# Eddy fluxes in baroclinic geostrophic turbulence

Larichev & Held JAS 1995 and H & L JAS 1996
as told by Cesar

PO Theory Seminar, SIO Fall 2015

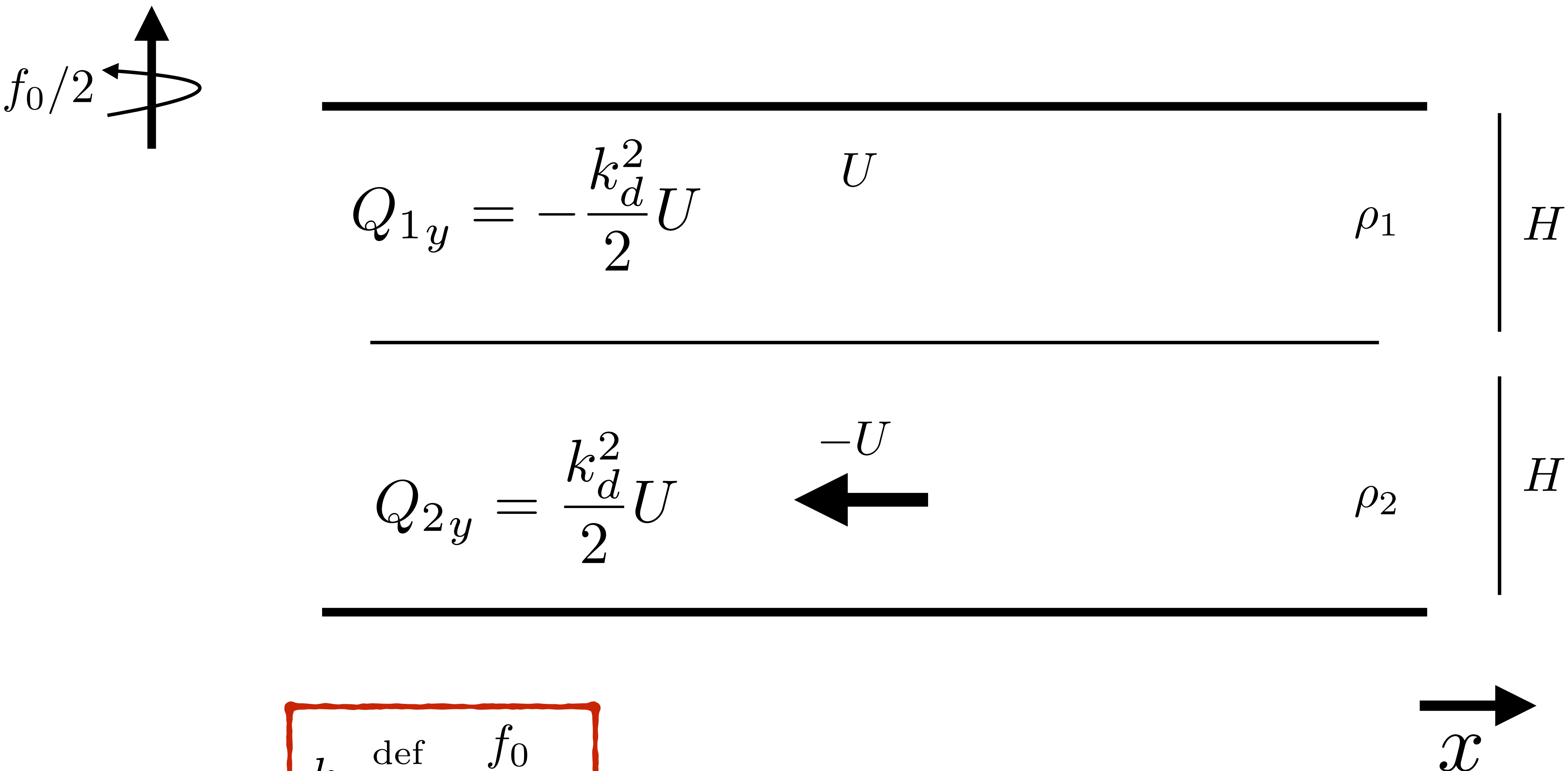
# Questions to be addressed

In strongly non-linear baroclinic systems…

1. Which scales dominate the PV flux or the conversion of background PE?

2. How does the PV flux and diffusivity scale with the background parameters?

# Two-layer QG model




$$k_d \stackrel{\text{def}}{=} \frac{f_0}{\sqrt{g'H}}$$

# Two-layer QG dynamics

Barotropic

$$\partial_t \nabla^2 \psi + \mathsf{J}(\psi, \nabla^2 \psi) + \mathsf{J}(\tau, \nabla^2 \tau) \\ + U \partial_x \nabla^2 \tau = -\nu_8 \nabla^8 (\nabla^2 \psi) - \mu \nabla^2 (\psi - \tau)/2$$

Baroclinic

$$\partial_t (\nabla^2 - k_d^2) \tau + \mathsf{J}(\psi, \nabla^2 \tau - k_d^2 \tau) + \mathsf{J}(\tau, \nabla^2 \tau - k_d^2 \tau) + \\ U \partial_x \nabla^2 \psi - k_d^2 U \psi_x = -\nu_8 \nabla^8 (\nabla^2 \tau - k_d^2 \tau) - \mu \nabla^2 (\tau - \psi)/2$$

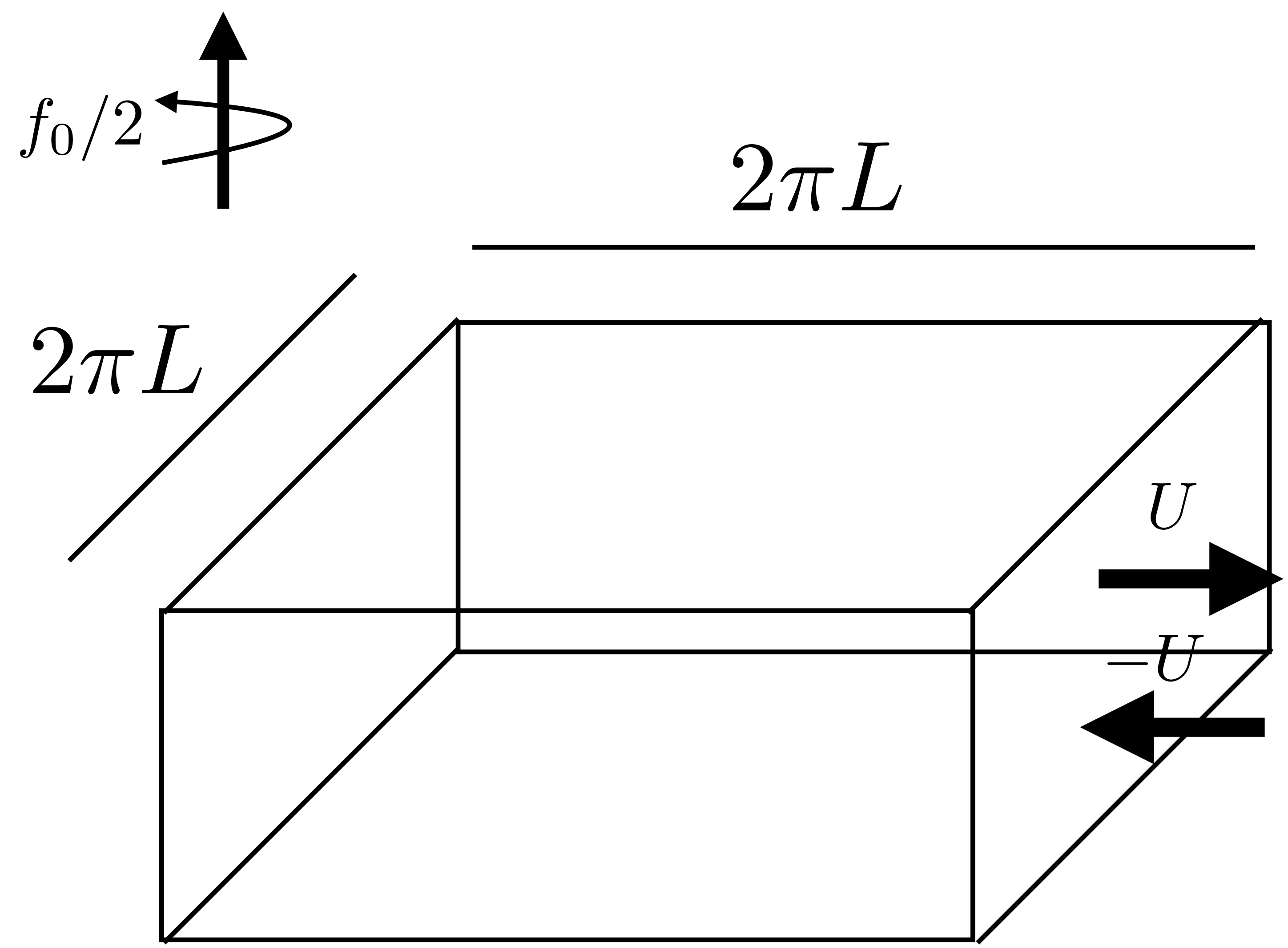
$$\nabla^2 \stackrel{\text{def}}{=} \partial_x^2 + \partial_y^2$$

