Modern diagnostics for the large-scale circulation: from transformed eulerian mean (TEM) to thickness weighted average (TWA)

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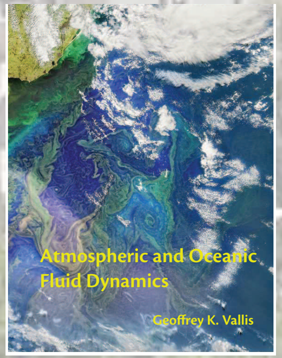
With help from Spencer Jones, Christopher Wolfe, WR Young and Vallis's book (2nd ed.)



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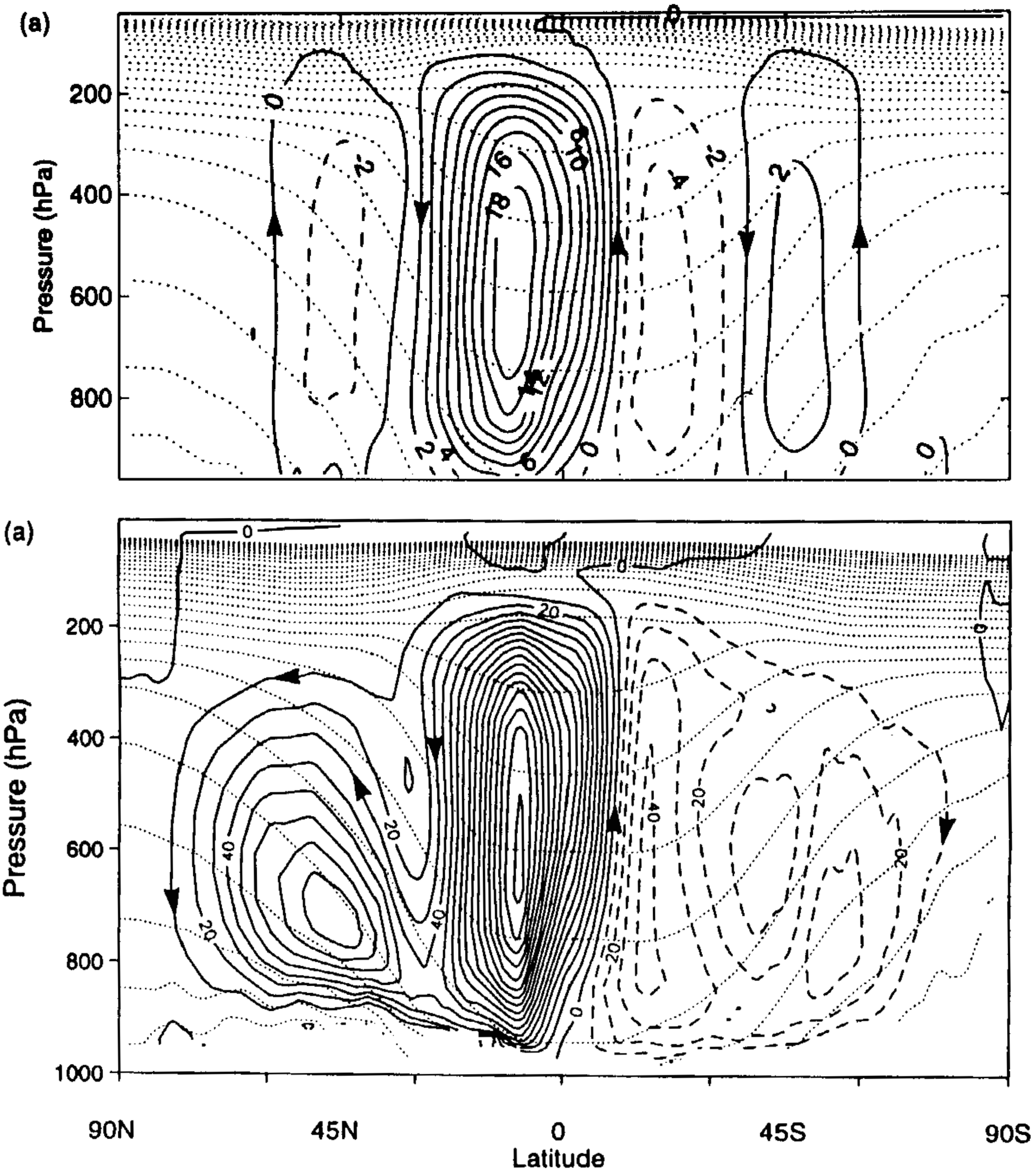
Historical puzzle in meteorology

