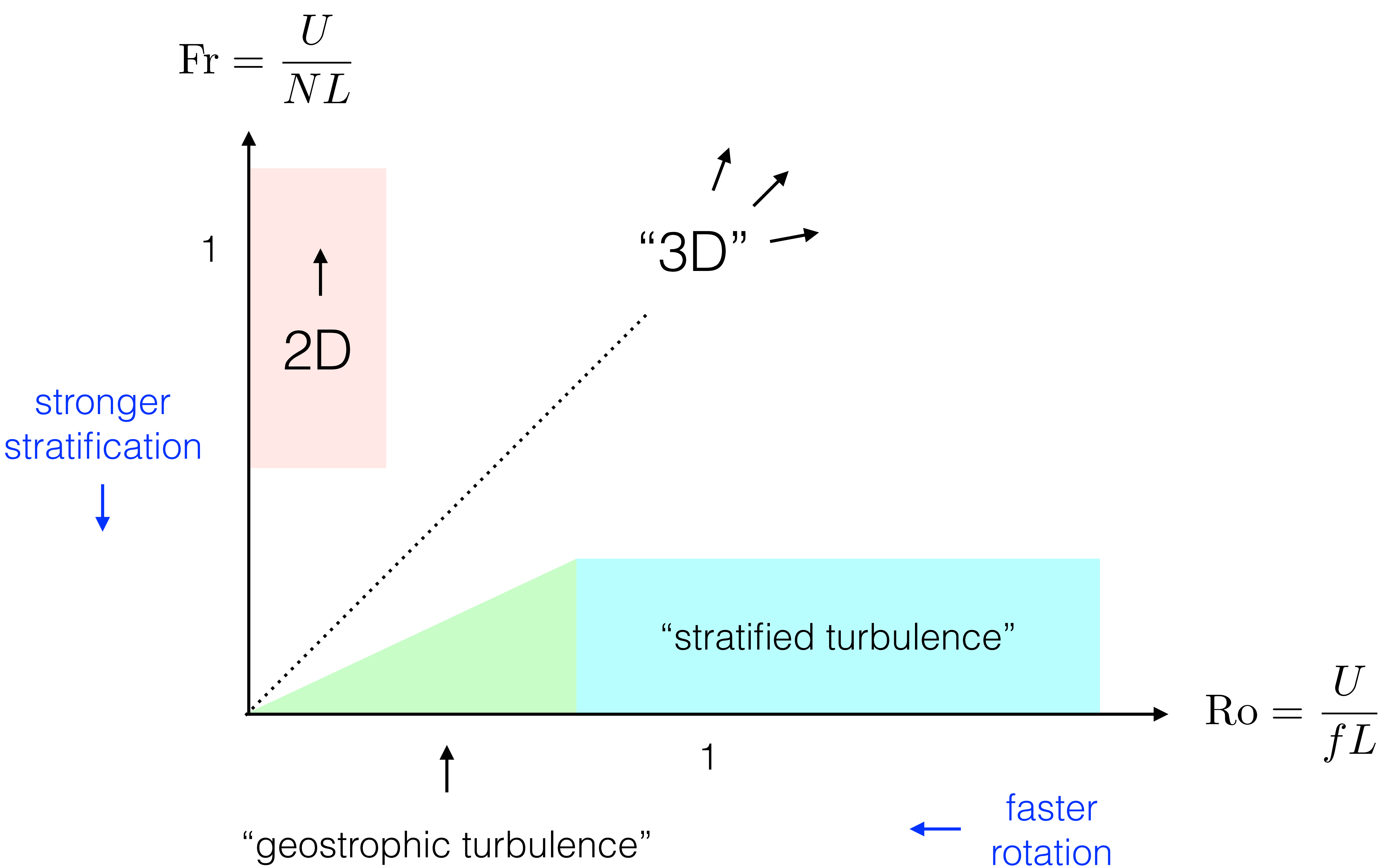
Notes:

- Increasing N/f implies increased forward cascade.



$$\partial_t E(k) + \int_0^k \Pi(p)\,\mathrm{d}p = -\epsilon_F - \epsilon_S$$

Rotating, stratified turbulence



Shear mode

"shear" $$E^{(S)} = \frac{1}{2}\sum_{k_h=0} |\hat{u}_{\boldsymbol{k}}|^2 + |\hat{v}_{\boldsymbol{k}}|^2 + |\hat{b}'_{\boldsymbol{k}}|^2/N^2$$