$$\psi = \frac{\psi_1 + \psi_2}{2} \qquad \tau = \frac{\psi_1 - \psi_2}{2}$$

# Doubly periodic box

$$Lk_d = 50$$

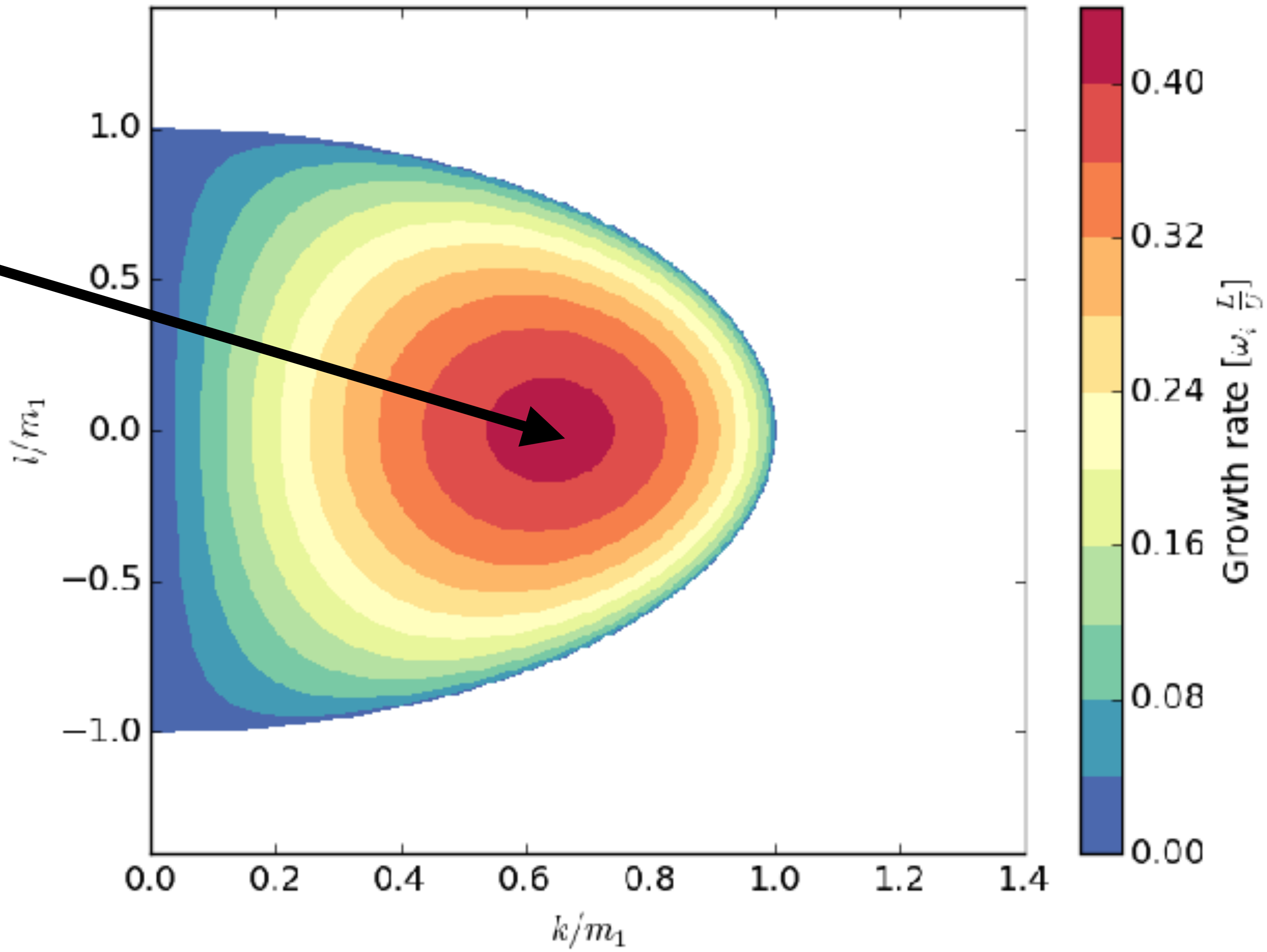
$$U = 0.005$$



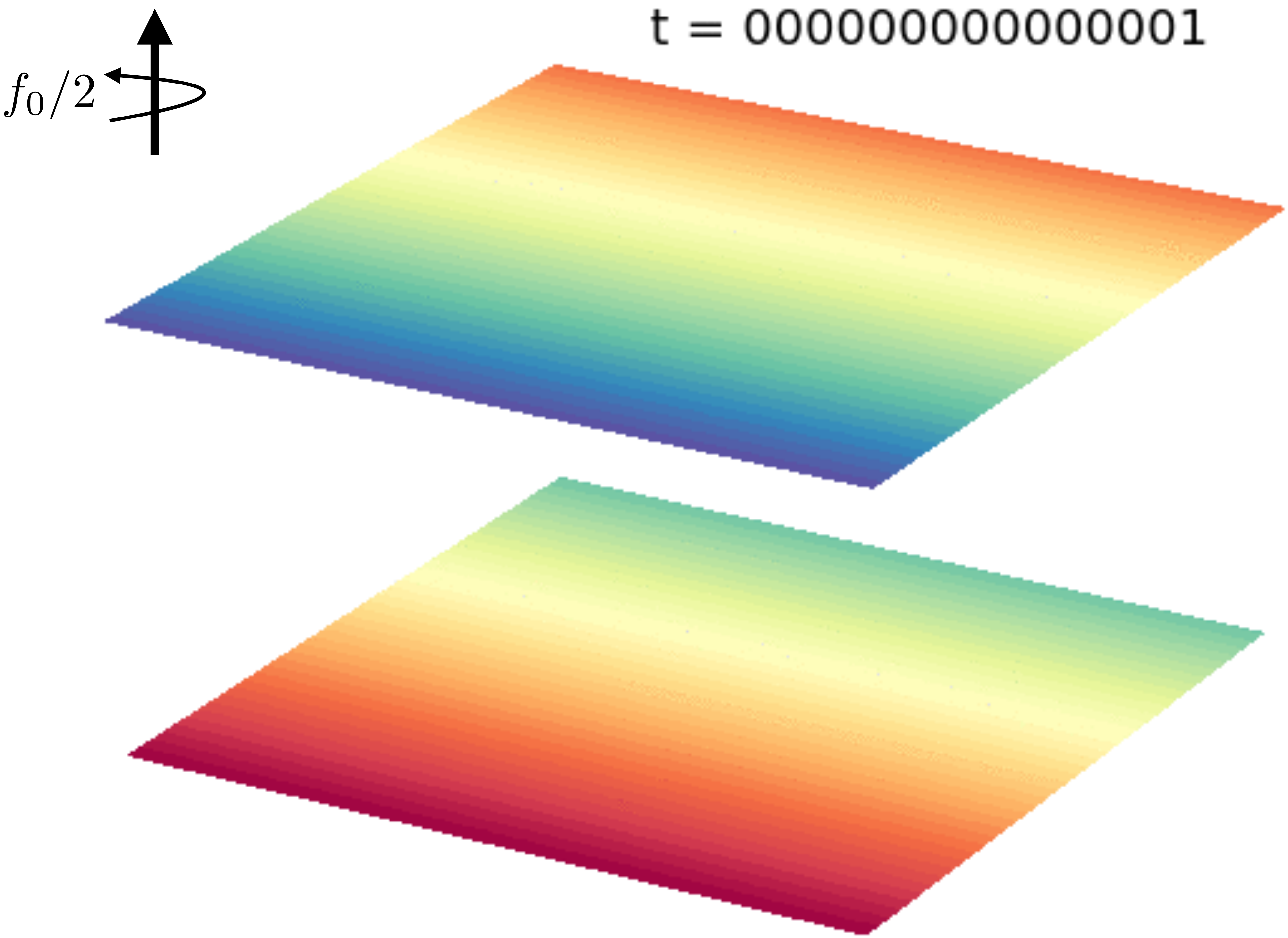
# Linear stability

$$(k, l) = (0.64 k_d, 0)$$

$$\text{e-folding scale} : 2.8 \, \frac{L}{U}$$

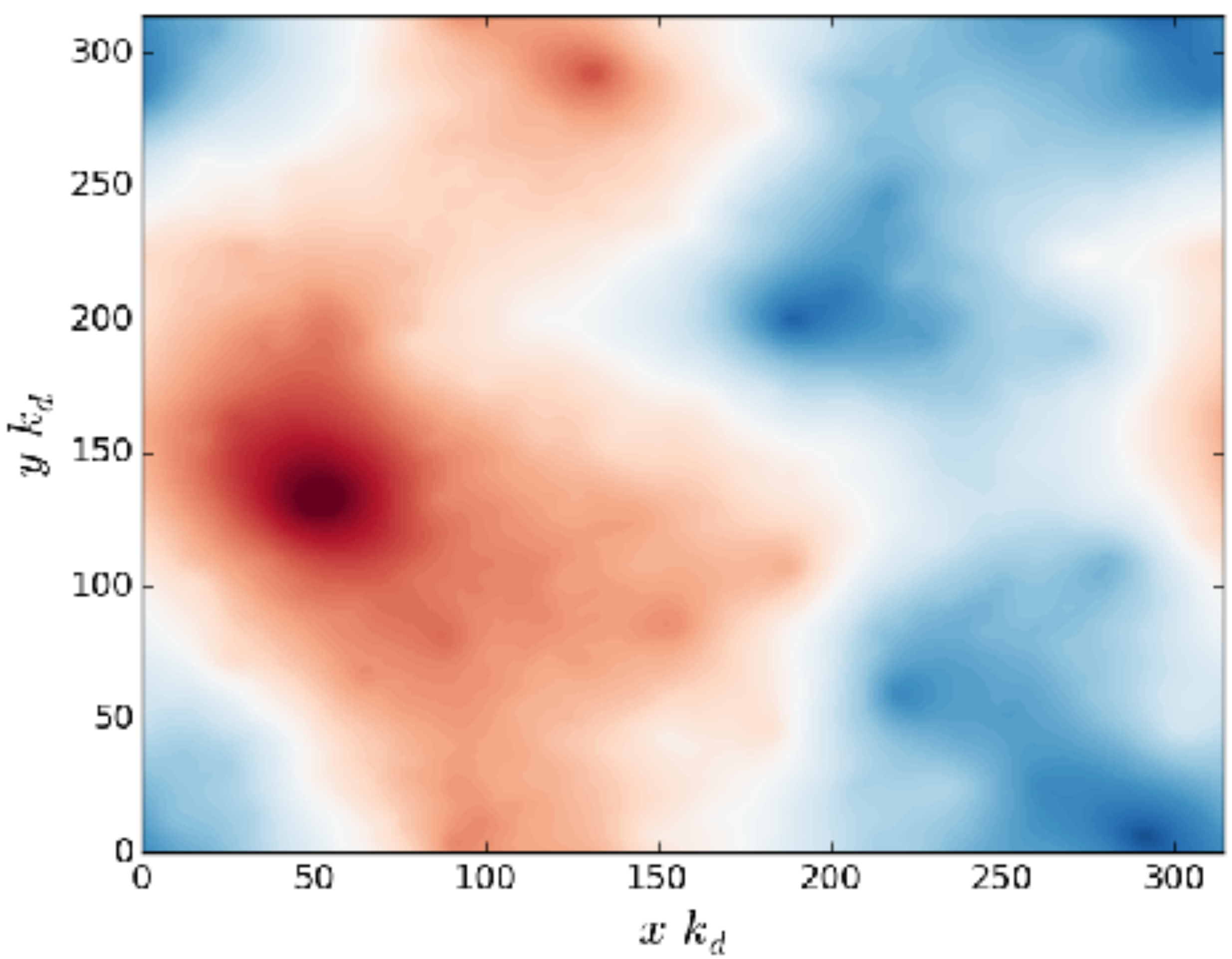


# Evolution of PV



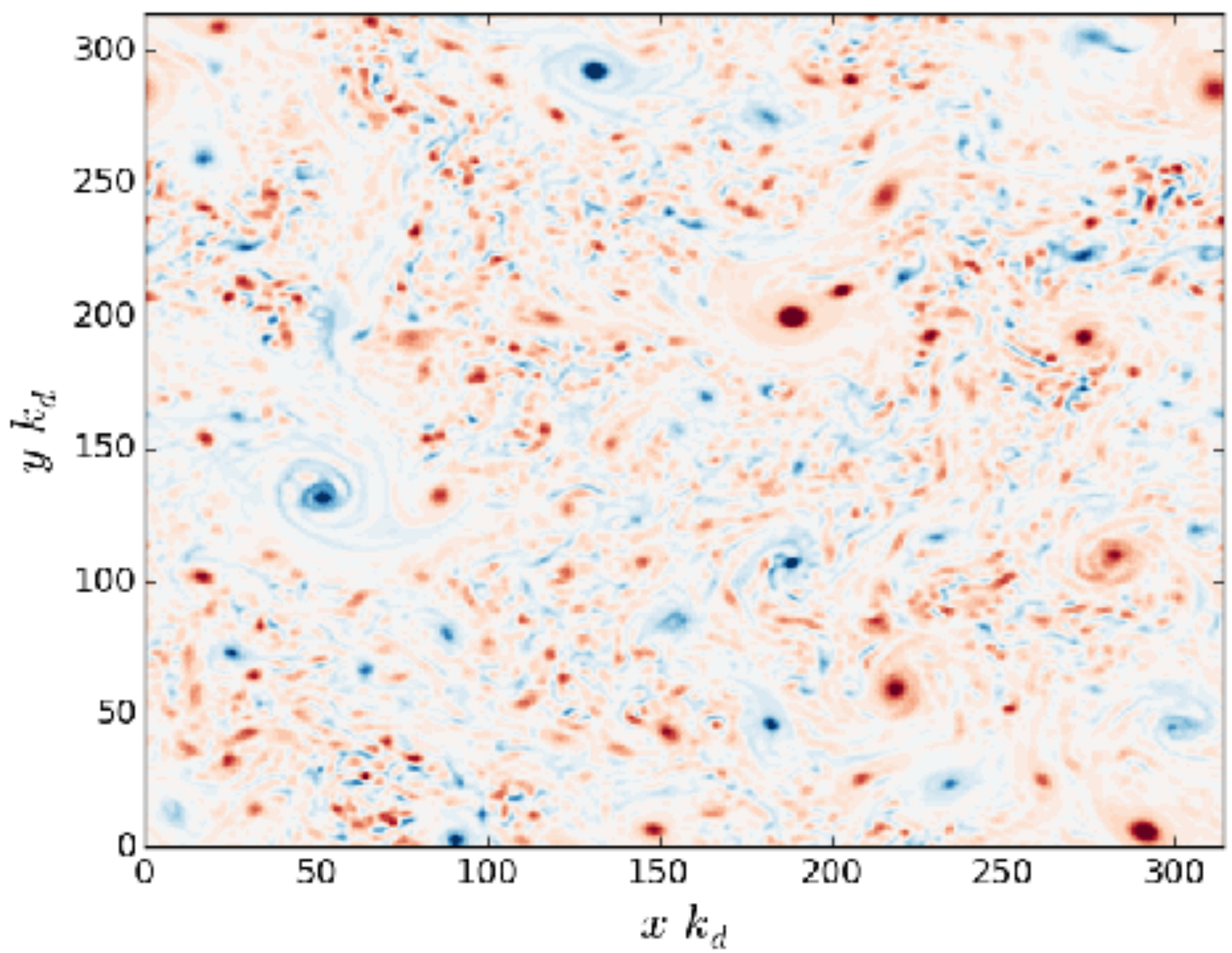
# Snapshot

$\psi$




Streamfunction

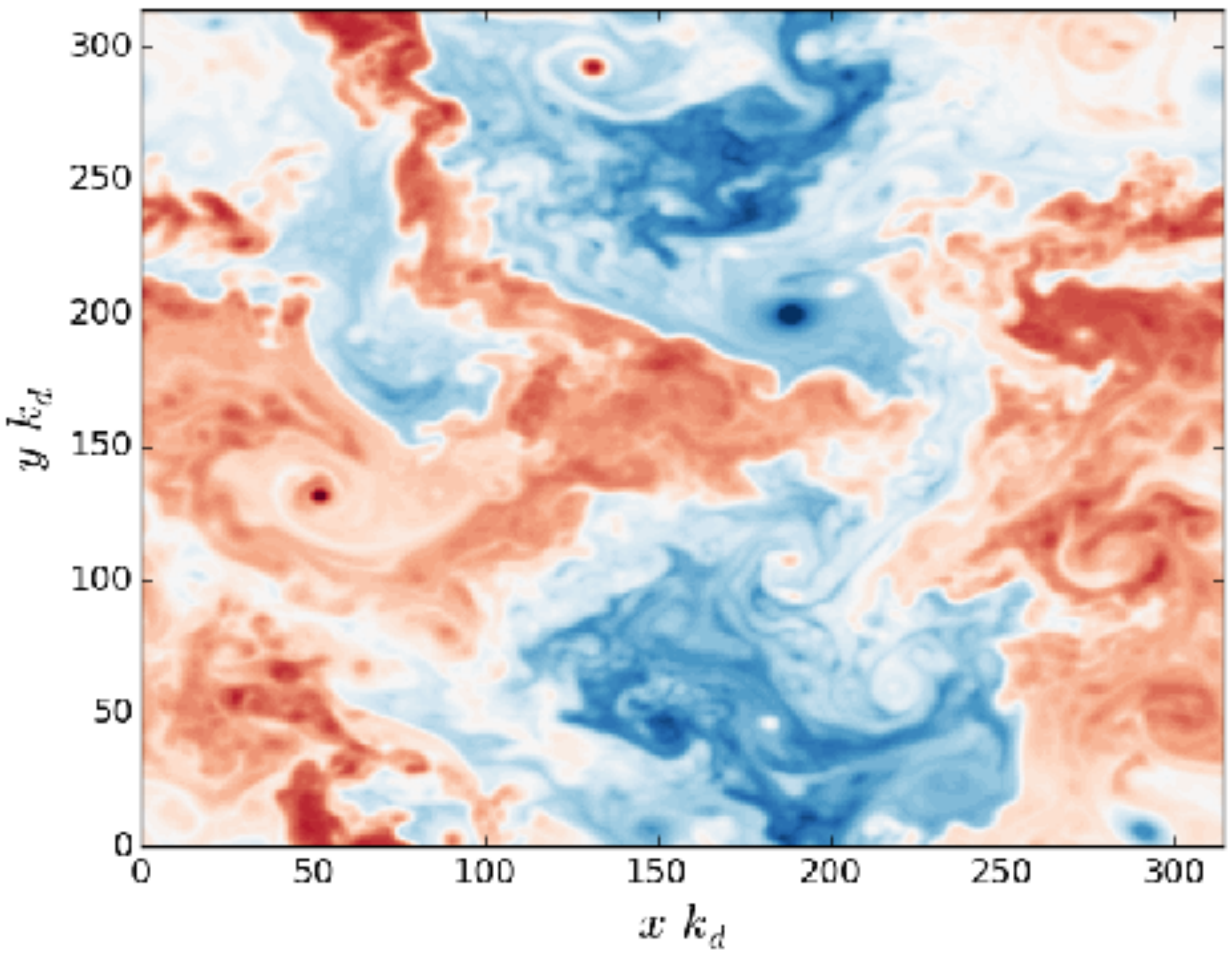
$\nabla^2\psi$




PV

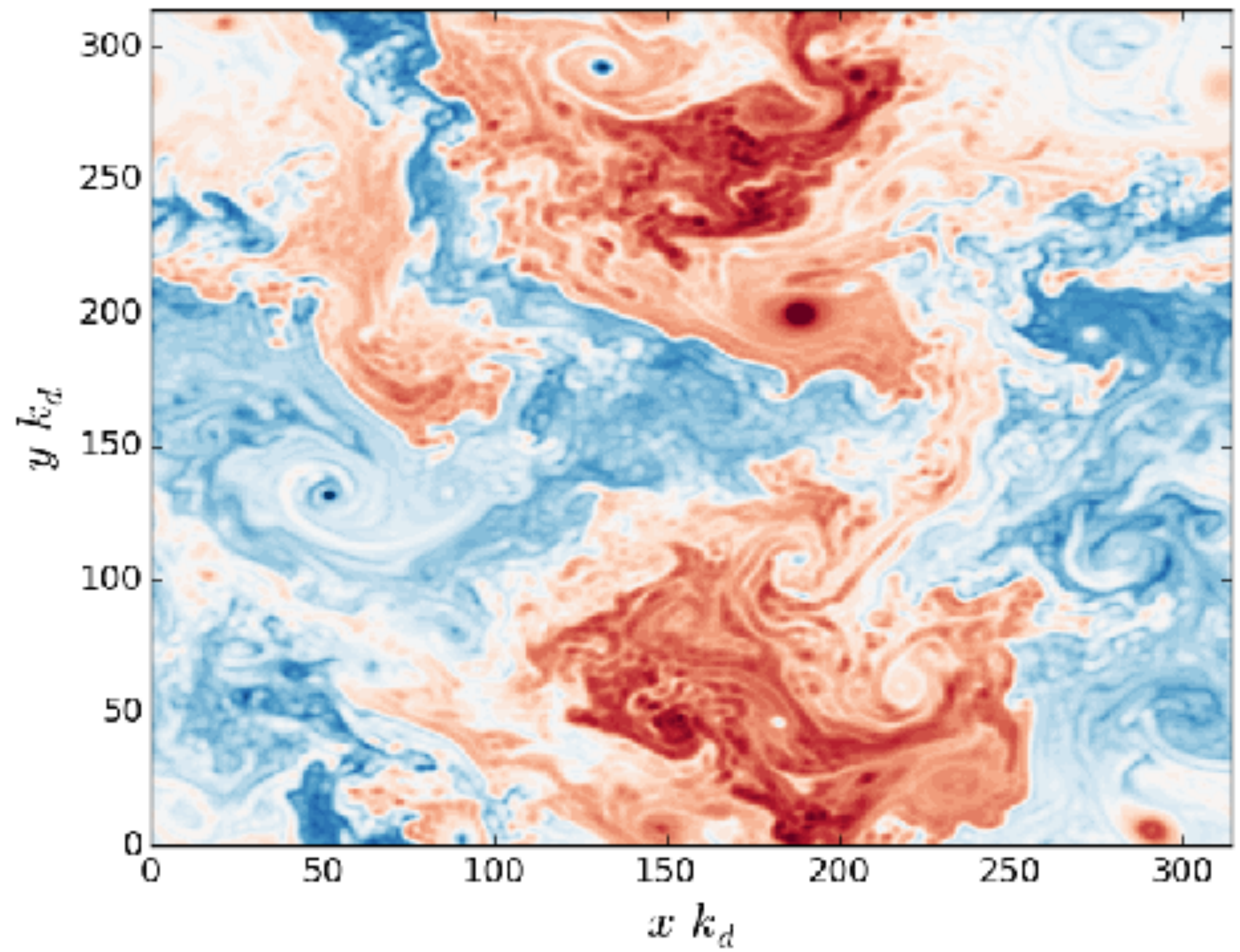
# Snapshot

$\tau$




Streamfunction

$(\nabla^2 - k_d^2)\tau$