Atmosphere Dec-Feb



Eulerian zonal-mean (averaged in longitude) meridional velocity at constant pressure: thermally indirect midlatitude Ferrel cells, increasing pole-to-equator temperature gradient

Zonal-mean meridional velocity at constant isentropes (potential temperature), remapped in pressure

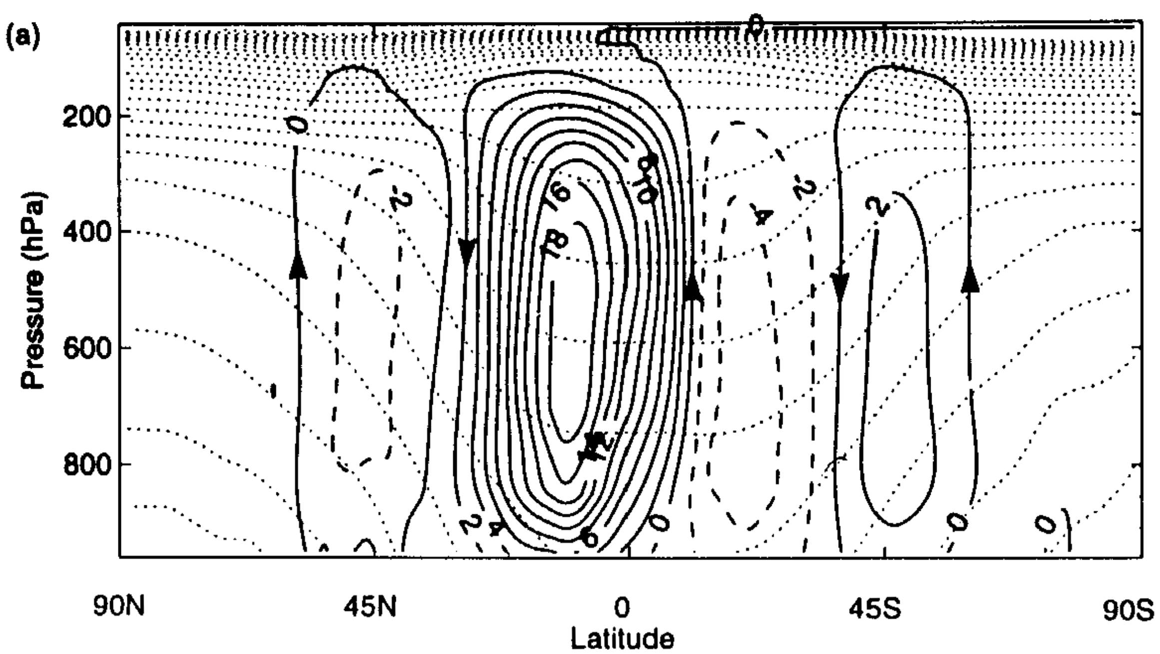
The thermally indirect cells (Ferrel) disappear in isentropic coordinates

Karoly, D.J., McIntosh, P.C., Berrisford, P., McDougall, T.J. and Hirst, A.C., 1997. Similarities of the Deacon cell in the Southern Ocean and Ferrel cells in the atmosphere. *Quarterly Journal of the Royal Meteorological Society*, *123*(538), pp.519-526.