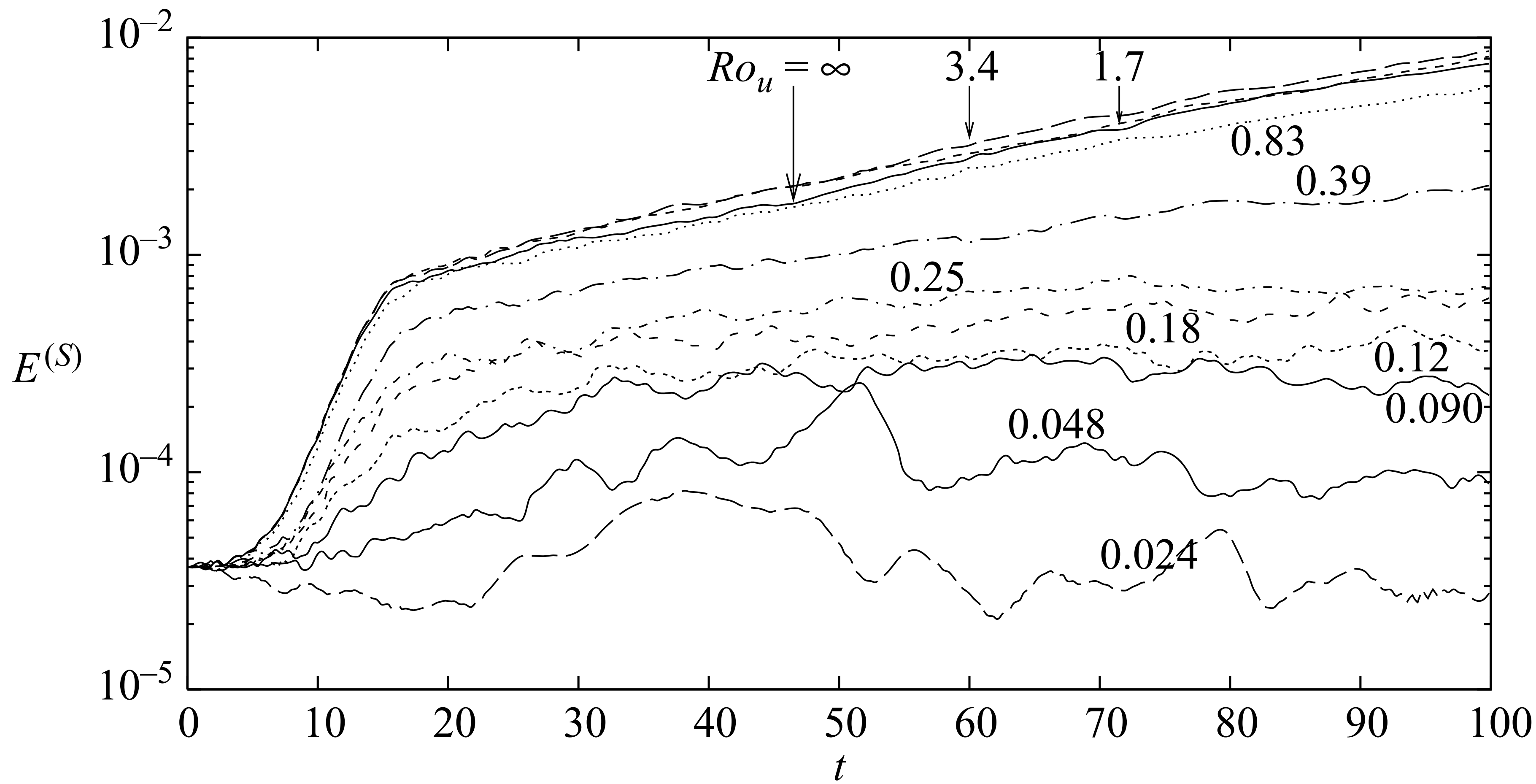


FIGURE 3. Time series of shear mode kinetic energy for different Rossby numbers without large-scale damping. When damping is employed, the growth is slightly enhanced.

“Dual cascades”