Atmospheric geostrophic turbulence:
Charney ’71 theory and Nastrom&Gage ’85 Obs.

as told by Paola Cessi

The observations: measurements from commercial Boeing 747's

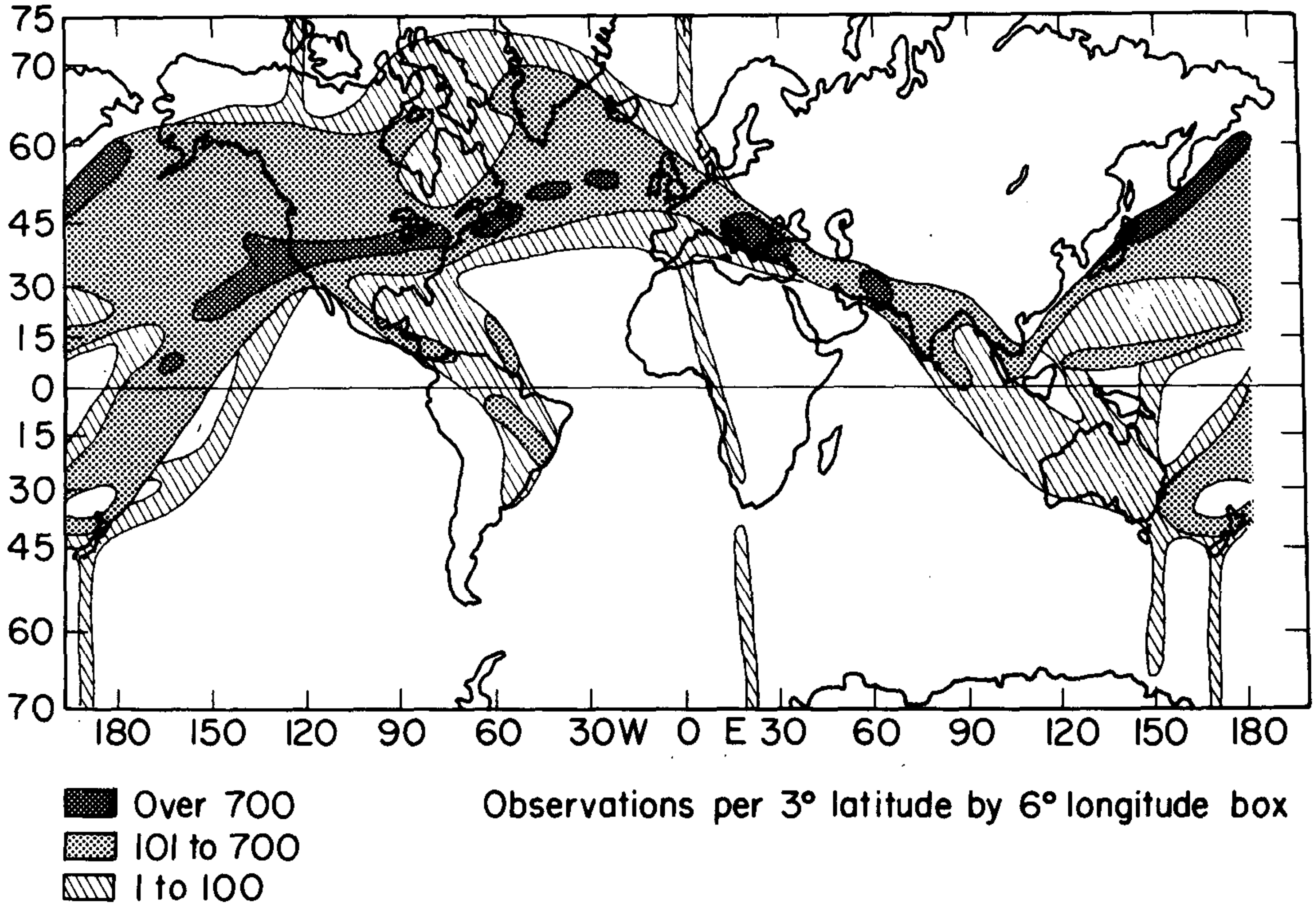


FIG. 1. Geographical distribution of GASP observations from flight segments at least 2400 km long; see text.