PV

# Two-layer QG Energetics

$$\partial_t E_\psi = \underbrace{\mathrm{Re}[\hat{\psi}^\star \hat{\mathrm{J}}(\psi, \nabla^2\psi)]}_{I} + \underbrace{\mathrm{Re}[\hat{\psi}^\star \hat{\mathrm{J}}(\tau, \nabla^2\tau)]}_{II} - \underbrace{Uk\kappa^2 \mathrm{Re}[\mathrm{i}\hat{\psi}^\star \hat{\tau}]}_{III} - \underbrace{\nu_8 \kappa^{10} |\hat{\psi}|^2}_{IV} - \underbrace{\mu\kappa^2(|\hat{\psi}|^2 - \mathrm{Re}[\hat{\psi}^\star \hat{\tau}])/2}_{V}$$

$$\partial_t E_\tau = \underbrace{\mathrm{Re}[\hat{\tau}^\star \hat{\mathrm{J}}(\tau, \nabla^2\psi)]}_{I} + \underbrace{\mathrm{Re}[\hat{\tau}^\star \hat{\mathrm{J}}(\psi, (\nabla^2 - k_d^2)\tau)]}_{II} - \underbrace{Uk(\kappa^2 - k_d^2)\mathrm{Re}[\mathrm{i}\hat{\tau}^\star \hat{\psi}]}_{III} - \underbrace{\nu_8 \kappa^8 (\kappa^2 + k_d^2)|\hat{\tau}|^2}_{IV} - \underbrace{\mu\kappa^2(|\hat{\tau}|^2 - \mathrm{Re}[\hat{\tau}^\star \hat{\psi}])/2}_{V}$$

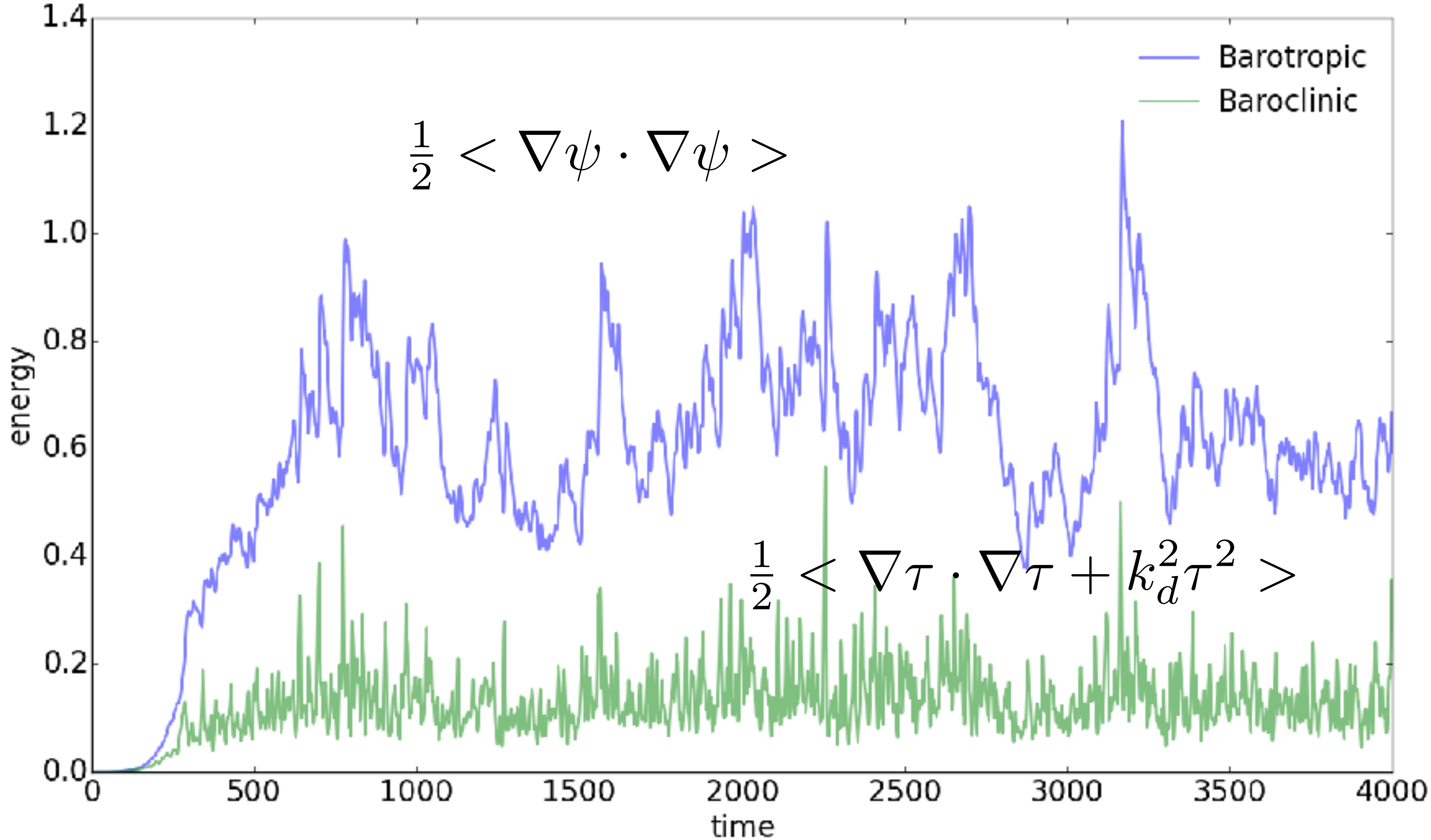
$$\kappa^2 \stackrel{\mathrm{def}}{=} k^2 + l^2$$

$$\tfrac{1}{2} < \nabla\psi \cdot \nabla\psi >= \int_0^\infty E_\psi(\kappa, t)\mathrm{d}\kappa$$

$$\tfrac{1}{2} < \nabla\tau \cdot \nabla\tau + m_1^2 \tau^2 >= \int_0^\infty E_\tau(\kappa, t)\mathrm{d}\kappa$$

