Eddy fluxes in baroclinic geostrophic turbulence

Larichev & Held JAS 1995 and H & L JAS 1996
as told by Cesar

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Questions to be addressed

In strongly non-linear baroclinic systems…

1. Which scales dominate the PV flux or the conversion of background PE?

2. How does the PV flux and diffusivity scale with the background parameters?
Two-layer QG model

\[ Q_{1y} = -\frac{k_d^2}{2} U \]

\[ Q_{2y} = \frac{k_d^2}{2} U \]

\[ k_d \overset{\text{def}}{=} \frac{f_0}{\sqrt{g'H}} \]
Two-layer QG dynamics

Barotropic

\[ \partial_t \nabla^2 \psi + J(\psi, \nabla^2 \psi) + J(\tau, \nabla^2 \tau) + U \partial_x \nabla^2 \tau = -\nu_8 \nabla^8 (\nabla^2 \psi) - \mu \nabla^2 (\psi - \tau)/2 \]

Baroclinic

\[ \partial_t (\nabla^2 - k_d^2) \tau + J(\psi, \nabla^2 \tau - k_d^2 \tau) + J(\tau, \nabla^2 \tau - k_d^2 \tau) + U \partial_x \nabla^2 \psi - k_d^2 U \psi_x = -\nu_8 \nabla^8 (\nabla^2 \tau - k_d^2 \tau) - \mu \nabla^2 (\tau - \psi)/2 \]

\[ \nabla^2 \text{ def } \partial_x^2 + \partial_y^2 \]

\[ \psi = \frac{\psi_1 + \psi_2}{2} \quad \tau = \frac{\psi_1 - \psi_2}{2} \]
Doubly periodic box

\[ Lk_d = 50 \]

\[ U = 0.005 \]
Linear stability

\[(k, l) = (0.64k_d, 0)\]

e-folding scale: \(2.8 \frac{L}{U}\)
Evolution of PV

\[ f_0/2 \]

\[ t = 0000000000000001 \]
Snapshot

\( \psi \)

\( \nabla^2 \psi \)

Streamfunction

PV
Snapshot

\[ \mathcal{T} \]

\[ (\nabla^2 - k_d^2) \mathcal{T} \]

Streamfunction

PV
Two-layer QG Energetics

\[ \partial_t E_\psi = \text{Re}[\hat{\psi}^* \hat{J}(\psi, \nabla^2 \psi)] + \text{Re}[\hat{\psi}^* \hat{J}(\tau, \nabla^2 \tau)] - U k \kappa^2 \text{Re}[i \hat{\psi}^* \hat{\tau}] \]

\[ - \nu_8 \kappa^{10} |\hat{\psi}|^2 - \mu \kappa^2 (|\hat{\psi}|^2 - \text{Re}[\hat{\psi}^* \hat{\tau}]) / 2 \]

\[ \partial_t E_\tau = \text{Re}[\hat{\tau}^* \hat{J}(\tau, \nabla^2 \psi)] + \text{Re}[\hat{\tau}^* \hat{J}(\psi, (\nabla^2 - k_d^2) \tau)] - U k (\kappa^2 - k_d^2) \text{Re}[i \hat{\tau}^* \hat{\psi}] \]

\[ - \nu_8 \kappa^8 (\kappa^2 + k_d^2) |\hat{\tau}|^2 - \mu \kappa^2 (|\hat{\tau}|^2 - \text{Re}[\hat{\tau}^* \hat{\psi}]) / 2 \]

\[ \kappa^2 \overset{\text{def}}{=} k^2 + l^2 \]

\[ \frac{1}{2} < \nabla \psi \cdot \nabla \psi > = \int_0^\infty E_\psi(\kappa, t) d\kappa \]

\[ \frac{1}{2} < \nabla \tau \cdot \nabla \tau + m_1^2 \tau^2 > = \int_0^\infty E_\tau(\kappa, t) d\kappa \]
Energy evolution

Barotropic/baroclinic partition

\[ \frac{1}{2} \langle \nabla \psi \cdot \nabla \psi \rangle \]

\[ \frac{1}{2} \langle \nabla \tau \cdot \nabla \tau + k_d \kappa^2 \tau^2 \rangle \]
Energy evolution