Most measurements in midlatitudes between 9 and 14 km in height

Only a subset of about 1500 flights are used in spectral analysis

The famous spectrum

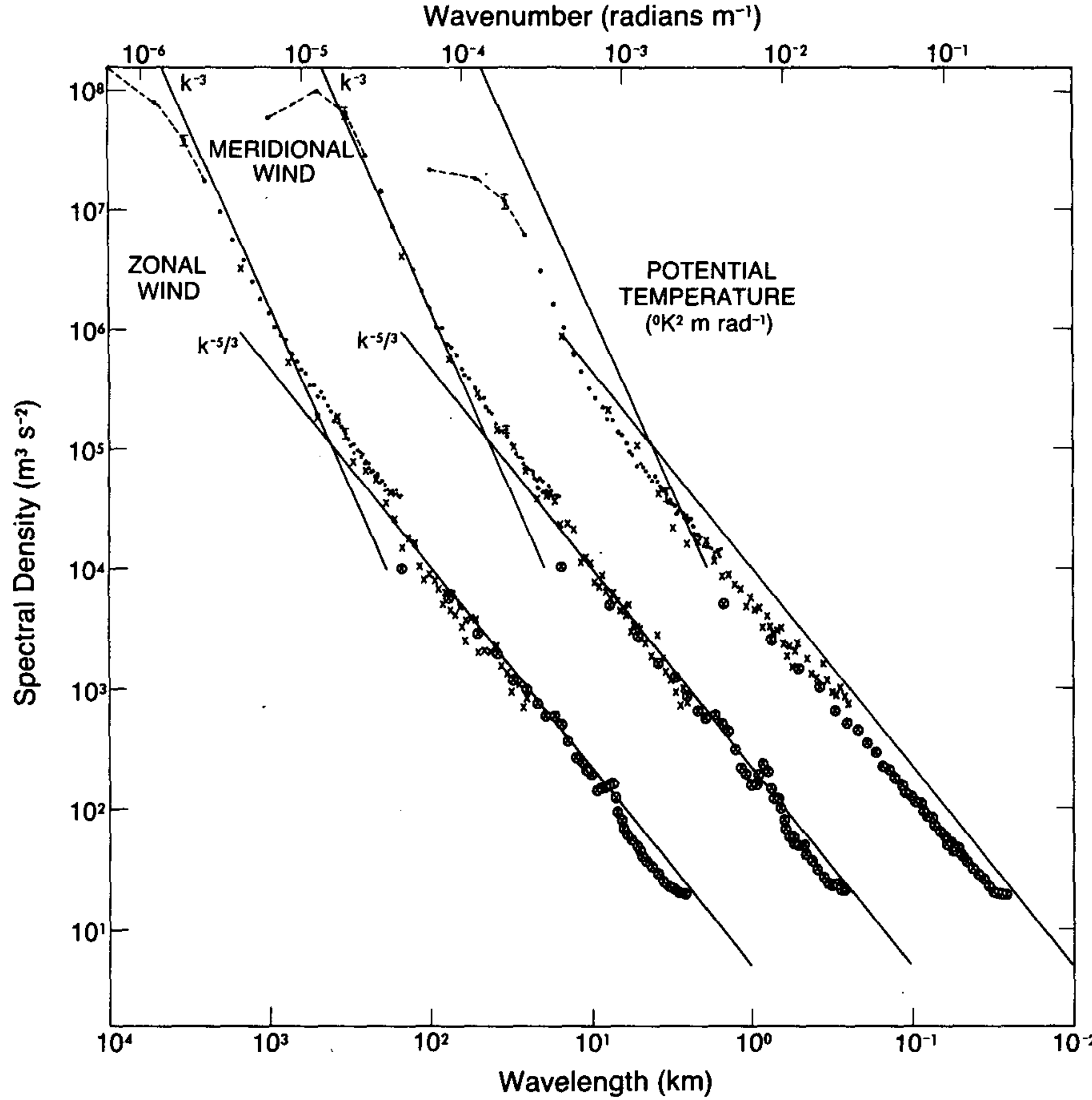


FIG. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes −3 and −5/3 are entered at the same relative coordinates for each variable for comparison.

velocity is isotropic

k^{-3} spectrum for velocity with scales > 1000km and $k^{-5/3}$ for scales < 300km

Charney's explanation for the k^{-3} spectrum

Use the continuously stratified QGPV approximation on large, but not planetary, scales

Restrictions:

$Ro = \frac{k_H U}{f_o} < 1$	Advective time less than one day
$\frac{f_0 k_V}{N k_H} < 1$	Horizontal scales less than baroclinic deformation radius
$\frac{k_V}{k_H} < Ro^2$	Horizontally nondivergent flow (small *w*)
$(a k_H)^{-1} < Ro$	Scales less than Earth's radius
$k_v > H$	Vertical scales smaller than the scale of *N(z)*