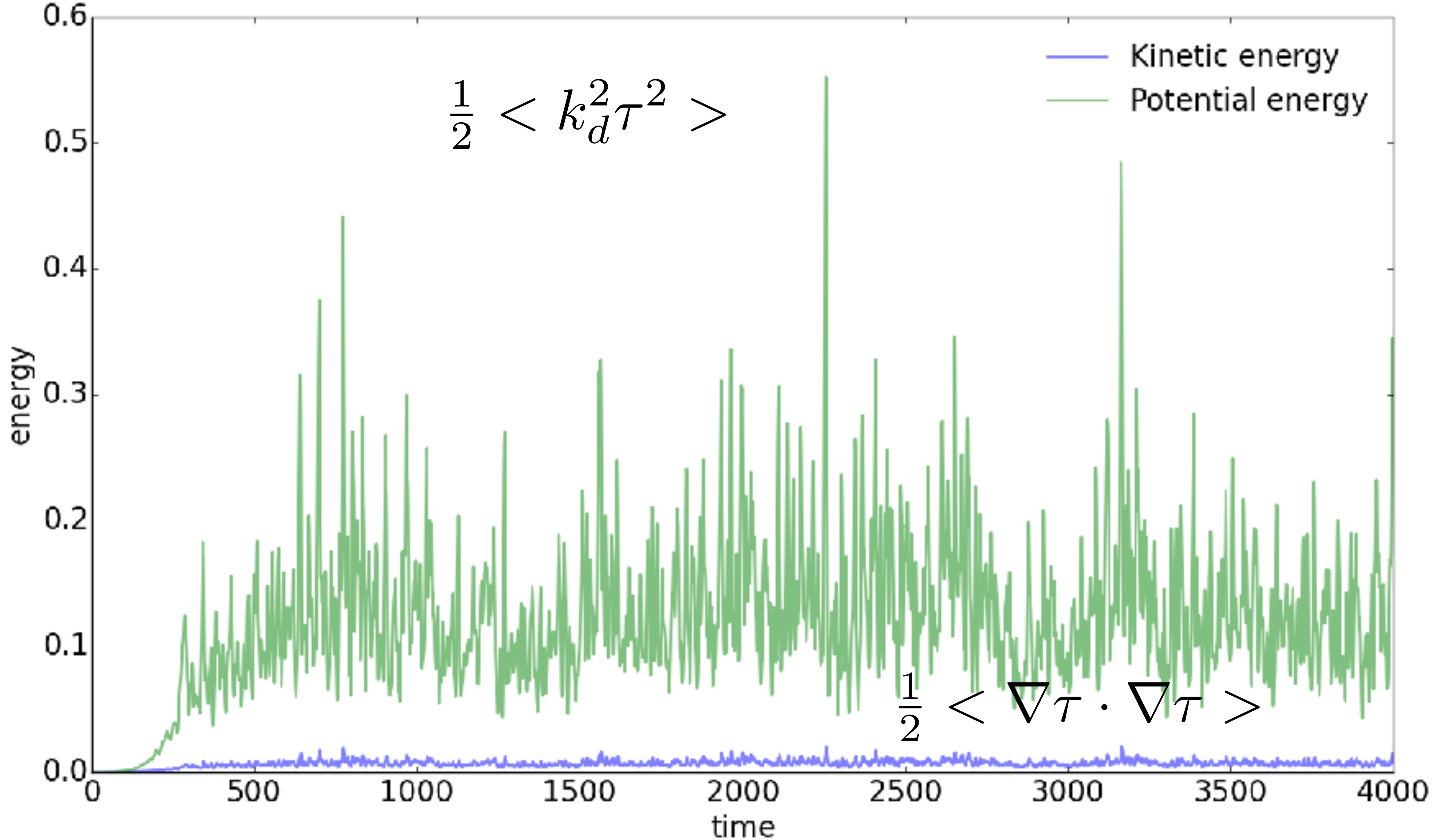
# Energy evolution

Barotropic/baroclinic partition



# Energy evolution

Baroclinic: KE/PE partition



# Two-layer QG Energetics

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# Two-layer QG Energetics

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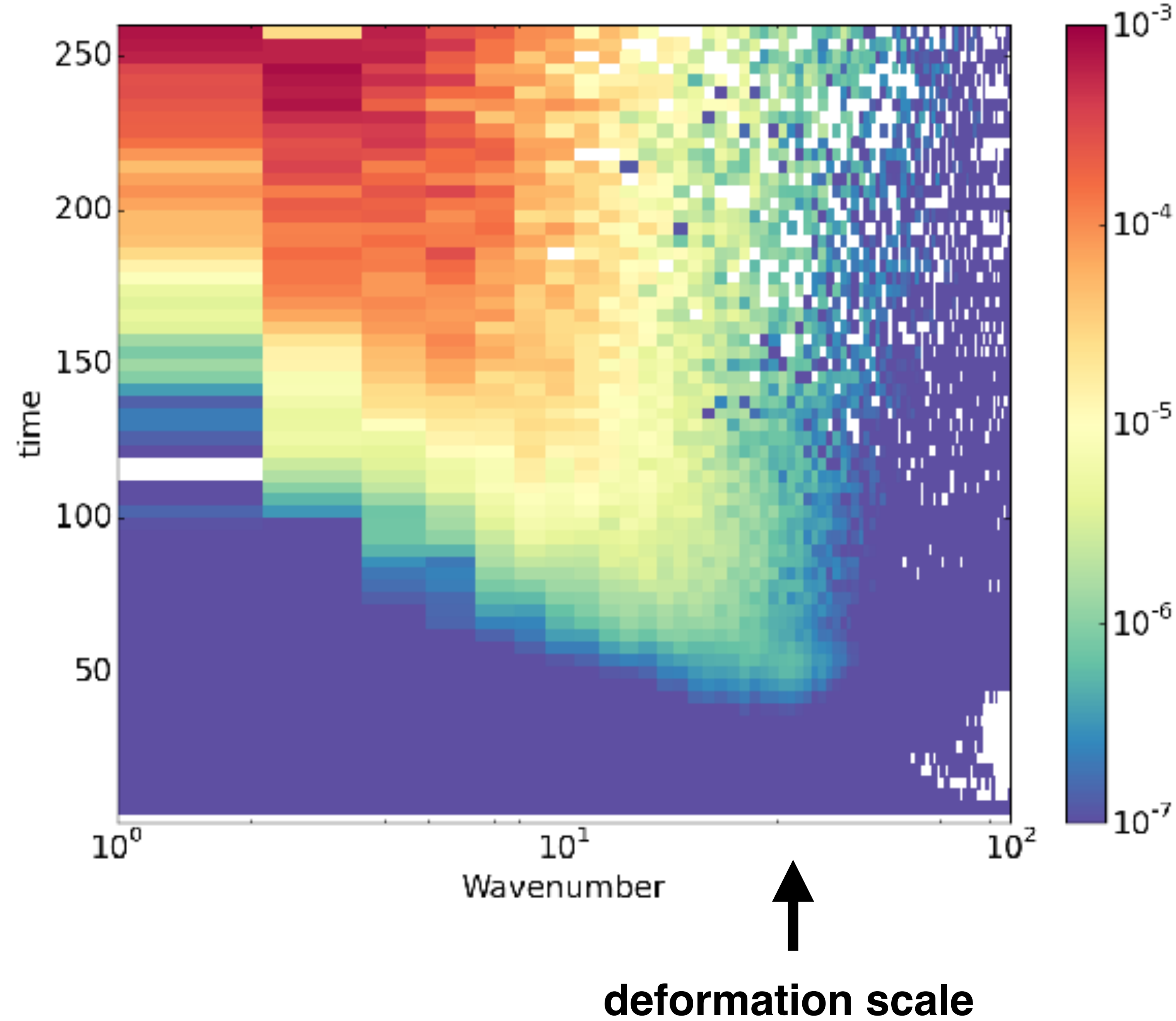
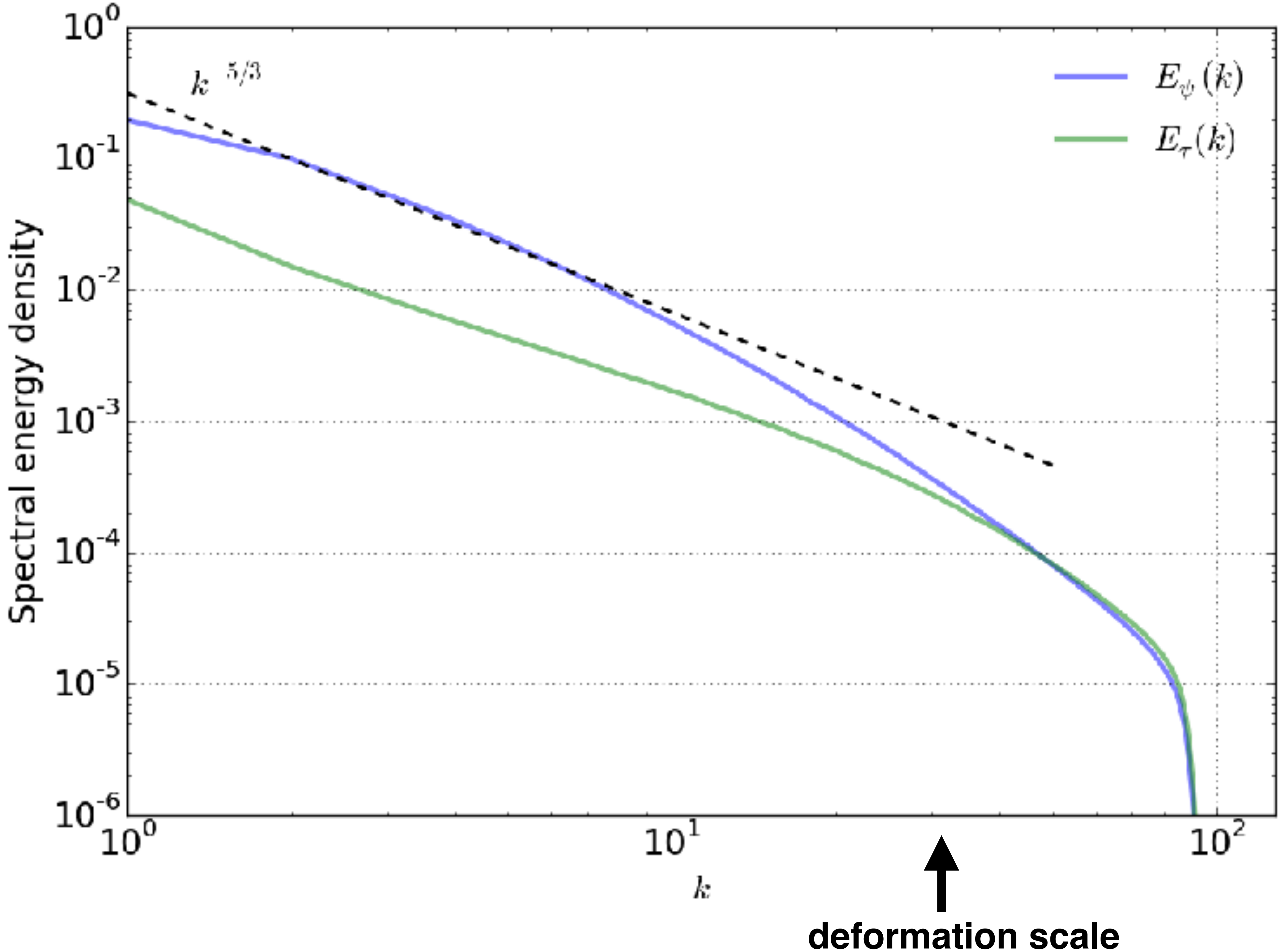
# “Production” in spinup stage

