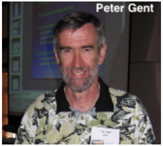
The Gent-McWilliams parameterization of eddy buoyancy fluxes

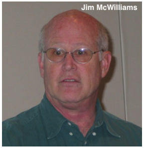
(as told by Cesar)

slides at tinyurl.com/POTheory-GM

Gent & **McWilliams**, *JPO*, 1990
(The most cited JPO paper ⋆)

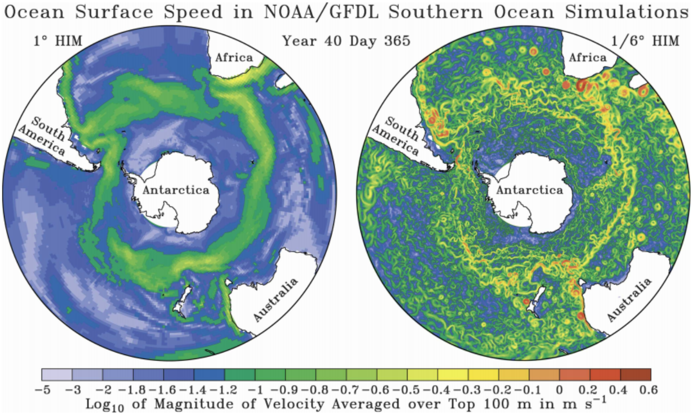
Griffies, *JPO*, 1998





Climate models are devoid of mesoscale eddies

In 1990, ocean models had "coarse resolution" (many still do today)



Source: Hallberg & Gnanadesikan.

A stroll through GM

A stroll through GM

Along-isopynal eddy fluxes

$$\overline{\mathbf{u}'\rho'} \cdot \nabla\bar{\rho} = \overline{\mathbf{u}'_h\rho'} \cdot \nabla_h\bar{\rho} + \overline{w'\rho'}\ \bar{\rho}_z \approx 0\,.$$

A stroll through GM

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The parameterization

$$\textbf{Ansatz}: \quad \overline{\mathbf{u}'_h \rho'} = -\kappa \nabla_h \bar{\rho} \quad \Longrightarrow \quad \overline{w'\rho'} = \kappa \frac{|\nabla_h \bar{\rho}|^2}{\bar{\rho}_z^2} \bar{\rho}_z\,.$$

A stroll through GM

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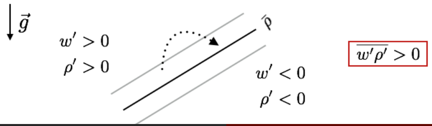
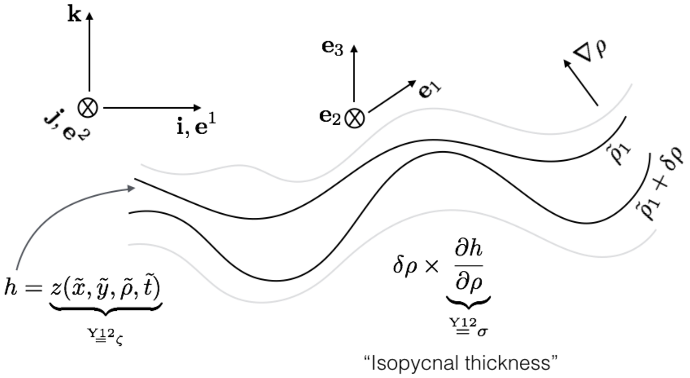
Properties of the parameterization

1. Preserves moments of ρ.
2. Releases available potential energy (cf. baroclinic instability).



Density (buoyancy) coordinates



$$f(x, y, z, t) = f(\tilde{x}, \tilde{y}, \tilde{\rho}, \tilde{t}) , \qquad f_x = f_{\tilde{x}} + \rho_x f_{\tilde{\rho}} , \qquad f_z = \rho_z f_{\tilde{\rho}} , \qquad \cdots$$

(cf. Young, *JPO*, 2012.)

Eddy-resolving models

The adiabatic thickness equation [cf. Young's σ equation (Y37)]

$$\frac{\partial}{\partial t}\frac{\partial h}{\partial \rho} + \nabla_\rho \cdot \left(\frac{\partial h}{\partial \rho}\mathbf{u}\right) = 0\,. \qquad \text{(GM1)}$$

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The adiabatic tracer τ equation

$$\underbrace{\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_\rho\right)}_{\stackrel{\text{def}}{=} D/Dt}\tau = \underbrace{\left(\frac{\partial h}{\partial \rho}\right)^{-1}\nabla_\rho\cdot\left(\mu\frac{\partial h}{\partial \rho}\mathbf{J}\cdot\nabla_\rho\tau\right)}_{\stackrel{\text{def}}{=} R(\tau)}\,, \tag{GM2}$$

with the matrix $\mathbf{J}$ defined in (GM4).