QGPV equation

$$\partial_t q + J(\psi, q) = 0 \quad u = -\psi_y, \quad v = \psi_x$$

Conservation of PV following horizontal flow at all z-levels

$$q(x, y, z, t) \equiv \nabla^2 \psi + \frac{f_0^2}{\rho} \partial_z \left(\frac{\rho \psi_z}{N^2} \right) + \beta y$$

Potential Vorticity

Multiply QGPV by $-\psi$ to get $dE/dt=0$.

$$E = \iiint \frac{1}{2} \rho(z) \left(|\nabla \psi|^2 + \frac{f_0^2}{N^2} \psi_z^2 \right) dx\, dy\, dz$$

Energy

Multiply QGPV by q to get $dF/dt=0$

$$F = \iiint \rho(z) q^2 \, dx\, dy\, dz$$

Potential enstrophy

Need to assume $\psi_z = 0$ at top and bottom boundary: no SQG!

All other invariants are ignored

Analogy with 2-D turbulence

Go into spectral space for both E and F:

$$2E = \sum_1^\infty \lambda_m a_m^2 \equiv \sum_1^\infty b_m \qquad 2F = \sum_1^\infty \lambda_m^2 a_m^2 \equiv \sum_1^\infty \lambda_m b_m \qquad \lambda_m > 0$$

By definition the λ_m are increasing with m

$$\sum_M^\infty b_m < \frac{1}{\lambda}_M \sum_M^\infty \lambda_m b_m < \frac{F}{\lambda_m} \underset{m\to\infty}{\to} 0$$

This proves that there cannot be a forward cascade in energy.

Charney remarks that the proof does not hold if the top or bottom boundaries are not isothermal and form fronts, but he proceeds on the assumption that fronts would be weak.

The spectrum of fronts

Assuming that fronts lead to a discontinuity in velocities implies that the velocity would have a spectrum of k^{-1} and hence the kinetic energy would have a spectrum of k^{-2}

We can understand that a discontinuous field has a spectrum k^{-1} because a δ- function has a spectrum of 1 and it is the derivative of a discontinuous function.

The k^{-3} spectrum

Rescale ψ and the vertical coordinate to make the $q - \psi$ relation into a Poisson equation

$$d\zeta \equiv \frac{N}{f_0} dz \qquad \chi \equiv \left[\frac{\rho(z)}{\rho(0)}\right]^{1/2} \psi$$

Make the following assumptions:

$k_H^{-1} \ll HN/f_0$ Horizontal scale less than deformation radius

$k_h^2 U_k \gg \beta$ Relative vorticity much larger than planetary vorticity

$k_v \gg D^{-1}$ Fluctuations on a vertical scale smaller than the scale of N

Horizontally homogenous and isotropic turbulence

Local transfer in wavenumber space

Existence of an inertial range

$$\partial_t \Delta\chi + J(\chi, \Delta\chi) = 0 \qquad \Delta\chi \equiv \chi_{xx} + \chi_{yy} + \chi_{zz}$$

The k^{-3} spectrum

The expression for energy is:

$$E = \iiint \frac{f_0}{2N} \rho(z) \left(\chi_x^2 + \chi_y^2 + \chi_\zeta^2 \right) \, dx \, dy \, d\zeta$$

Charney invokes homogeneity and isotropy in all 3 dimensions, and that energy depends only on the transfer of potential enstrophy η