$$\mathrm{P}(k,l) = U k_d^2 \mathrm{Re}[\breve{v}_\psi \hat{\tau}^\star]$$

- Conversion starts near the deformation scale
- But moves towards larger scales
- In statistical steady state, the production at deformation scales is negligible

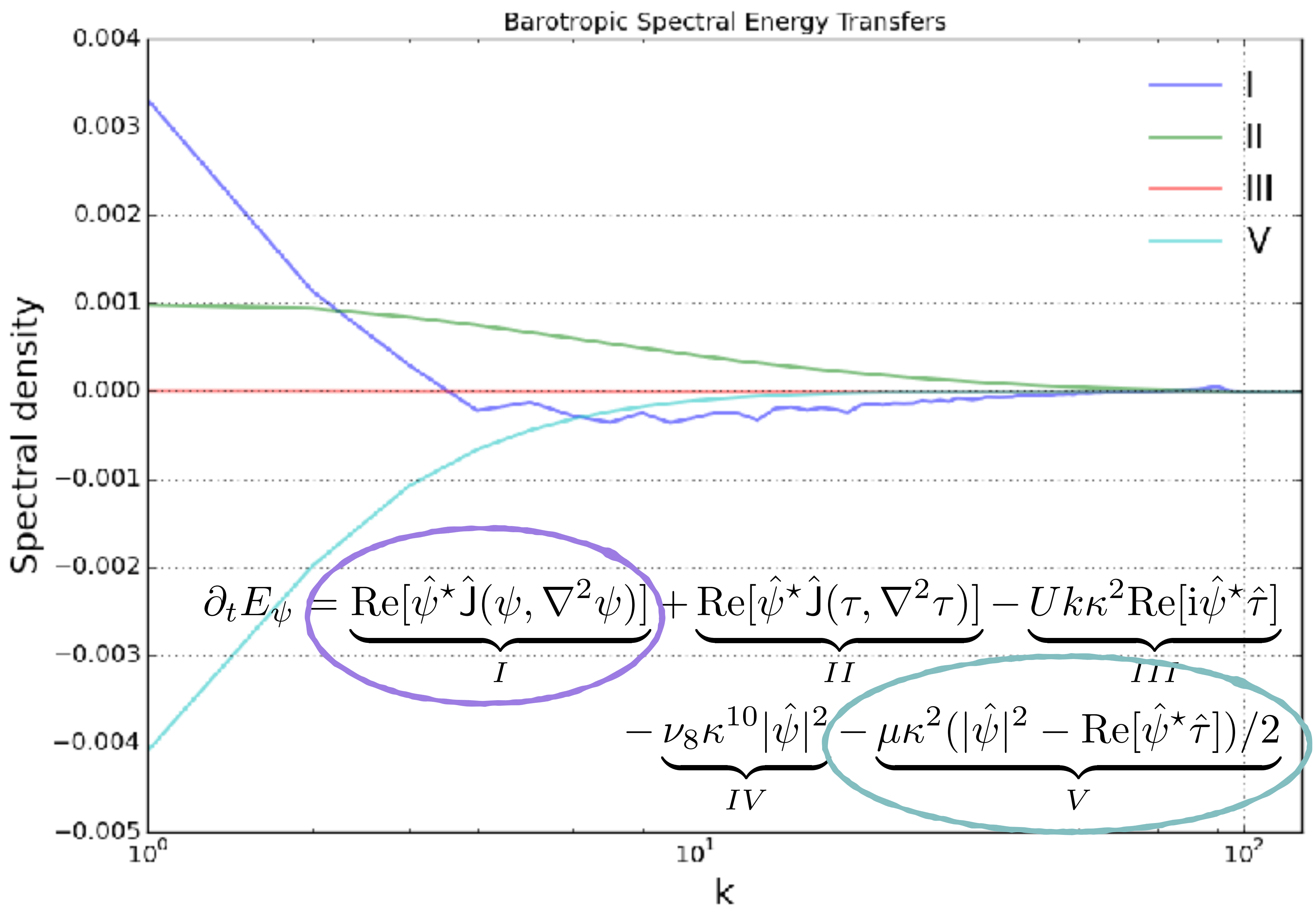


# Spectra



k 5/3
E_ψ(k)
E_τ(k)
Spectral energy density
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^0
10^1
10^2
k
deformation scale

# Spectral fluxes

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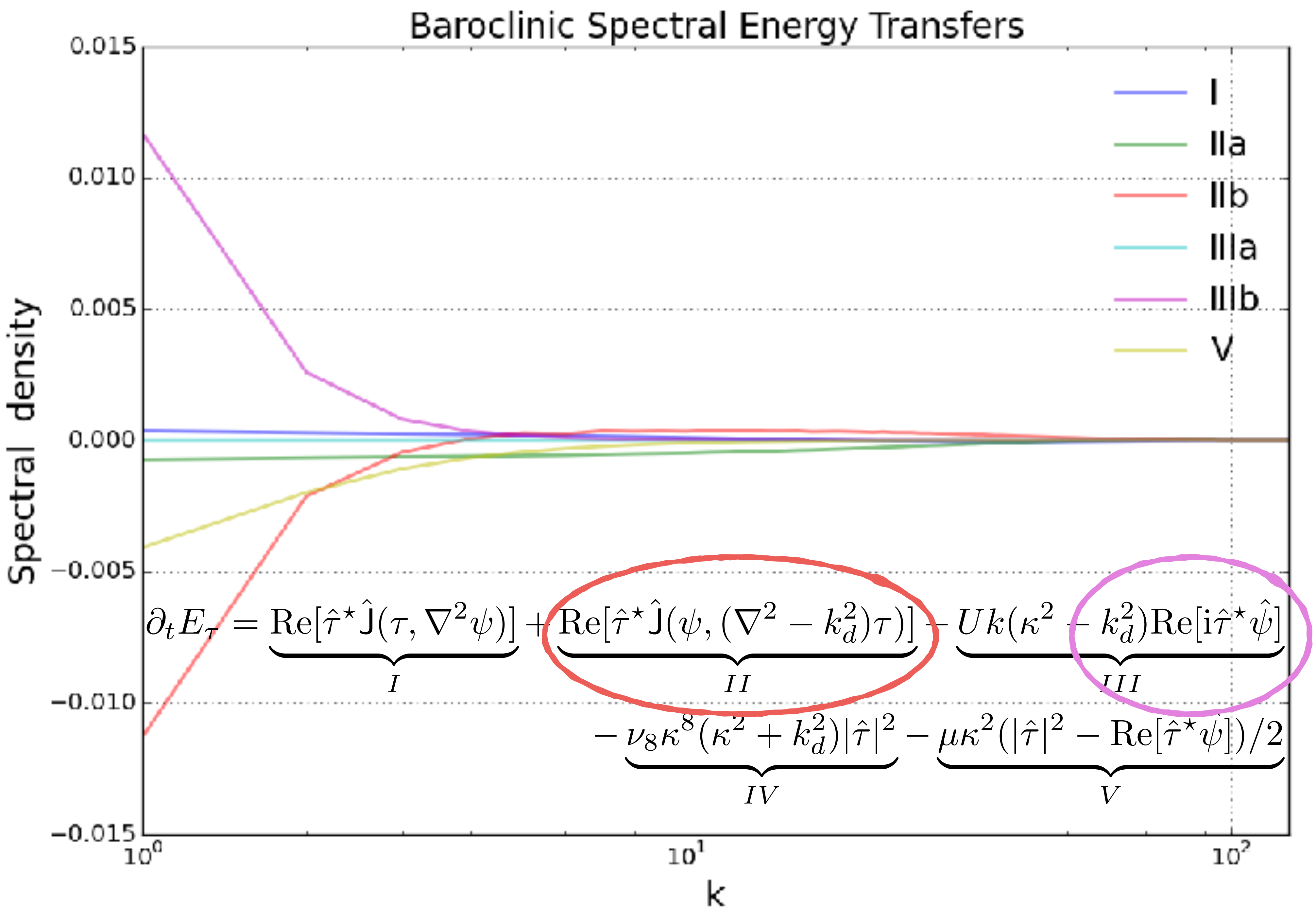
(This helped me catch a bug in my code!)

