Living with buoyancy coordinates

An Exact Thickness-Weighted Average Formulation of the Boussinesq Equations

William R. Young

Scripps Institution of Oceanography, University of California, San Diego, La Jolla, California

William R Young
(in 2-D)

A 3-D representation of William R Young is available at an office near you!

Why would I use this paper?

Whenever you do anything in buoyancy coordinates, for example:

- If you want to plot the residual overturning in z-coordinates and overlay the buoyancy field
- If you want to plot the divergence of the heat flux on a buoyancy surface

If you don't, then you risk falling into traps that mean volume conservation is violated.

TEM in Isentropic Coordinates

$$\overline{v}^* \equiv \overline{v} + \frac{1}{H}\overline{v'h'},$$

$$\overline{v}_* = \overline{v} + \frac{1}{\overline{h}}\overline{v'h'},$$

is the meridional thickness flux

More generally expressed as:

$$\overline{v}_* \equiv \frac{\overline{hv}}{\overline{h}}.$$

$$\sigma \stackrel{\text{def}}{=} \zeta_{\tilde{b}}$$

Think of this like h, it's just a continuous field

Residual velocities

$$(\hat{u}, \hat{v}) \stackrel{\text{def}}{=} (\overline{\sigma u}, \overline{\sigma v})/\overline{\sigma}$$

$$\overline{\sigma}\hat{u} = \overline{u\sigma}, \quad \text{and} \quad \overline{\sigma}\hat{v} = \overline{v\sigma}.$$

Thickness weighted average

$$b_t + ub_x + vb_y + wb_z = \varpi.$$

diabatic effects

Buoyancy is chosen because if stability is assumed, there is a single-valued value of z for every b

$$\tilde{x} = x, \qquad \tilde{y} = y, \qquad \tilde{b} = b(x, y, z, t), \qquad \tilde{t} = t.$$

→

$$\partial_x = \partial_{\tilde{x}} + b_x \partial_{\tilde{b}}, \qquad \partial_y = \partial_{\tilde{y}} + b_y \partial_{\tilde{b}}, \qquad \partial_z = b_z \partial_{\tilde{b}}, \qquad \partial_t = \partial_{\tilde{t}} + b_t \partial_{\tilde{b}}.$$

→

$$\partial_x = \partial_{\tilde{x}} - \zeta_{\tilde{x}} \sigma^{-1} \partial_{\tilde{b}}, \qquad \partial_y = \partial_{\tilde{y}} - \zeta_{\tilde{y}} \sigma^{-1} \partial_{\tilde{b}}, \qquad \partial_z = \sigma^{-1} \partial_{\tilde{b}}, \qquad \partial_t = \partial_{\tilde{t}} - \zeta_{\tilde{t}} \sigma^{-1} \partial_{\tilde{b}}.$$

$$z = \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t})$$

$$\sigma(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \overset{\text{def}}{=} \zeta_{\tilde{b}} = 1/b_z,$$

Thickness weighted average

$$\partial_x = \partial_{\tilde{x}} - \zeta_{\tilde{x}} \sigma^{-1} \partial_{\tilde{b}},$$

$$\partial_y = \partial_{\tilde{y}} - \zeta_{\tilde{y}} \sigma^{-1} \partial_{\tilde{b}},$$

$$\partial_z = \sigma^{-1} \partial_{\tilde{b}},$$

$$\partial_t = \partial_{\tilde{t}} - \zeta_{\tilde{t}} \sigma^{-1} \partial_{\tilde{b}}.$$

From the buoyancy equation

$$w = \zeta_{\tilde{t}} + u\zeta_{\tilde{x}} + v\zeta_{\tilde{y}} + \varpi\zeta_{\tilde{b}}.$$

$$\frac{D}{Dt} \overset{\text{def}}{=} \partial_t + u\partial_x + v\partial_y + w\partial_z,$$

$$\downarrow$$

$$\frac{D}{Dt} = \partial_{\tilde{t}} + u\partial_{\tilde{x}} + v\partial_{\tilde{y}} + \varpi\partial_{\tilde{b}}.$$

This allows us to recast all sorts of equations in buoyancy coordinates

Advection of a passive scalar

$$c_t + uc_x + vc_y + wc_z = \gamma.$$

$$\downarrow$$

$$c_{\tilde{t}} + uc_{\tilde{x}} + vc_{\tilde{y}} + \varpi c_{\tilde{b}} = \gamma$$

Volume conservation

$$\nabla \cdot \mathbf{u} = 0$$

$$\downarrow$$

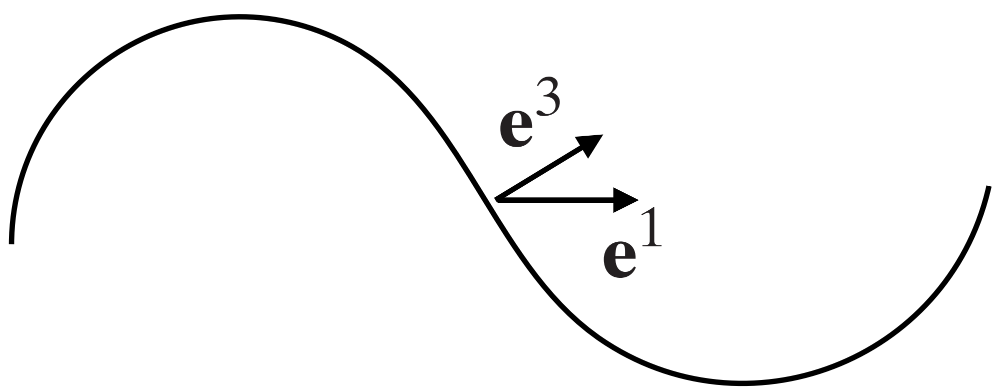
$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\sigma\varpi)_{\tilde{b}} = 0.$$

$$z = \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \qquad \sigma(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \overset{\text{def}}{=} \zeta_{\tilde{b}}$$

Non-orthogonal coordinate systems

Covariant coordinates

$$\mathbf{e}^1 \overset{\text{def}}{=} \mathbf{i}, \quad \mathbf{e}^2 \overset{\text{def}}{=} \mathbf{j}, \quad \mathbf{e}^3 \overset{\text{def}}{=} \nabla b$$