η has dimension of T^{-3} because potential enstrophy has dimensions T^{-2}

$E(k)$ has dimension of $L^3 T^{-2}$ because $E = \int E(k) \, dk$

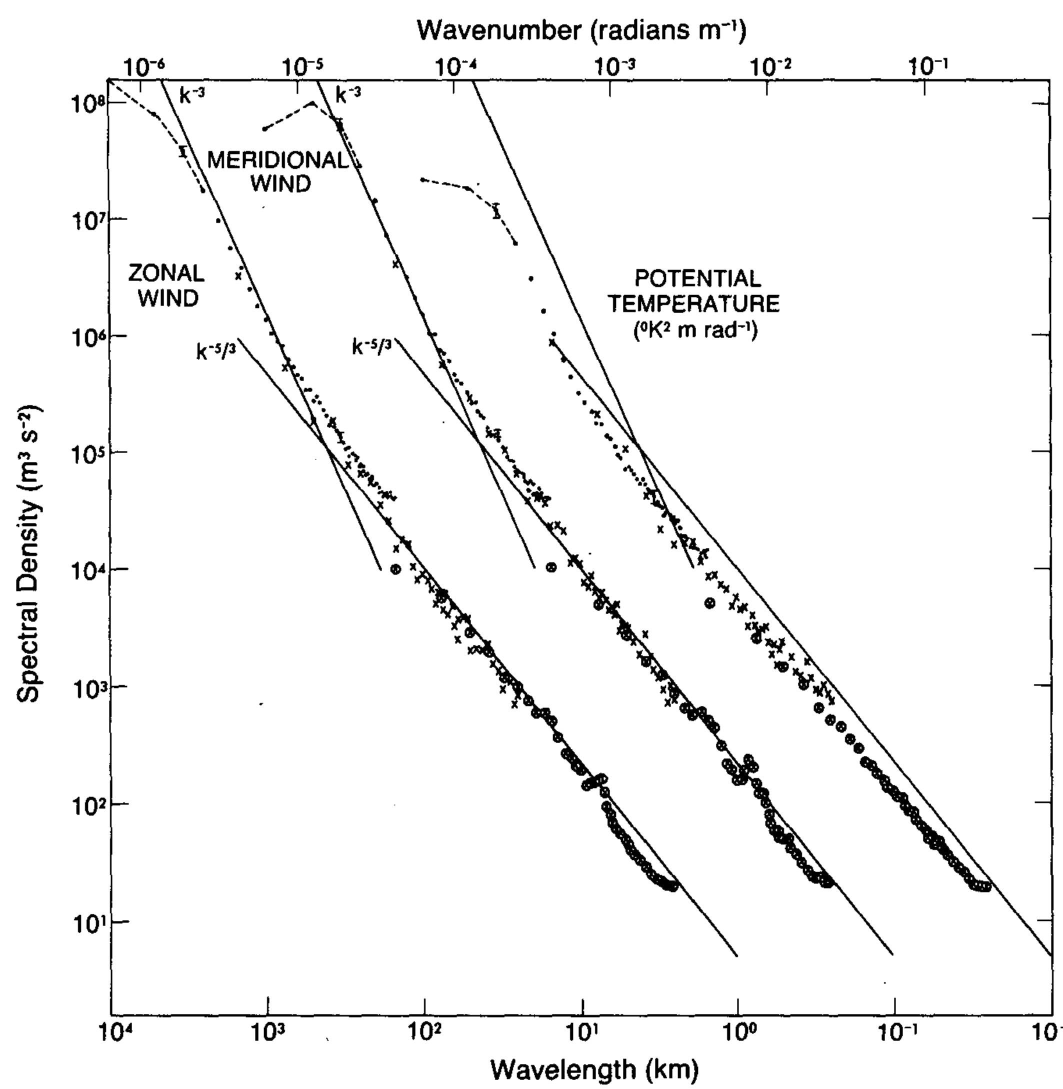
Dimensional analysis gives

$$E(k) = C \eta^{2/3} k^{-3}$$

Implications of the energy spectrum

Dissipation is effected by Ekman drag at large scale, as in 2-D turbulence.

Available potential energy is half of kinetic energy (equal contribution from u^2 and v^2)



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The k^{-3} spectrum - an alternative explanation (Batchelor via Rick)

Consider the scalar variance between k_1 and $k_2 \sim k_1$ is by definition

$S(k_1)(k_2 - k_1)$ where $S(k)$ is the variance spectrum

Assume the scalar to be advected by a uniform shear γ

the scalar's wavenumber grows like $e^{\gamma t}$

For weak diffusion, scalar variance is conserved:

$$S(k_1)(k_2 - k_1) = S(e^{\gamma t})(k_2 e^{\gamma t} - k_1 e^{\gamma t})$$

Only possible if $S(k) = 1/k$

If QGPV **can be considered a passive scalar**, strained by large-scale shear, the spectrum of q is $S(q) = 1/k$

$-k^2 \psi_k = q_k$ $k^4 \psi_k^2 = S(q) \sim 1/k$ $E(k) \sim k^{-3}$

Criticisms to Charney's scaling

The k^{-3} spectrum should apply to scales smaller than deformation radius, which instead show a shallower slope (perhaps due to GW turbulence).

Conservation of PV or enstrophy occurs **at every level in z,** while energy requires volume integration: isotropy in 3-D for enstrophy is not justified.

The existence of quantized modes does not apply in the infinite domain considered by Charney, but it applies in the finite domain (Tung & Welch, 2001)

Neglecting the surface contribution of temperature (SQG) is not justified (Navid's presentation next week).

Others?