Baroclinic: KE/PE partition

\[ \frac{1}{2} \left< k_d^2 \nabla^2 \right> \]
Two-layer QG Energetics

\[
\partial_t E_\psi = \underbrace{\text{Re}[\hat{\psi}^* \hat{J}(\psi, \triangle \psi)]}_{I} + \underbrace{\text{Re}[\hat{\psi}^* \hat{J}(\tau, \triangle \tau)]}_{II} - U k \kappa^2 \text{Re}[i \hat{\psi}^* \hat{\tau}]_{III} \\
- \nu_8 \kappa^{10} |\hat{\psi}|^2 - \mu \kappa^2 (|\hat{\psi}|^2 - \text{Re}[\hat{\psi}^* \hat{\tau}])/2_{IV} \\
\partial_t E_\tau = \underbrace{\text{Re}[\hat{\tau}^* \hat{J}(\tau, \triangle \psi)]}_{I} + \underbrace{\text{Re}[\hat{\tau}^* \hat{J}(\psi, (\triangle - m_1^2) \tau)]}_{II} - U k (\kappa^2 + m_1^2) \text{Re}[i \hat{\tau}^* \hat{\psi}]_{III} \\
- \nu_8 \kappa^8 (\kappa^2 + m_1^2) |\hat{\tau}|^2 - \mu \kappa^2 (|\hat{\tau}|^2 - \text{Re}[\hat{\tau}^* \hat{\psi}])/2_{IV}
\]

\[
\frac{1}{2} < \nabla \psi \cdot \nabla \psi > = \int_0^\infty E_\psi(\kappa, t) d\kappa \\
\frac{1}{2} < \nabla \tau \cdot \nabla \tau + m_1^2 \tau^2 > = \int_0^\infty E_\tau(\kappa, t) d\kappa
\]
Two-layer QG Energetics

\[ \partial_t E_\psi = \text{Re}[\hat{\psi}^* \hat{J}(\psi, \nabla^2 \psi)] + \text{Re}[\hat{\psi}^* \hat{J}(\tau, \nabla^2 \tau)] - U k \kappa^2 \text{Re}[i \hat{\psi}^* \hat{\tau}] \]

\[ \partial_t E_\tau = \text{Re}[\hat{\tau}^* \hat{J}(\tau, \nabla^2 \psi)] + \text{Re}[\hat{\tau}^* \hat{J}(\psi, (\nabla^2 - k_d^2) \tau)] - U k (\kappa^2 - k_d^2) \text{Re}[i \hat{\tau}^* \hat{\psi}] \]

\[ \kappa^2 \overset{\text{def}}{=} k^2 + l^2 \]

\[ \frac{1}{2} < \nabla \psi \cdot \nabla \psi > = \int_0^\infty E_\psi(\kappa, t) d\kappa \]

\[ \frac{1}{2} < \nabla \tau \cdot \nabla \tau + m_1^2 \tau^2 > = \int_0^\infty E_\tau(\kappa, t) d\kappa \]
“Production” in spinup stage

- Conversion starts near the deformation scale
- But moves towards larger scales
- In statistical steady state, the production at deformation scales is negligible

\[ P(k, l) = U k_d^2 \Re[\tilde{v}_\psi \hat{\tau}^*] \]
Spectra

$E_\psi(k)$

$E_\tau(k)$

$k^{5/3}$

Spectral energy density

deformation scale
Spectral fluxes

\[ \partial_t E_{\psi} = \Re[\hat{\psi}^* \hat{J}(\psi, \nabla^2 \psi)] + \Re[\hat{\psi}^* \hat{J}(\tau, \nabla^2 \tau)] - U k \kappa^2 \Re[i\hat{\psi}^* \hat{\tau}] - \nu_8 \kappa^{10} |\hat{\psi}|^2 - \mu k^2 (|\hat{\psi}|^2 - \Re[\hat{\psi}^* \hat{\tau}]) / 2 \]

(This helped me catch a bug in my code!)
Spectral fluxes

\[ \partial_t E_\tau = \text{Re}[\hat{\tau}^*(\tau, \nabla^2 \psi)] + \text{Re}[\hat{\tau}^*(\psi, (\nabla^2 - k_d^2)\tau)] - U k (\kappa^2 - k_d^2) \text{Re}[i\hat{\tau}^* \hat{\psi}] \]

\[ - \nu_8 \kappa^8 (\kappa^2 + k_d^2) |\hat{\tau}|^2 - \mu \kappa^2 (|\hat{\tau}|^2 - \text{Re}[\hat{\tau}^* \hat{\psi}]) / 2 \]

Baroclinic Spectral Energy Transfers

- I
- II
- III
- IV
- V

Spectral density

k

deformation scale
The “Salmonian” energy cycle

\[ U \lambda^{-2} \langle \psi_x \tau \rangle \]

Baroclinic Energy

Barotropic Energy

\[ L^{-1} \]
\[ k_0 \]
\[ \lambda^{-1} \]
\[ S80 \]
\[ HH80 \]