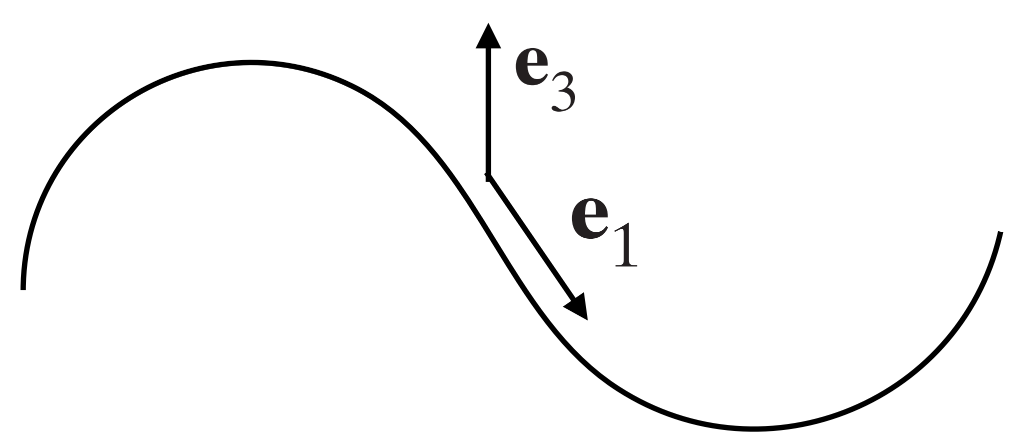


Contravariant coordinates

$$\mathbf{e}_1 \overset{\text{def}}{=} \sigma \mathbf{e}^2 \times \mathbf{e}^3 = \mathbf{i} + \zeta_{\tilde{x}} \mathbf{k},$$

$$\mathbf{e}_2 \overset{\text{def}}{=} \sigma \mathbf{e}^3 \times \mathbf{e}^1 = \mathbf{j} + \zeta_{\tilde{y}} \mathbf{k},$$

$$\mathbf{e}_3 \overset{\text{def}}{=} \sigma \mathbf{e}^1 \times \mathbf{e}^2 = \sigma \mathbf{k}.$$



$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j,$$

$$\partial_{\tilde{x}} = \mathbf{e}_1 \cdot \nabla, \quad \partial_{\tilde{y}} = \mathbf{e}_2 \cdot \nabla, \quad \partial_{\tilde{b}} = \mathbf{e}_3 \cdot \nabla.$$

u and v do double duty

For an arbitrary vector field

$$\mathbf{q} = q\mathbf{i} + r\mathbf{j} + s\mathbf{k},$$

$$= q^1\mathbf{e}_1 + q^2\mathbf{e}_2 + q^3\mathbf{e}_3,$$

$$= q_1\mathbf{e}^1 + q_2\mathbf{e}^2 + q_3\mathbf{e}^3.$$

$$\mathbf{q} = \underbrace{q}_{=q^1}\,\mathbf{e}_1 + \underbrace{r}_{=q^2}\,\mathbf{e}_2 + \underbrace{\sigma^{-1}(s - \zeta_{\tilde{x}}q - \zeta_{\tilde{y}}r)\mathbf{e}_3}_{=q^3}$$

$$\mathbf{q} = \underbrace{(q + s\zeta_{\tilde{x}})}_{=q_1}\mathbf{e}^1 + \underbrace{(r + s\zeta_{\tilde{y}})}_{=q_2}\mathbf{e}^2 + \underbrace{\sigma s}_{=q_3}\,\mathbf{e}^3.$$

For velocity

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

$$\mathbf{u} = u\mathbf{e}_1 + v\mathbf{e}_2 + \sigma^{-1}(\zeta_{\tilde{t}} + \varpi\zeta_{\tilde{b}})\mathbf{e}_3.$$

$$\mathbf{e}_1 \stackrel{\text{def}}{=} \sigma\mathbf{e}^2 \times \mathbf{e}^3 = \mathbf{i} + \zeta_{\tilde{x}}\mathbf{k},$$

$$\mathbf{e}_2 \stackrel{\text{def}}{=} \sigma\mathbf{e}^3 \times \mathbf{e}^1 = \mathbf{j} + \zeta_{\tilde{y}}\mathbf{k},$$

$$\mathbf{e}_3 \stackrel{\text{def}}{=} \sigma\mathbf{e}^1 \times \mathbf{e}^2 = \sigma\mathbf{k}.$$

This is a feature of the non-orthogonal coordinate system

The third component of residual velocity

We define

$$w^{\sharp} \stackrel{\text{def}}{=} \overline{\zeta}_{\tilde{t}} + \hat{u}\overline{\zeta}_{\tilde{x}} + \hat{\upsilon}\overline{\zeta}_{\tilde{y}} + \hat{\varpi}\overline{\zeta}_{\tilde{b}}$$

such that

$$\mathbf{u}^{\sharp} \stackrel{\text{def}}{=} \hat{u}\mathbf{i} + \hat{\upsilon}\mathbf{j} + w^{\sharp}\mathbf{k}$$

$$= \hat{u}\overline{\mathbf{e}}_1 + \hat{\upsilon}\overline{\mathbf{e}}_2 + \overline{\boldsymbol{\sigma}}^{-1}(\overline{\zeta}_{\tilde{t}} + \hat{\varpi}\overline{\zeta}_{\tilde{b}})\overline{\mathbf{e}}_3.$$

and the flow is incompressible:

$$\nabla \cdot \mathbf{u}^{\sharp} = 0$$

and $w^{\sharp}$ is the velocity used to advect $b^{\sharp}$

$$b^{\sharp}_t + \mathbf{u}^{\sharp} \cdot \nabla b^{\sharp} = \hat{\varpi}.$$

Zonally averaging the overturning

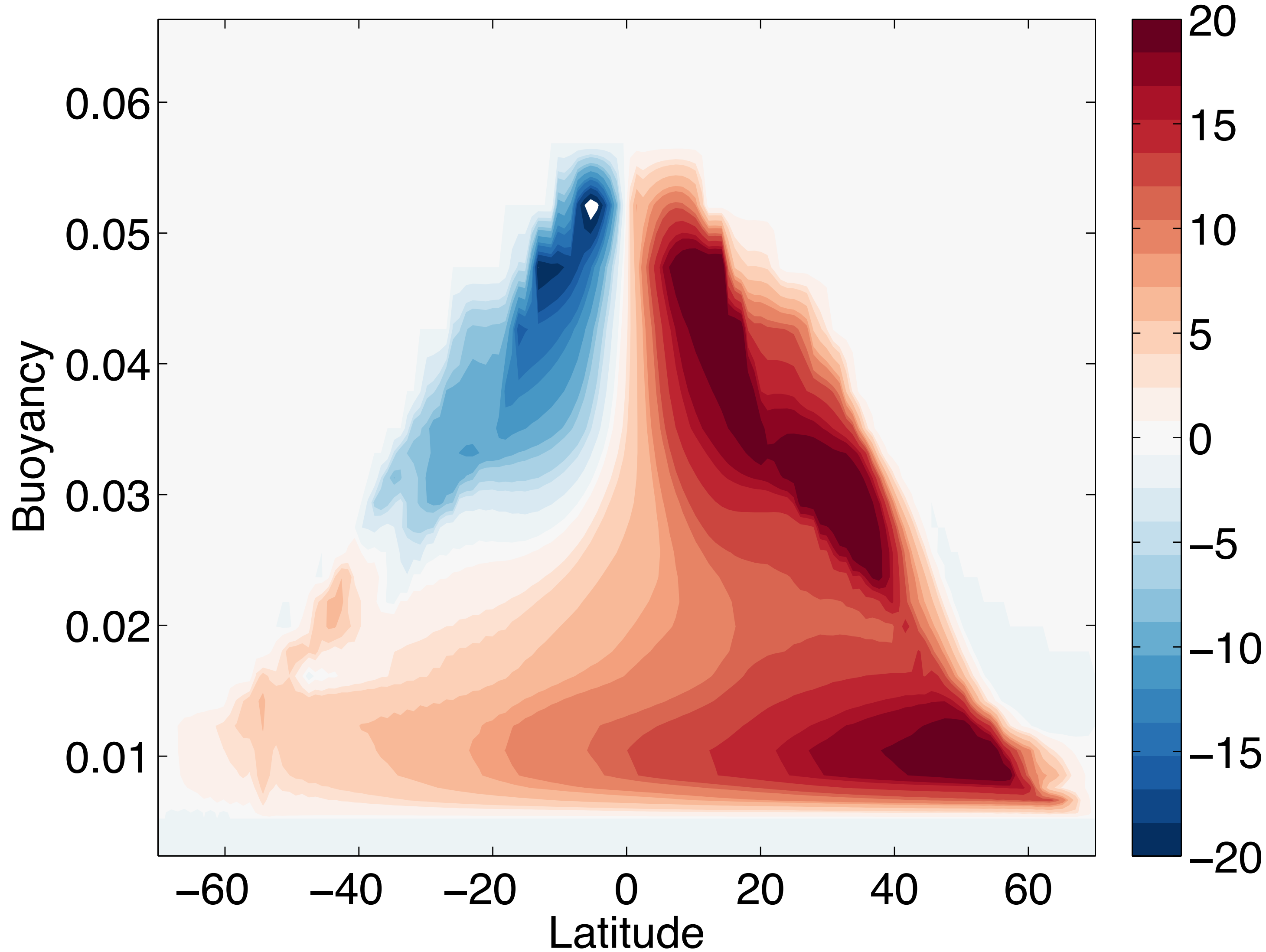
Residual Overturning Streamfunction

$$v^{\dagger} = \overline{v} + v'$$

$$\psi(y, \tilde{b}) = \overline{\int_{\tilde{b}(x,y,z)}^{b_s} v\sigma \, db'} = \int_0^{L_x} \int_{-H}^{0} v^{\dagger} \mathcal{H}\left[b(x,y,z) - \tilde{b}\right] dz \, dx$$

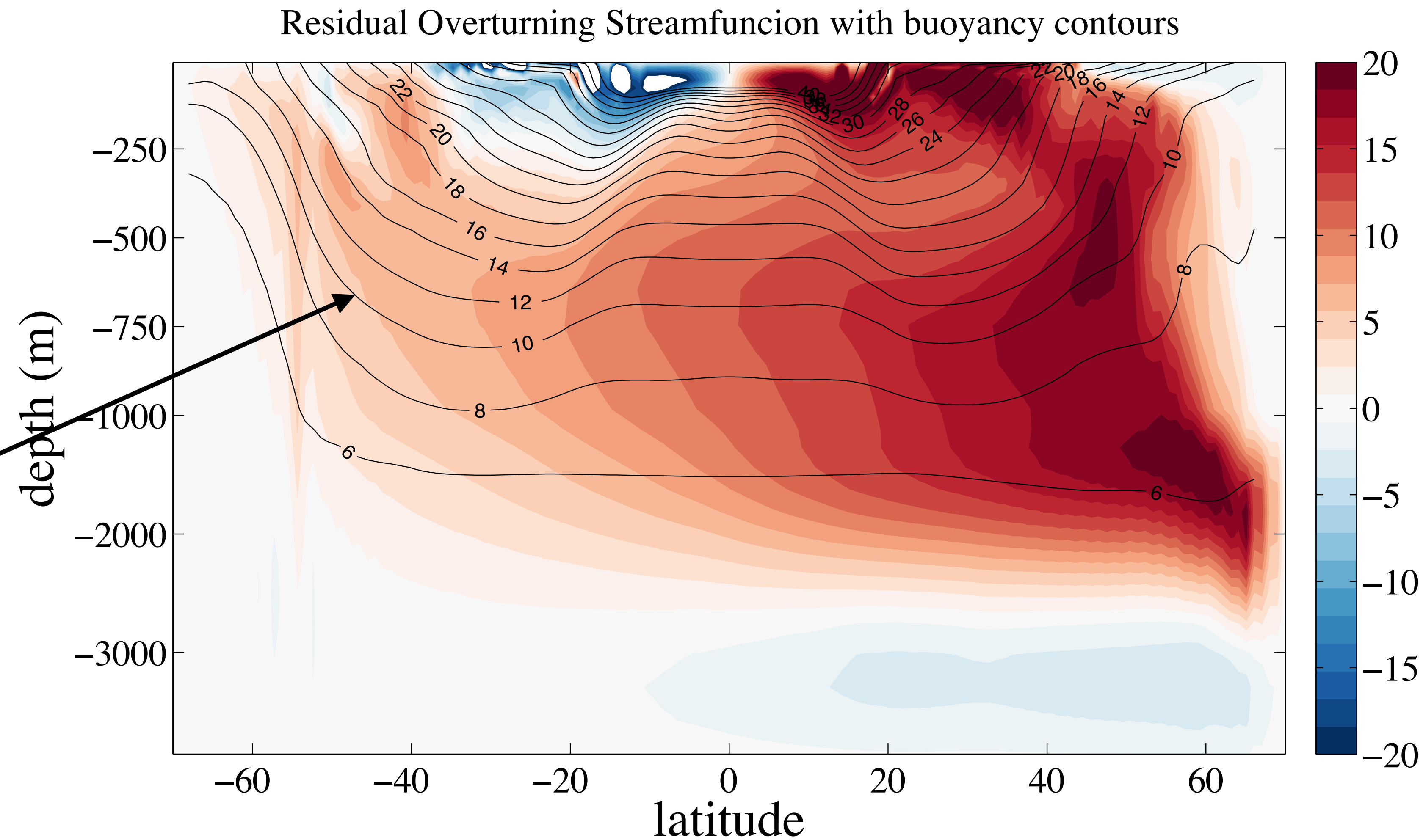
Heaviside function is only nonzero above $\tilde{b}$