**deformation scale**

# Spectral fluxes

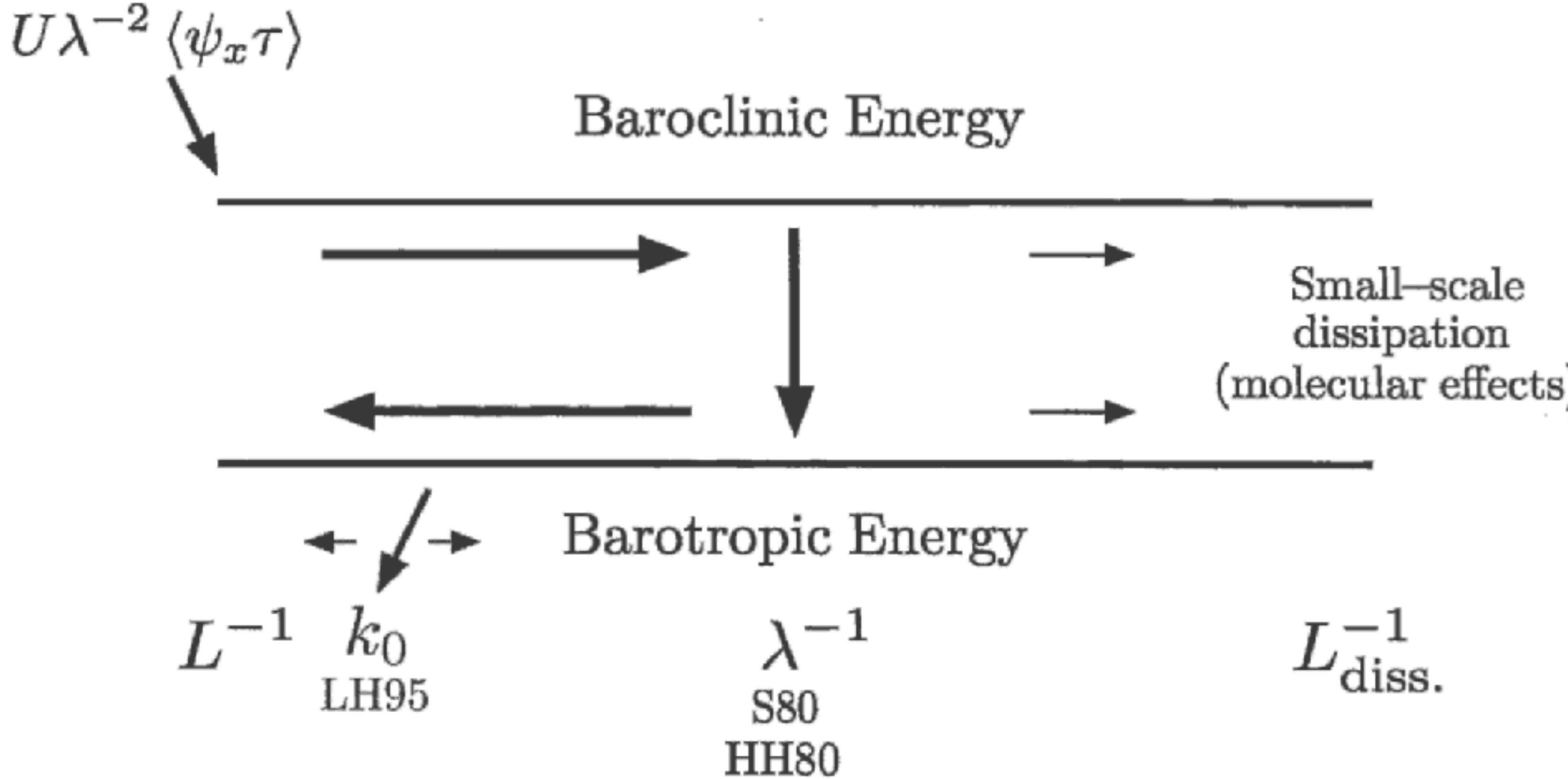



$$\partial_t E_\tau = \underbrace{\mathrm{Re}[\hat{\tau}^\star \hat{\mathsf{J}}(\tau, \nabla^2\psi)]}_{I} + \underbrace{\mathrm{Re}[\hat{\tau}^\star \hat{\mathsf{J}}(\psi, (\nabla^2 - k_d^2)\tau)]}_{II} - \underbrace{Uk(\kappa^2 - k_d^2)\mathrm{Re}[\mathrm{i}\hat{\tau}^\star \hat{\psi}]}_{III}$$

$$- \underbrace{\nu_8 \kappa^8 (\kappa^2 + k_d^2)|\hat{\tau}|^2}_{IV} - \underbrace{\mu\kappa^2(|\hat{\tau}|^2 - \mathrm{Re}[\hat{\tau}^\star \hat{\psi}])/2}_{V}$$

**deformation scale**

# The "Salmonian" energy cycle



# Scaling the PV flux

Using Kolmogovian arguments the barotropic energy spectrum in the energy inertial range is

$$E_\psi = C_\psi \epsilon_\psi^{2/3} \kappa^{-5/3}$$

$$\kappa^2 \stackrel{\text{def}}{=} k^2 + l^2$$

# Scaling the PV flux

At scales larger than the deformation radius,
the baroclinic PV equation reduces to the scalar equation