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Sloppy notation ALERT

$$\nabla_\rho\theta = \theta_{\tilde{x}}\ \mathbf{e}_1 + \theta_{\tilde{y}}\ \mathbf{e}_2\,, \qquad \nabla\cdot\mathbf{f} = \left(\frac{\partial h}{\partial \rho}\right)^{-1}\nabla_\rho\cdot\left[\left(\frac{\partial h}{\partial \rho}\right)\mathbf{f}\right]\,.$$

Eddy-resolving models

Three important properties (GM90)

Between any two isopycnals, the system conserves

A. All moments of density ρ and the volume.

B. The domain-averaged tracer concentration τ.

$R(\rho) = 0$ (no isopycnal mixing of density), so that

C. The density identically satisfies the tracer equation:

$$\frac{D\rho}{Dt} = \rho_t + u\rho_{\tilde{x}} + v\rho_{\tilde{y}} = 0\,.$$

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The thickness balance of eddy-resolving models (GM90)

In statistical steady state:

$$\nabla_\rho \cdot \left(\frac{\partial \bar{h}}{\partial \rho}\bar{\mathbf{u}}\right) + \overline{\nabla_\rho \cdot \left(\frac{\partial h'}{\partial \rho}\mathbf{u}'\right)} \approx 0\,, \qquad \text{(GM5)}$$

Non-eddy-resolving models

The non-eddy-resolving thickness equation

$$\frac{\partial}{\partial t}\frac{\partial h}{\partial \rho} + \nabla_\rho \cdot \left(\frac{\partial h}{\partial \rho}\mathbf{u}\right) + \nabla_\rho \cdot \mathbf{F} = 0\,. \qquad \text{(GM6)}$$

with the eddy thickness "flux" **F**.

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with the eddy thickness “flux” $\mathbf{F}$.

Choices for parameterizing $\mathbf{F}$

1. **Adiabatic** (but compressible) **flow**.
2. **Incompressible** (but diabatic) **flow**.

Non-eddy-resolving models

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The GM90 choice: (2) incompressible flow

$$\frac{D\rho}{Dt} = Q\,, \qquad \text{(GM7)}$$

with the non-conservative density source Q.

Non-eddy-resolving models

The GM90 choice: (2) incompressible (quasi-adiabatic) flow

$$\frac{\partial h}{\partial \rho} Q = \int^{\tilde{\rho}} \nabla_\rho \cdot \mathbf{F} \;\; \mathrm{d}\tilde{\rho} \,. \tag{GM8}$$

The GM90 choice: non-conservative source in the tracer equation

$$\underbrace{\left(\partial_t + u\partial_x + v\partial_y + Q\partial_{\tilde{\rho}}\right)}_{\stackrel{\text{def}}{=} D/Dt} \tau = R(\tau) + \left(\frac{\partial h}{\partial \rho}\right)^{-1} E(\tau) \,. \tag{GM10}$$

Non-eddy-resolving models

The GM90 choice: non-conservative source in the tracer equation

$$(\partial_t + u\partial_x + v\partial_y + Q\partial_\rho)\tau = R(\tau) + \left(\frac{\partial h}{\partial \rho}\right)^{-1} E(\tau)\,. \qquad \text{(GM10)}$$

The choice that satisfies property B

$$E(\tau) = \frac{\partial}{\partial \rho}\left[\left(\frac{\partial h}{\partial \rho}\right) Q\tau\right] + \nabla_\rho \cdot \mathbf{G}\,. \qquad \text{(GM11)}$$

The choice that satisfies property C

$$\left(\frac{\partial h}{\partial \rho}\right) Q = E(\rho)\,. \qquad \text{(GM12)}$$

Non-eddy-resolving models

The "flux" G satisfy

$$\nabla_\rho \cdot [\rho \mathbf{F} + \mathbf{G}(\rho)] = 0\,, \tag{GM13}$$

so that the simplest solution is

$$\mathbf{G}(\tau) = -\tau \mathbf{F}\,.$$

The non-eddy-resolving tracer equation

$$(\partial_t + \mathbf{u} \cdot \nabla_\rho)\,\tau + \underbrace{\left(\frac{\partial h}{\partial \rho}\right)^{-1} \mathbf{F}}_{\text{Eddy velocity}} \cdot \nabla_\rho \tau = R(\tau)\,. \tag{GM14}$$

Recall: $\mathbf{F} = \overline{\left(\frac{\partial h}{\partial \rho}\right)' \mathbf{u}'}$.