$b^{\sharp}$



Zonally averaging the buoyancy surfaces

People like to transform the ROC and display it in z-coordinates



$b^\sharp$ contours

We have

$$\psi(y,b) \qquad \overline{\zeta}(y,b)$$

So we can express $\psi(y,\overline{\zeta})$

Then interpolate onto z-grid to get $\psi(y,z)$

Zonally averaging the buoyancy surfaces

To get $b^\sharp$:

Take the full 3-D buoyancy field

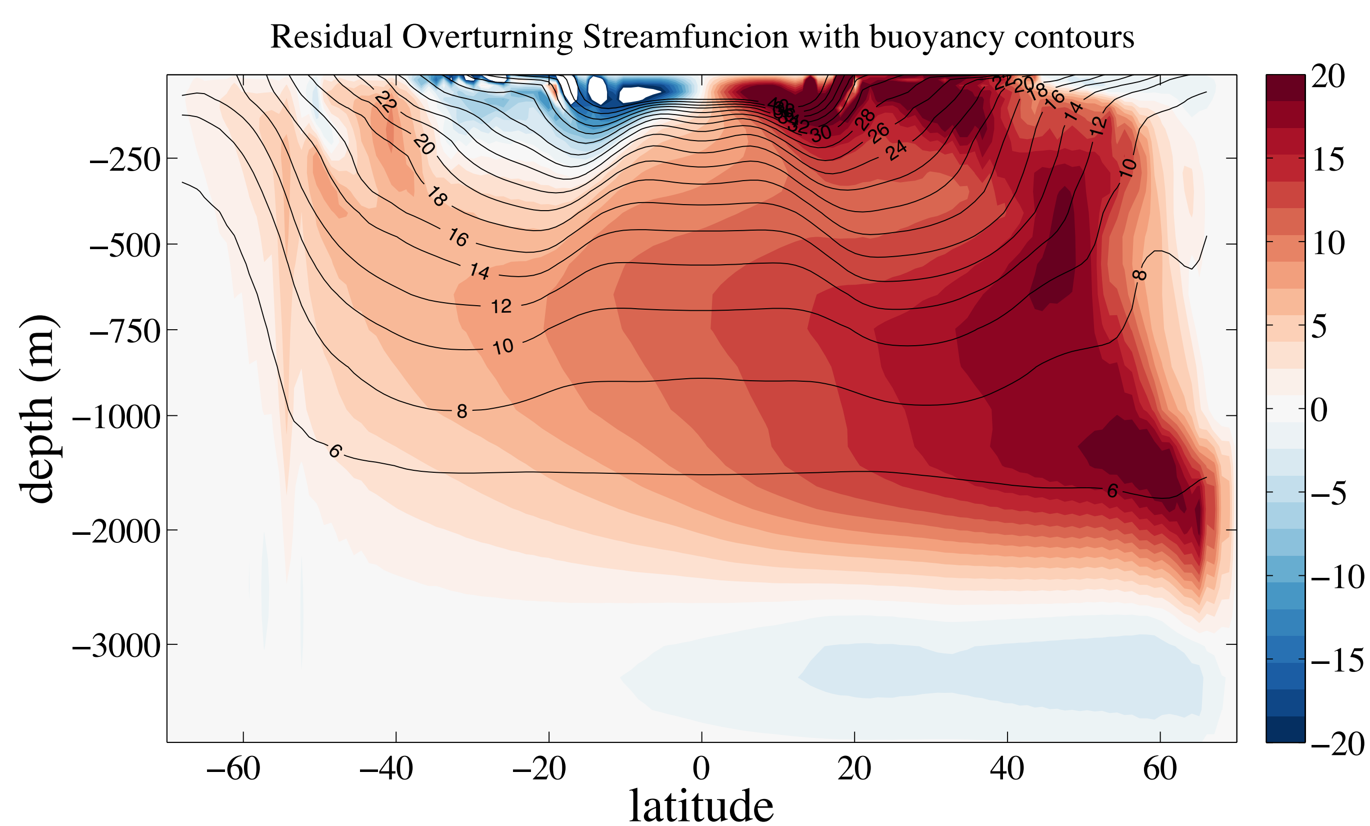
Calculate ζ using $\zeta(x,y,\tilde{b}) = \int_{-H}^{0} \mathcal{H}\left[b(x,y,z) - \tilde{b}\right]\, dz$

Zonally average for $\overline{\zeta}(y, b^\sharp)$ and invert for $b^\sharp(y, \overline{\zeta})$

(I do this by interpolation)

and $w^\sharp$ is the velocity used to advect $b^\sharp$

$$b_t^\sharp + \mathbf{u}^\sharp \cdot \nabla b^\sharp = \hat{\varpi}.$$

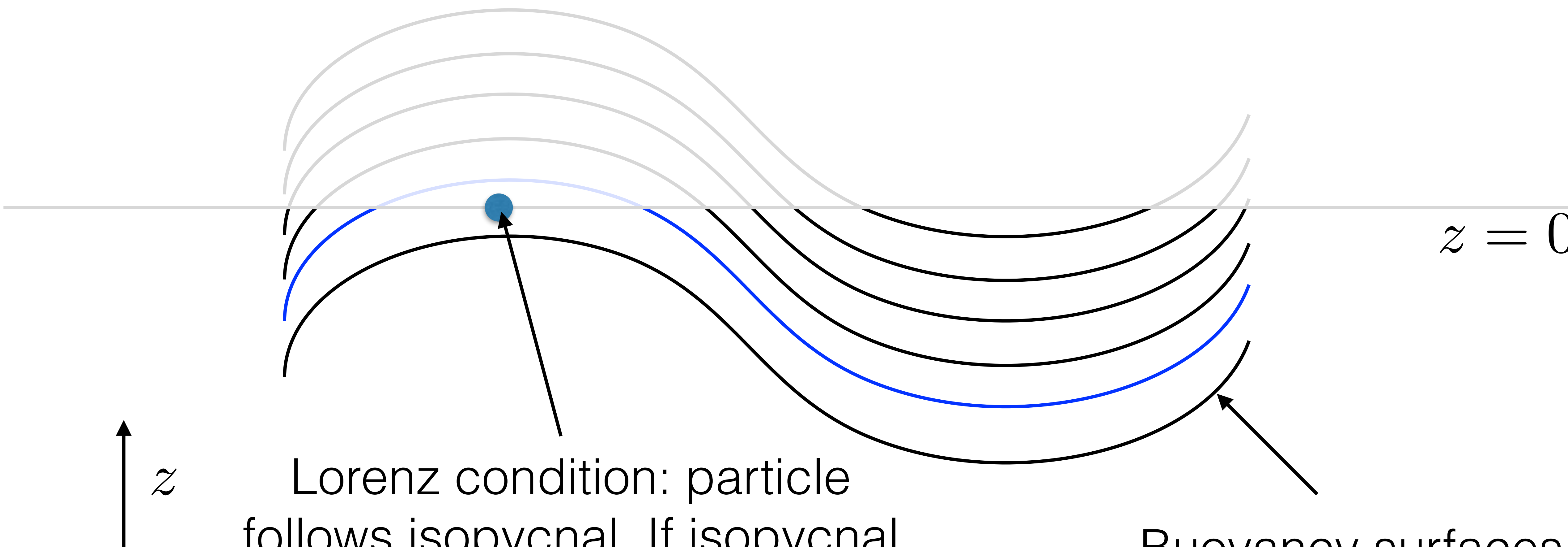


Boundary conditions/beyond the boundary

The actual boundary condition is

$$\mathbf{u}^{\sharp} \cdot \mathbf{n} = 0,$$

There are places where the isopycnals outcrop, and the buoyancy surface does not exist. How do we deal with this when averaging?