Two different measures of zonally averaged transport

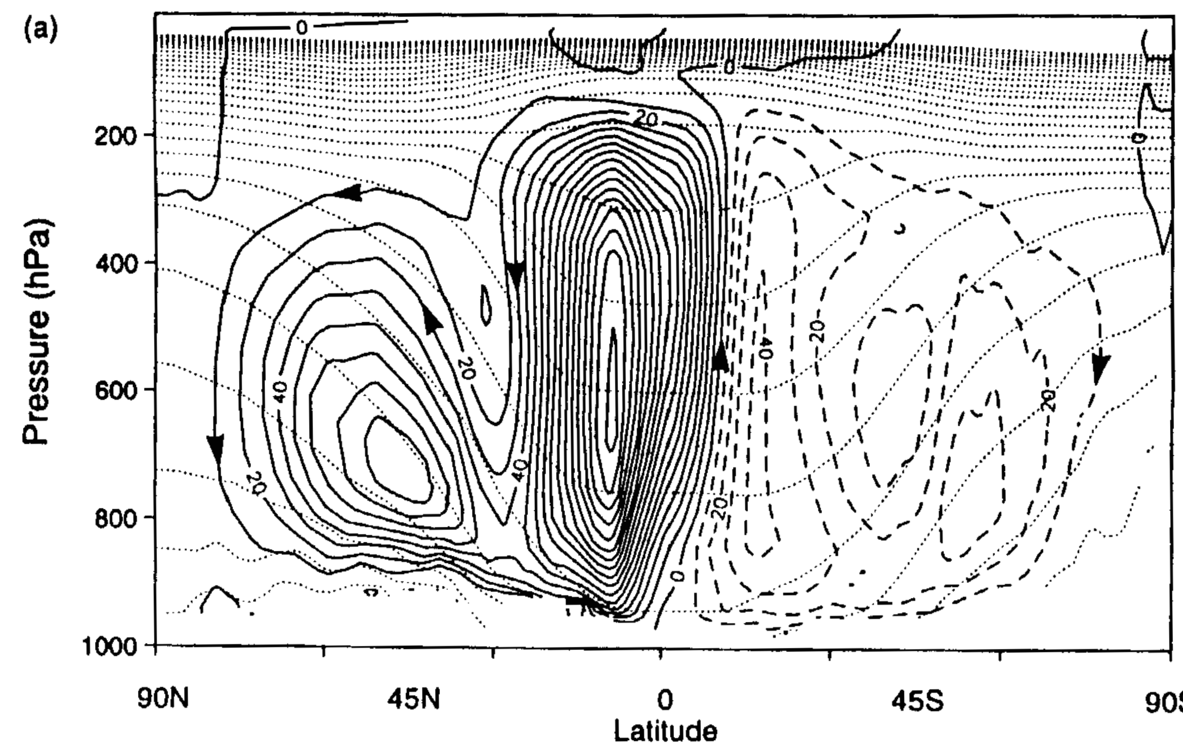
Volume transport

$$\nabla \cdot \bar{\mathbf{v}} = 0$$



Potential temperature transport

$$\partial_t \bar{\rho} + \nabla \cdot \overline{\rho \mathbf{v}} = \text{Diabatic terms}$$



Temperature, salinity, humidity, CO2 transports are also interesting
(Potential) temperature is special: it is stably stratified, a good vertical coordinate