Non-eddy-resolving models

The simple choice for **F**

$$\mathbf{F} = -\frac{\partial}{\partial \rho}\left(\kappa \nabla_\rho h\right), \tag{GM15}$$

with thickness diffusivity κ.

It is simple, but is it justified?

If anything, this choice makes Q a local function:

$$\left(\frac{\partial h}{\partial \rho}\right) Q = -\nabla_\rho \cdot \left(\kappa \nabla_\rho h\right). \tag{GM16}$$

The GM skew flux

Tracer equation

$$(\partial_t + \mathbf{u} \cdot \nabla) T = R(T) \,, \tag{G1}$$

with tracer T (temperature, salinity, or passive), and the mixing operator

$$R(T) = \partial_m (J^{mn} \partial_n T) \,. \tag{G2}$$

The second-order mixing tensor **J**

K^{mn} Symmetric (diffusive) part: $K^{mn} = (J^{mn} + J^{nm})/2.$

A^{mn} Anti-symmetric (advective) part: $A^{mn} = (J^{mn} - J^{nm})/2.$

The GM skew flux

Two forms of the stirring operator

$$R_A(T) = \partial_m(\underbrace{A^{mn}\partial_n T}_{\stackrel{\text{def}}{=} -F^m_{skew}}) = (\partial_m A^{mn})\partial_n T + \cancelto{0}{A^{mn}\partial_n\partial_m T} \quad \text{(G7)}$$

$$= \partial_n[\underbrace{(\partial_m A^{mn})T}_{\stackrel{\text{def}}{=} -F^n_{adv}}] - \cancelto{0}{T\partial_m\partial_n A^{mn}} \quad \text{(G3)}$$

The GM skew flux

The advective flux $\mathbf{F}_{adv}$

$$F^n_{adv} = U^n_\star A^{mn}\,, \tag{G4}$$

with the non-divergent eddy velocity $U^n_\star$: $\partial_n U^n_\star = -\partial_n\partial_m A^{mn} = 0$.

$$\mathbf{F}_{adv} = T(\nabla \times \boldsymbol{\psi})\,, \tag{G6}$$

with the vector streamfunction $A^{nm} \stackrel{\text{def}}{=} \epsilon^{mnp}\psi_p$.

The GM skew flux

The skew flux $\mathbf{F}_{skew}$

The skew flux

$$F^m_{skew} = -A^{mn}\partial_n T \tag{G8}$$

is perpendicular to the tracer surfaces

$$\nabla T \cdot \mathbf{F}_{skew} = -(\partial_m T)A^{mn}(\partial_n T) = 0\,. \tag{G9}$$

Thus

$$F^m_{skew} = -\nabla T \times \boldsymbol{\psi}\,, \tag{G10}$$

and

$$\mathbf{F}_{adv} = \mathbf{F}_{skew} + \nabla \times (T\boldsymbol{\psi})\,. \tag{G11}$$

The GM skew flux

The GM choice [Recall, in density coordinates, $\mathbf{F} = -\frac{\partial}{\partial \rho}(\kappa \nabla_\rho h)$]

$$\mathbf{A} = A^{mn} = \kappa \begin{bmatrix} 0 & 0 & -S_x \\ 0 & 0 & -S_y \\ S_x & S_y & 0 \end{bmatrix}, \tag{G14}$$

with the isopycnal slope $\mathbf{S} \stackrel{\text{def}}{=} -\nabla_h \rho / \partial_z \rho$. Thus

$$\mathbf{F}_{adv} = T \nabla \times (\mathbf{k} \times \kappa S) = T \mathbf{U}_\star \,, \tag{G15}$$

$$\mathbf{F}_{skew} = -\nabla T \times (\mathbf{k} \times \kappa \mathbf{S}) = \kappa \mathbf{S} \partial_z T - \mathbf{k}(\kappa \mathbf{S} \cdot \nabla_h T) \,. \tag{G16}$$

GM + Redi

GM stirring + Redi diffusion (with diffusivity A)

$$\mathbf{J} = J^{mn} = \begin{bmatrix} A & 0 & (A-\kappa)S_x \\ 0 & A & (A-\kappa)S_y \\ (A+\kappa)S_x & AS_y & A \end{bmatrix} \tag{G27}$$

$$= \underbrace{\begin{bmatrix} A & 0 & AS_x \\ 0 & A & AS_y \\ AS_x & AS_y & A|\mathbf{S}|^2 \end{bmatrix}}_{=K^{mn}} + \underbrace{\begin{bmatrix} 0 & 0 & -\kappa S_x \\ 0 & 0 & -\kappa S_y \\ \kappa S_x & \kappa S_y & 0 \end{bmatrix}}_{=A^{mn}},$$

The tracer equation

$$(\partial_t + \mathbf{u}\cdot\nabla)T = \underbrace{\partial_m(K^{mn}\partial_n T)}_{\text{Redi diffusion}} + \underbrace{\partial_m(A^{mn}\partial_n T)}_{\text{GM stirring}}.$$

Take home (or dump)

Summary

- Quasi-adiabatic parameterization.
- Tracer stirring performed by total (mean + eddy) velocity.
- Tracer mixing can be equivalently represented by advective or skew fluxes.
- (GM90: unclear, if not inconsistent, paper.)