Lorenz condition: particle follows isopycnal. If isopycnal disappears, particle must remain at surface until the isopycnal reappears

Buoyancy surfaces

Boundary conditions and beyond

$\sigma \overset{\text{def}}{=} \zeta_{\tilde{b}}$

$\overline{\sigma} \to 0$ at the boundaries

Naturally $(u, v) = 0$ beyond the boundaries, so $\dfrac{\overline{v\sigma}}{\overline{\sigma}} \to 0$

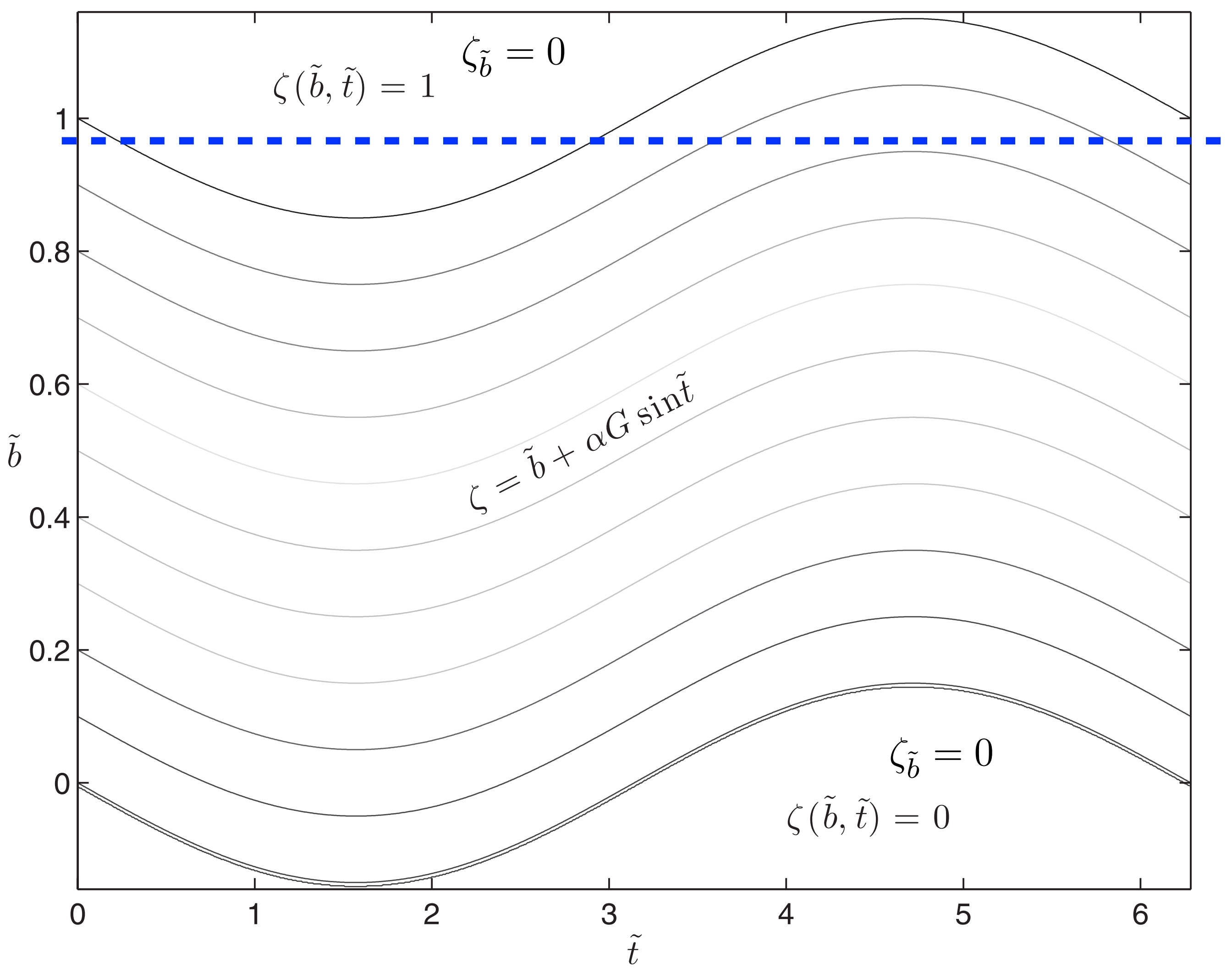


FIG. 1. The isopycnal depth $\zeta(\tilde{b}, \tilde{t})$ in (99) at $(x, y) = 0$ as function of $\tilde{b}$ and $\tilde{t}$. In z coordinates the ocean depth is $0 < z < 1$ and ζ is extended with the constant value $\zeta = 1$ for isopycnals “above” the sea surface and $\zeta = 0$ for isopycnals “below” the bottom.