$$\tau_t + \mathsf{J}(\psi, \tau) = 0$$

An in the inertial range, the energy production scales as

$$\epsilon_\tau = \mathrm{const.} \sim E_\tau \kappa / T_\kappa$$

# Scaling the PV flux

But a local estimate of the eddy turn around time-scale is

$$T_\kappa \sim (E_\psi k^3)^{1/2}$$

And therefore

$$E_\tau = C_\tau \epsilon_\tau \epsilon_\psi^{-1/3} \kappa^{-5/3}$$

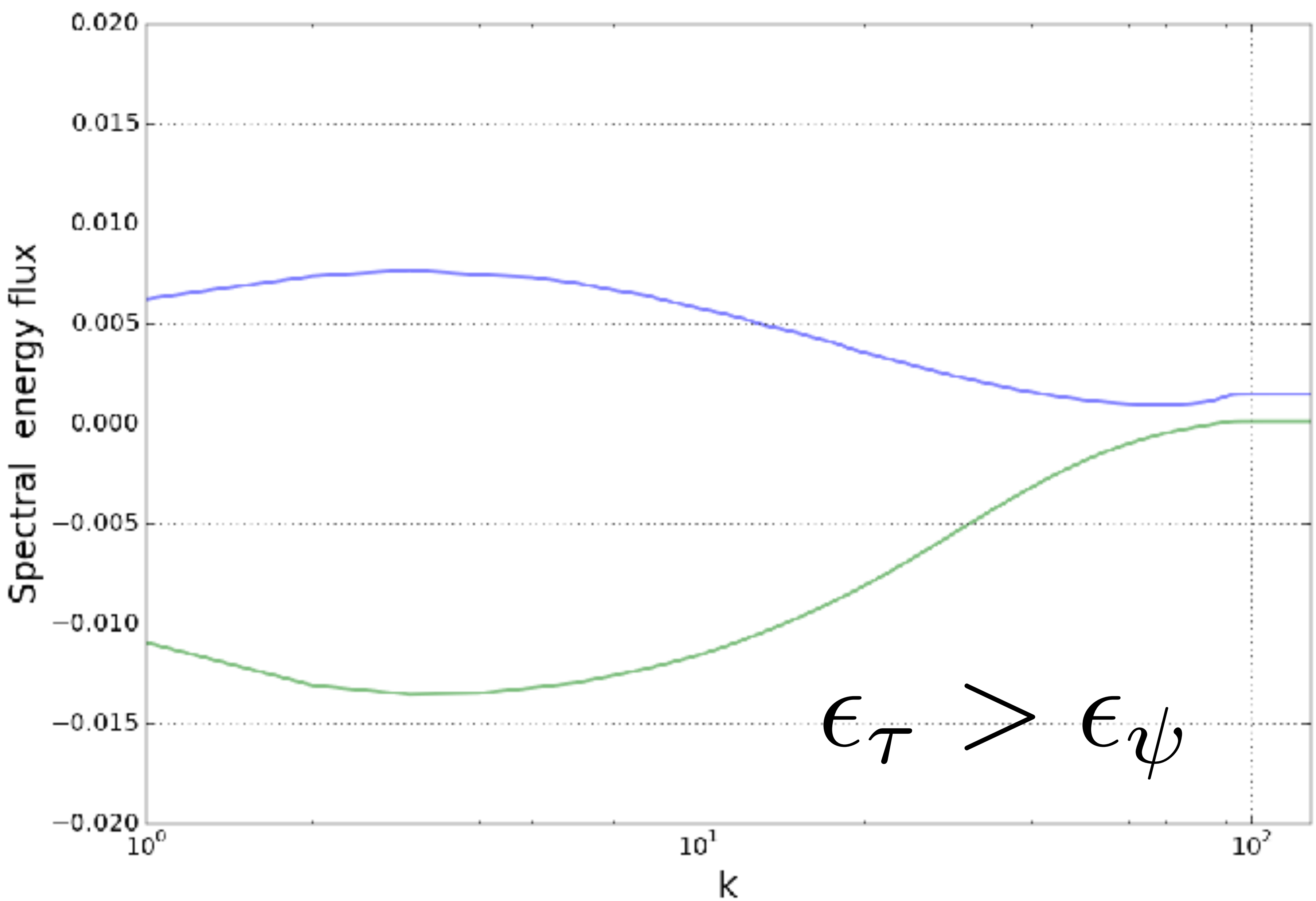
# Scaling the PV flux

In statistical steady state, the “Salmonian” phenomenological picture of energy transformations gives

$$\epsilon_{\tau} = \epsilon_{\psi}$$

# Scaling the PV flux

But this is not quantitatively true in the simulations



# Scaling the PV flux

"Accepting this relationship as a qualitative guide…"

$$\epsilon_\tau = \epsilon_\psi$$

And therefore

$$E_\tau = C_\tau \epsilon^{2/3} \kappa^{-5/3}$$

$$E_\psi = C_\psi \epsilon^{2/3} \kappa^{-5/3}$$

# Scaling the PV flux

Now, the numerical calculations show that in the "inertial range" the baroclinic energy is largely potential. Hence

$$\hat{\psi} \sim \frac{k_d}{\kappa}\,\hat{\tau}$$

And in physical space

$$\psi \sim (k_d/\kappa_0)\tau$$

where $\kappa_0^{-1}$ is the energy containing length scale

# Scaling the PV flux

Now in equilibrium the baroclinic energy production is

$$\epsilon_\tau = U k_d^2 < \overline{\tau v_\psi} >$$

Which is proportional to the meridional PV flux

$$\overline{v_i q_i} = (-1)^i k_d^2 < \overline{\tau v_\psi} >$$

# Scaling the PV flux

With the mean PV gradient, the baroclinic streamfunction equation at scales larger than the deformation radius is

$$\tau_t + \mathsf{J}(\psi, \tau - Uy) = 0$$

Turbulent diffusion arguments give

$$\tau \sim U\kappa_0^{-1} \quad \Longrightarrow \quad \psi \sim \frac{k_d}{\kappa_0^2} U$$

# Scaling the PV flux

Putting all this together gives the energy production

$$\epsilon_\tau \sim U k_d^2 \kappa_0 \psi \tau \sim U^3 \frac{k_d^3}{\kappa_0^2}$$

# Turbulent diffusivity

Using mixing-length arguments, the eddy diffusivity is

$$D = V\kappa_0^{-1}$$

$$V \stackrel{\text{def}}{=} \text{RMS velocity}$$

$$V = \kappa_0\psi \sim k_d\tau = U\frac{k_d}{\kappa_0}$$

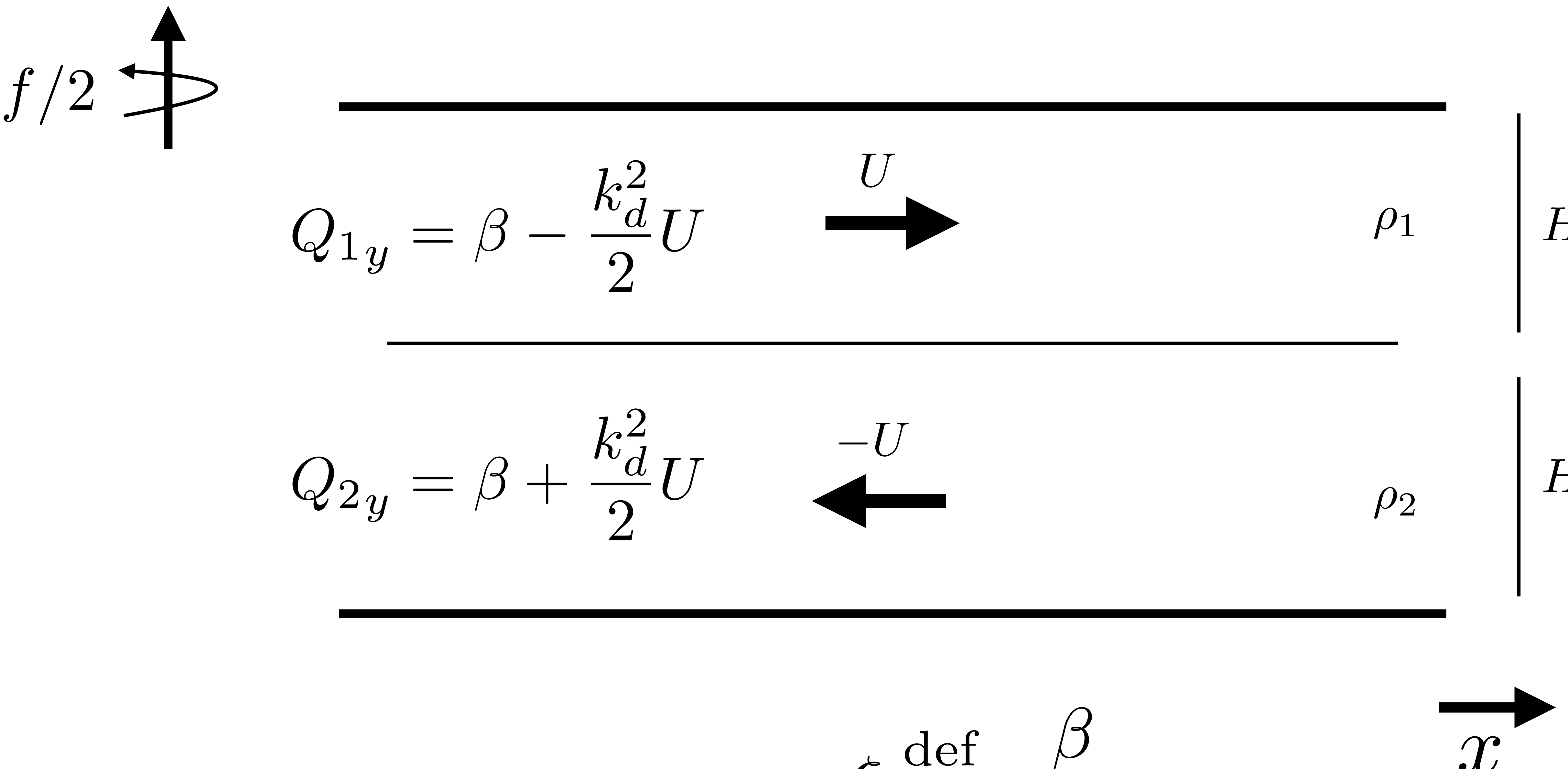
$$D = \frac{k_d}{\kappa_0^2}U$$

# What determines $\kappa_0$

- Size of the domain
- Planetary vorticity gradient (halting of the cascade)
- Bottom friction
- Topography
- etc.

# Adding beta

# Two-layer QG model




Unstable if $\xi \stackrel{\text{def}}{=} \frac{\beta}{U k_d^2}$

# On a $\beta - \text{plane}$

The upscale cascade is halted at (e.g, see Rhines 1975)

$$\kappa_0 = \left(\frac{\beta}{V}\right)^{1/2}$$

Thus

$$V \sim \xi U \qquad \xi \sim \frac{k_d}{\kappa_0} \qquad D \sim U\frac{\xi^2}{k_d}$$

These results are qualitatively consistent with simulations

# In summary…

- Numerical simulations and analysis indicate that the energy production (PV flux, heat flux, etc) is contained at the largest scales of the flow.

- Kolmogorovian-type assumptions + turbulent diffusion of a scalar lead to simple scaling for the eddy amplitude, energy flux, and eddy diffusivity in terms of the basic state shear and energy containing scale.

- On a beta plane, the cascade is assumed to be halted at the Rhines scale, and the scaling is simply written in terms of the criticality.

# Questions

- Are these results applicable to the oceanic mesoscales?
- If so, are the results of this type of study being used to parameterize mesoscale eddies in coarse resolution models?