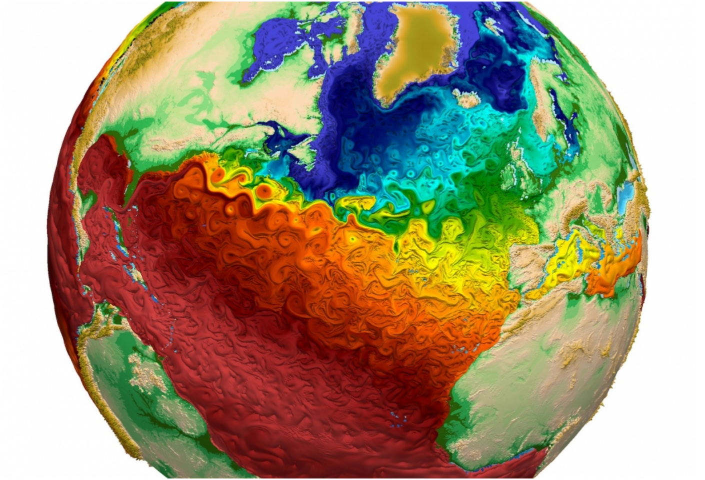
Long live GM

There once was an ocean model called MOM,
That occasionally used to bomb,
But eddy advection, and much less convection,
Turned it into a stable NCOM.

(Limerick by Peter Gent)

The future of climate modeling: GM *1990 — †2020?

The ocean component will fully resolve mesoscales eddies

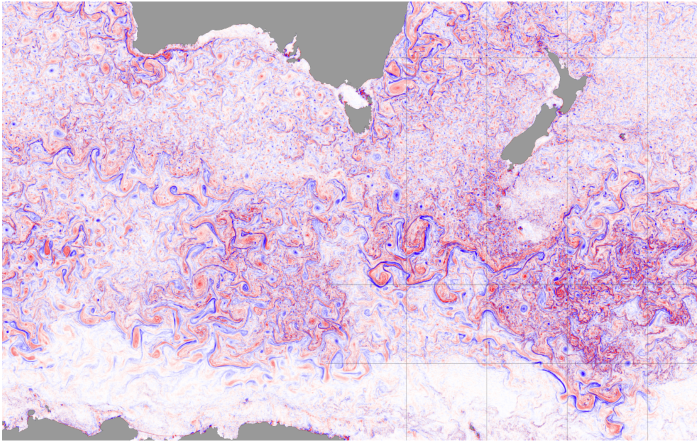


Source: Los Alamos National Laboratory.

But...

There is always something to parameterize

Snapshot of vorticity in a “fine-scale-resolving” (1 km) model



Visualization credit: Ryan Abernathey.

Topics of discussion

(Or things to think about in the privacy of your own study)

How to determine the GM coefficient κ?

i. Is it spatially variable? (Yes)
ii. It is sign definite?
iii. How does it relate to eddy diffusivities estimated from data?

Why was GM successful?

Can we transfer some of the GM experience to parameterizations of finer scales?

GM does not parameterize momentum fluxes and $\overline{b'^2}/N^2$...

$$\overline{u'q'} = \left(\overline{u'v'}\right)_x + \frac{1}{2}\left[\left(\overline{v'^2 - u'^2}\right) - \frac{\overline{b'^2}}{N^2}\right]_y + \left(f_0\,\frac{\overline{u'b'}}{N^2}\right)_z$$

$$\overline{v'q'} = -\left(\overline{u'v'}\right)_y + \frac{1}{2}\left[\left(\overline{v'^2 - u'^2}\right) + \frac{\overline{b'^2}}{N^2}\right]_x + \left(f_0\,\frac{\overline{v'b'}}{N^2}\right)_z$$

Useful references

Clearer than the original GM paper

Redi diffusion

Redi, *JPO*, 1982, Isopycnal mixing by coordinate rotation.

Better interpretations of the GM parameterization

- Gent et al., *JPO*, 1996, Parameterizing eddy-induced tracer transports in ocean circulation models.
- Gent, *OM*, 2011, The GM parameterization: 20/20 hindsight.

Beyond GM

- Marshall et al., *JPO*, 2012, A Framework for parameterizing eddy potential vorticity fluxes.