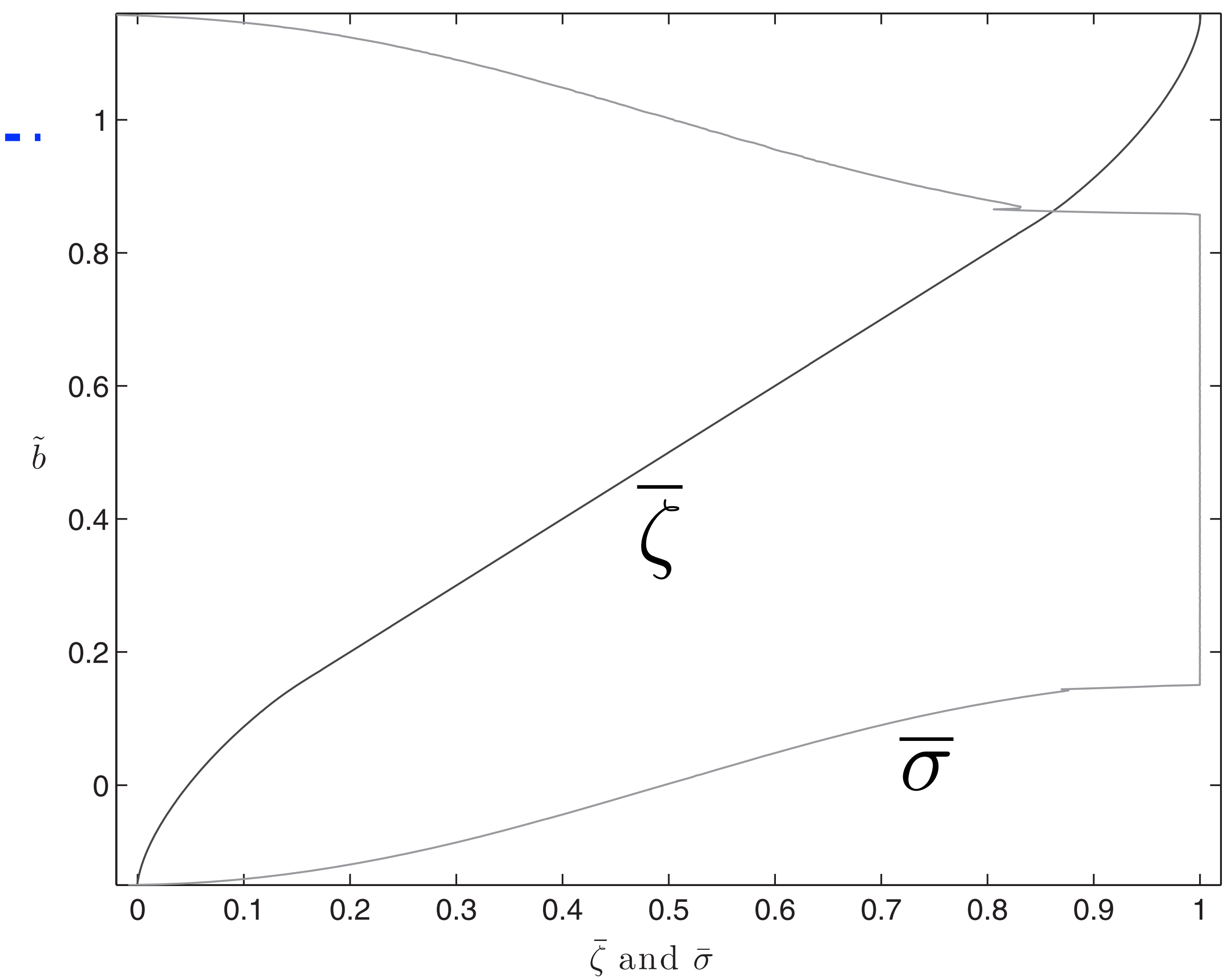


FIG. 2. The average isopycnal depth $\tilde{\zeta}(\tilde{b})$ and the average thickness $\overline{\sigma} = \overline{\zeta}_{\tilde{b}}$ at $(x, y) = 0$ as function of $\tilde{b}$. The function $b^{\sharp}$ is the inverse of $\overline{\zeta}(\tilde{b})$ above and is defined on the original domain $0 < z < 1$. In the central part of the domain, $\alpha G < \tilde{b} < 1 - \alpha G$, the average depth is obtained from (99) as $\tilde{\zeta} = \tilde{b}$, and therefore $\overline{\sigma} = 1$.

Gradient and divergence

Make sure you take the gradient in the correct coordinate system:

$$\begin{aligned}\nabla f(\tilde{x},\tilde{y},\tilde{b},\tilde{t}) &= f_{\tilde{x}}\nabla\tilde{x} + f_{\tilde{y}}\nabla\tilde{y} + f_{\tilde{b}}\nabla b,\\ &= f_{\tilde{x}}\mathbf{e}^1 + f_{\tilde{y}}\mathbf{e}^2 + f_{\tilde{b}}\mathbf{e}^3.\end{aligned}$$

Divergence must be thickness weighted:

$$\nabla\cdot\mathbf{q} = \sigma^{-1}(\sigma q^1)_{\tilde{x}} + \sigma^{-1}(\sigma q^2)_{\tilde{y}} + \sigma^{-1}(\sigma q^3)_{\tilde{b}}.$$

Decomposition

$$\frac{D^\sharp}{Dt} \stackrel{\text{def}}{=} \partial_{\tilde{t}} + \hat{u}\partial_{\tilde{x}} + \hat{\upsilon}\partial_{\tilde{y}} + \hat{\varpi}\partial_{\tilde{b}}$$

$$= \partial_t + \hat{u}\partial_x + \hat{\upsilon}\partial_y + w^\sharp\partial_z.$$

We already know

$$\zeta = \overline{\zeta} + \zeta' \quad \text{and} \quad \sigma = \overline{\sigma} + \sigma'$$

But we can also define

$$\theta = \hat{\theta} + \theta''$$

where $\overline{\sigma\theta''} = 0.$

$$\hat{\theta} \stackrel{\text{def}}{=} \frac{\overline{\sigma\theta}}{\overline{\sigma}}.$$

therefore

$$\overline{\sigma\phi\theta} = \overline{\sigma}(\hat{\phi}\hat{\theta} + \widehat{\phi''\theta''}).$$

$$\overline{\sigma\nabla\cdot q} = \overline{\sigma}\nabla\cdot\widehat{q^j}\overline{\mathbf{e}}_j.$$

You can move σ outside the brackets, so

$$\sigma\frac{D\theta}{Dt} = (\sigma\theta)_{\tilde{t}} + (\sigma u\theta)_{\tilde{x}} + (\sigma\upsilon\theta)_{\tilde{y}} + (\sigma\varpi\theta)_{\tilde{b}}.$$

Can be split into mean and eddy terms:

$$\mathbf{J}^\theta \stackrel{\text{def}}{=} \widehat{u''\theta''}\overline{\mathbf{e}}_1 + \widehat{\upsilon''\theta''}\overline{\mathbf{e}}_2 + \widehat{\varpi''\theta''}\overline{\mathbf{e}}_3,$$

$$\overline{\sigma\frac{D\theta}{Dt}} = \overline{\sigma}\left(\frac{D^\sharp\hat{\theta}}{Dt} + \nabla\cdot\mathbf{J}^\theta\right),$$

Eliassen-Palm Flux

Montgomery potential

$$m(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}) \stackrel{\text{def}}{=} p(x, y, \zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}), t) - \tilde{b}\zeta(\tilde{x}, \tilde{y}, \tilde{b}, \tilde{t}).$$

We can re-express the equations of motion

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + m_{\tilde{x}} = \mathcal{X},$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu + m_{\tilde{y}} = \mathcal{Y},$$

Adiabatic processes

$$\zeta + m_{\tilde{b}} = 0, \longrightarrow \overline{\zeta} = -\overline{m}_{\tilde{b}},$$

$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\varpi\sigma)_{\tilde{b}} = 0,$$