Small-scale dissipation (molecular effects)

\[ L_{\text{diss.}}^{-1} \]
Scaling the PV flux

Using Kolmogovian arguments the barotropic energy spectrum in the energy inertial range is

\[ E_\psi = C_\psi \epsilon_\psi^{2/3} \kappa^{-5/3} \]

\[ \kappa^2 \overset{\text{def}}{=} k^2 + l^2 \]
At scales larger than the deformation radius, the baroclinic PV equation reduces to the scalar equation

\[ \tau_t + J(\psi, \tau) = 0 \]

An in the inertial range, the energy production scales as

\[ \epsilon_{\tau} = \text{const.} \sim \frac{E_{\tau} \kappa}{T_{\kappa}} \]
Scaling the PV flux

But a local estimate of the eddy turn around time-scale is

$$T_\kappa \sim (E_\psi k^3)^{1/2}$$

And therefore

$$E_\tau = C_\tau \epsilon_\tau \epsilon_\psi^{-1/3} \kappa^{-5/3}$$
Scaling the PV flux

In statistical steady state, the “Salmonian” phenomenological picture of energy transformations gives

\[ \varepsilon_T = \varepsilon_\psi \]
Scaling the PV flux

But this is not quantitatively true in the simulations

$$\epsilon_\tau > \epsilon_\psi$$
Scaling the PV flux

“Accepting this relationship as a qualitative guide…”

\[ \epsilon_\tau = \epsilon_\psi \]

And therefore

\[ E_\tau = C_\tau \epsilon^{2/3} \kappa^{-5/3} \]

\[ E_\psi = C_\psi \epsilon^{2/3} \kappa^{-5/3} \]
Scaling the PV flux

Now, the numerical calculations show that in the “inertial range” the baroclinic energy is largely potential. Hence

$$\hat{\psi} \sim \frac{k_d}{\kappa} \hat{T}$$

And in physical space

$$\psi \sim \left(\frac{k_d}{\kappa_0}\right)\tau$$

where $\kappa_0^{-1}$ is the energy containing length scale
Scaling the PV flux

Now in equilibrium the baroclinic energy production is

$$\epsilon_{\tau} = U k_d^2 \langle \tau \nu_\psi \rangle$$

Which is proportional to the meridional PV flux

$$\nu_i q_i = (-1)^i k_d^2 \langle \tau \nu_\psi \rangle$$
Scaling the PV flux

With the mean PV gradient, the baroclinic streamfunction equation at scales larger than the deformation radius is

$$\tau_t + J(\psi, \tau - U y) = 0$$

Turbulent diffusion arguments give

$$\tau \sim U \kappa_0^{-1} \quad \Rightarrow \quad \psi \sim \frac{k_d}{\kappa_0^2 U}$$
Scaling the PV flux

Putting all this together gives the energy production

$$\epsilon_\tau \sim U k_d^2 \kappa_0 \psi \tau \sim U^3 \frac{k_d^3}{\kappa_0^2}$$
Turbulent diffusivity

Using mixing-length arguments, the eddy diffusivity is

\[ D = V \kappa_0^{-1} \]

\[ V \overset{\text{def}}{=} \text{RMS velocity} \]

\[ V = \kappa_0 \psi \sim k_d \tau = U \frac{k_d}{\kappa_0} \]

\[ D = \frac{k_d}{\kappa_0^2} U \]
What determines $\kappa_0$

- Size of the domain
- Planetary vorticity gradient (halting of the cascade)
- Bottom friction
- Topography
- etc.
Adding beta
Two-layer QG model

\[ Q_{1y} = \beta - \frac{k_d^2}{2} U \]
\[ Q_{2y} = \beta + \frac{k_d^2}{2} U \]

Unstable if \( \xi \overset{\text{def}}{=} \frac{\beta}{U k_d^2} \)
On a $\beta$–plane

The upscale cascade is halted at (e.g., see Rhines 1975)

$$\kappa_0 = \left(\frac{\beta}{V}\right)^{1/2}$$

Thus

$$V \sim \xi U \quad \xi \sim \frac{k_d}{\kappa_0} \quad D \sim U \frac{\xi^2}{k_d}$$

These results are qualitatively consistent with simulations.
In summary...

• Numerical simulations and analysis indicate that the energy production (PV flux, heat flux, etc) is contained at the largest scales of the flow.

• Kolmogorovian-type assumptions + turbulent diffusion of a scalar lead to simple scaling for the eddy amplitude, energy flux, and eddy diffusivity in terms of the basic state shear and energy containing scale.

• On a beta plane, the cascade is assumed to be halted at the Rhines scale, and the scaling is simply written in terms of the criticality.
Questions

• Are these results applicable to the oceanic mesoscales?

• If so, are the results of this type of study being used to parameterize mesoscale eddies in coarse resolution models?