Therefore

$$\sigma m_{\tilde{x}} = -m_{\tilde{b}\tilde{b}} m_{\tilde{x}} = (\zeta m_{\tilde{x}})_{\tilde{b}} + \left(\frac{1}{2}\zeta^2\right)_{\tilde{x}}$$

$$\overline{\sigma m_{\tilde{x}}} = \overline{\sigma}\,\overline{m}_{\tilde{x}} + (\overline{\zeta' m'_{\tilde{x}}})_{\tilde{b}} + \left(\frac{1}{2}\overline{\zeta'^2}\right)_{\tilde{x}}.$$

$$\overline{\sigma}^{-1}\overline{\sigma m_{\tilde{x}}} = \overline{m}_{\tilde{x}} + \nabla \cdot \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\,\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\,\overline{\mathbf{e}}_3\right).$$

Eliassen-Palm Flux

Equations of motion

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv + m_{\tilde{x}} = \mathcal{X},$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu + m_{\tilde{y}} = \mathcal{Y},$$

$$\zeta + m_{\tilde{b}} = 0,$$

$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\varpi\sigma)_{\tilde{b}} = 0,$$

From before

$$\overline{\sigma\frac{D\theta}{Dt}} = \overline{\sigma}\left(\frac{D^{\sharp}\hat{\theta}}{Dt} + \nabla\cdot\mathbf{J}^{\theta}\right),$$

$$\overline{\sigma}^{-1}\overline{\sigma m_{\tilde{x}}} = \overline{m}_{\tilde{x}} + \nabla\cdot\overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\overline{\mathbf{e}}_3\right).$$

Putting these together

$$\frac{D^{\sharp}\hat{u}}{Dt} - f\hat{v} + \overline{m}_{\tilde{x}} + \nabla\cdot\mathbf{E}^{u} = \hat{\mathcal{X}}$$

$$\frac{D^{\sharp}\hat{v}}{Dt} + f\hat{u} + \overline{m}_{\tilde{y}} + \nabla\cdot\mathbf{E}^{v} = \hat{\mathcal{Y}}.$$

Adiabatic processes

E-P vectors

$$\mathbf{E}^{u} \stackrel{\text{def}}{=} \mathbf{J}^{u} + \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\overline{\mathbf{e}}_3\right) \qquad \mathbf{E}^{v} \stackrel{\text{def}}{=} \mathbf{J}^{v} + \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\overline{\mathbf{e}}_2 + \overline{\zeta' m'_{\tilde{y}}}\overline{\mathbf{e}}_3\right),$$

form drag

Conclusions

Doing a thickness weighted average can become very confusing - refer to Bill's paper whenever you do it!

This might mean you have to use a non-orthogonal coordinate system:

$$\mathbf{e}^1 \stackrel{\text{def}}{=} \mathbf{i}, \quad \mathbf{e}^2 \stackrel{\text{def}}{=} \mathbf{j}, \quad \mathbf{e}^3 \stackrel{\text{def}}{=} \nabla b$$

$$\mathbf{e}_1 \stackrel{\text{def}}{=} \sigma \mathbf{e}^2 \times \mathbf{e}^3 = \mathbf{i} + \zeta_{\tilde{x}} \mathbf{k},$$

$$\mathbf{e}_2 \stackrel{\text{def}}{=} \sigma \mathbf{e}^3 \times \mathbf{e}^1 = \mathbf{j} + \zeta_{\tilde{y}} \mathbf{k},$$

$$\mathbf{e}_3 \stackrel{\text{def}}{=} \sigma \mathbf{e}^1 \times \mathbf{e}^2 = \sigma \mathbf{k}.$$

Conclusions

Doing a thickness weighted average can become very confusing - refer to Bill's paper whenever you do it!

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\psi(y,\tilde{b})=\overline{\int^{b_s}_{\tilde{b}(x,y,z)}v \sigma
\, db'}=\int_0^{L_x}\int_{-H}^0 v^{\dagger} \mathcal{H}
\left[b(x,y,z)-\tilde{b} \right]\, dz \, dx
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Velocities and buoyancy contours

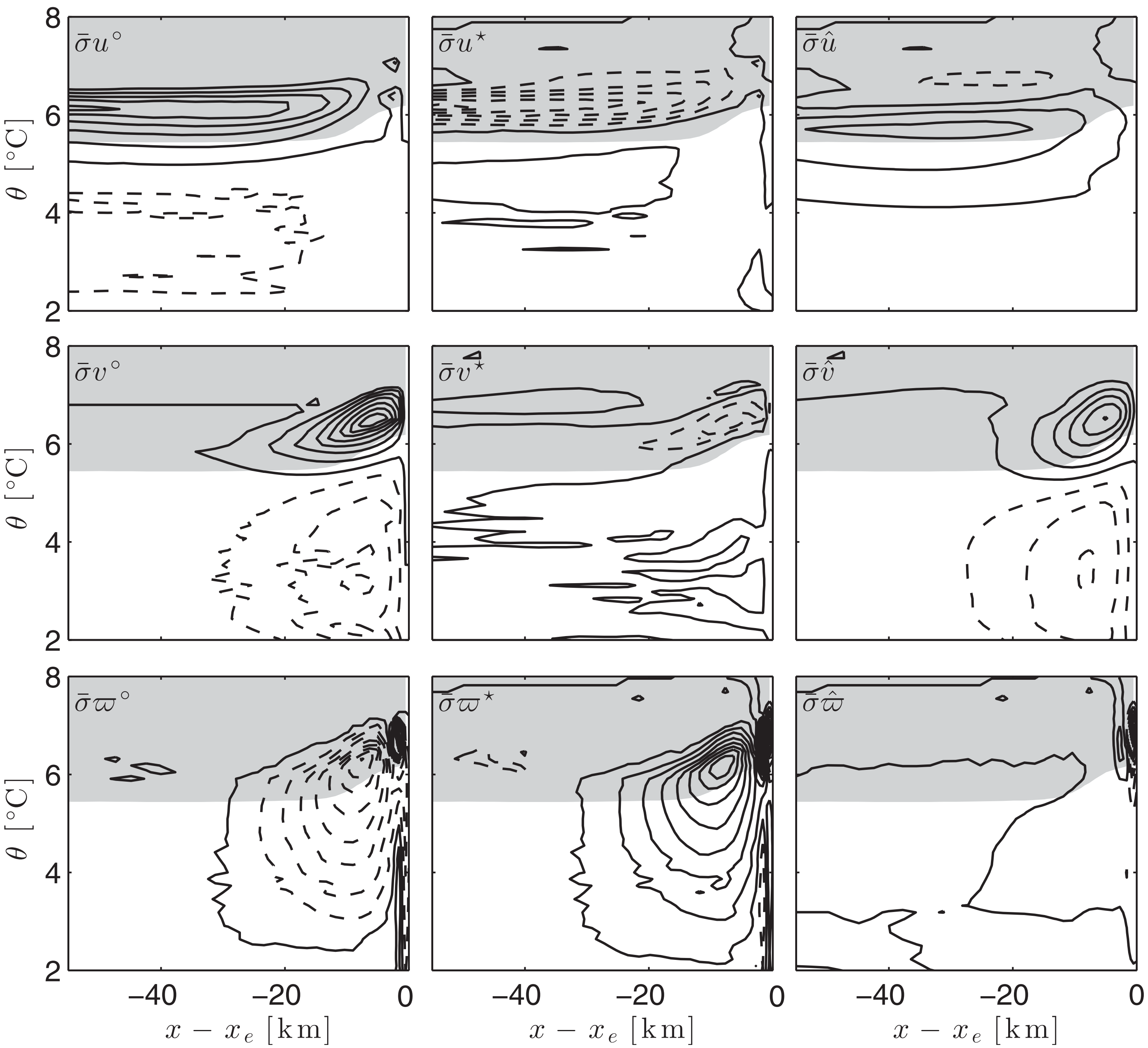

FIG. 6. The three components of $\overline{\sigma}\mathbf{u}^\sharp$ at $y = 3000$ km are shown as a function of $\tilde{x}$ and $\tilde{\theta}$. Here, (top row, rhs) $\overline{\sigma}\hat{u}$, (middle row, rhs) $\overline{\sigma}\hat{v}$, and (bottom row, rhs) $\overline{\sigma}\hat{\varpi} + \overline{\zeta}_{\tilde{t}}$ are shown. The lhs and central columns additionally show the contributions of the residual transport from the time-mean and eddy components respectively. The definitions of the fields are given in the text. The CI are 300 m s, 2000 m s, and 2×10^{-5} m s^{-1} for the top, middle, and bottom rows, respectively; negative contours are dashed. The swash is shaded gray.

Plotting mean and eddy parts

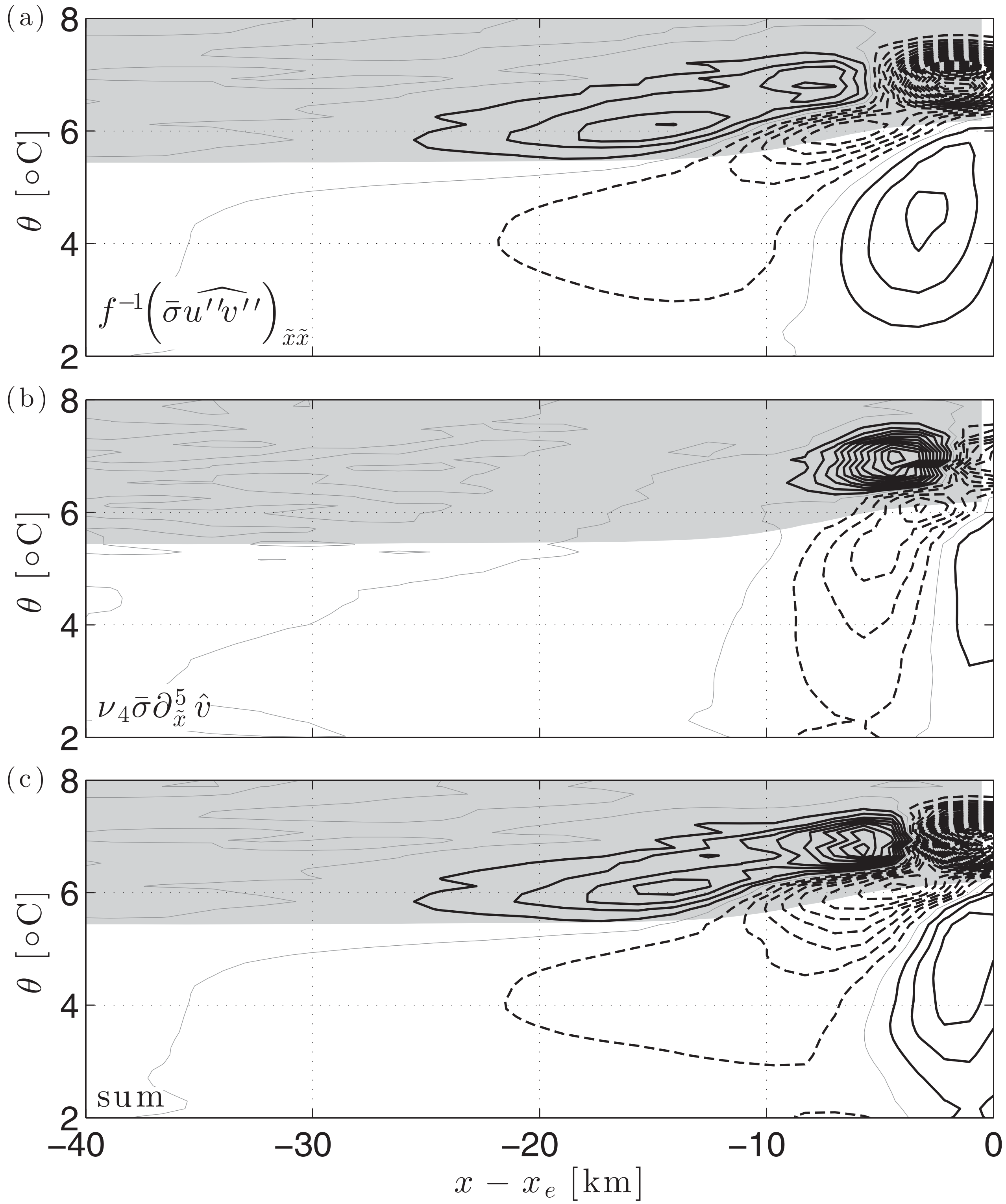

FIG. 11. The terms (a) $(\overline{\sigma}\widehat{u''v''})_{\tilde{x}\tilde{x}}/f$, (b) $\nu_4\hat{v}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$, and (c) their sum near the eastern boundary are shown as a function of x and θ at the nominal latitude $y = 3005$ km (all fields are also averaged in y over a 160-km swath). The CI is 0.01 s; negative contours are dashed and the zero contour is thin and gray. The swash is shaded gray.

$$\overline{m}_{\tilde{y}}(\overline{\sigma}/f)_{\tilde{x}} - \overline{m}_{\tilde{x}}(\overline{\sigma}/f)_{\tilde{y}} \approx -\left(\frac{\overline{\sigma' m'_{\tilde{y}}}}{f}\right)_{\tilde{x}} - \left[\frac{(\overline{\sigma}\widehat{u''v''})_{\tilde{x}} + \nu_4 \overline{\sigma}\,\overline{v}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}}}{f}\right]_{\tilde{x}}.$$