The Eulerian flow indicates equatorward heat transport in mid-latitudes

In isentropic coordinates there is poleward transport

Maintenance of the Eulerian Ferrell cell

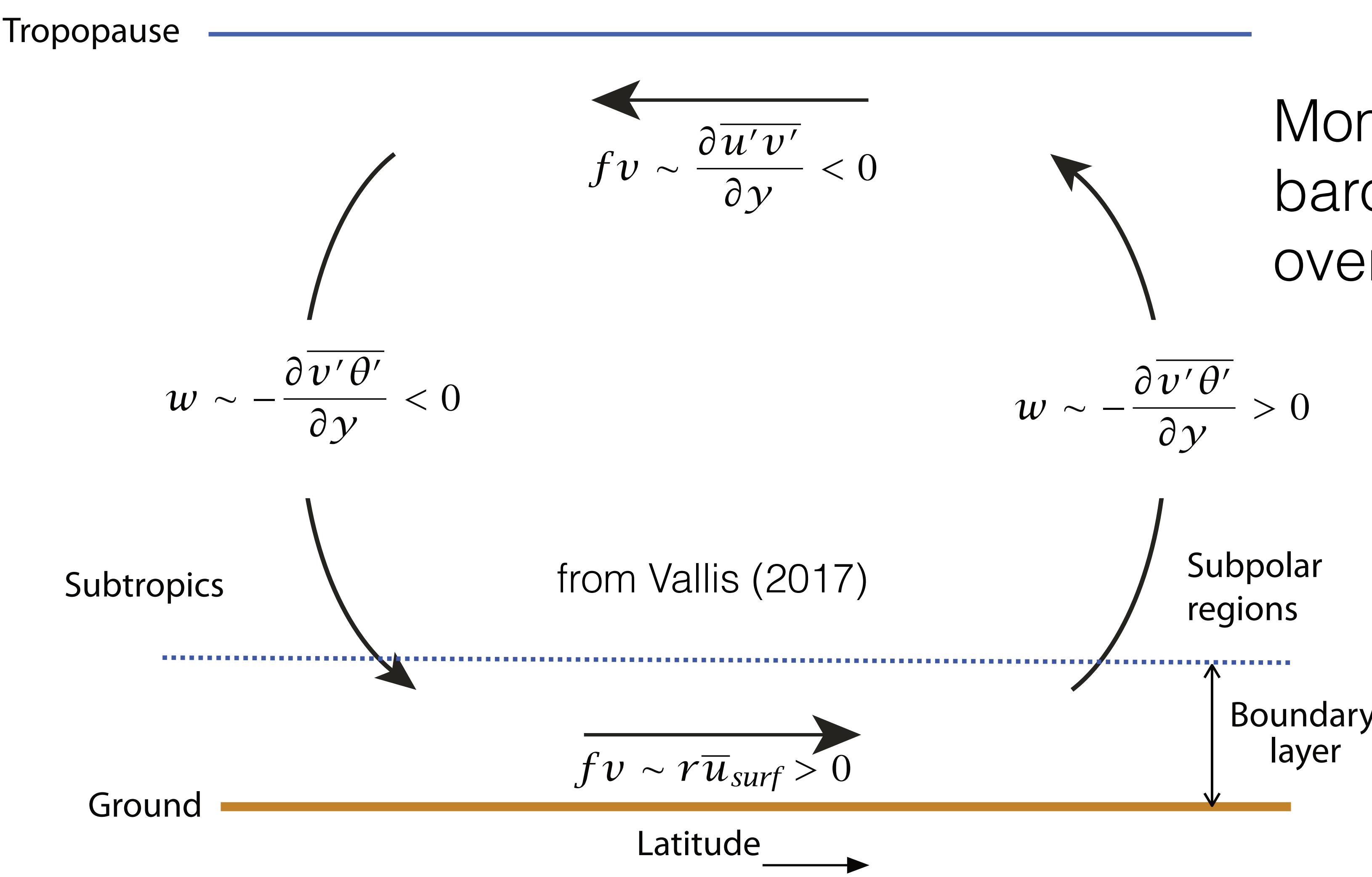
The zonally average $\overline{v}$ is ageostrophic $\qquad \bar{p}_x = 0$

$$-f\overline{v} = -\frac{1}{\cos^2\vartheta}\frac{\partial}{\partial\vartheta}(\cos^2\vartheta\,\overline{u'v'}) + \frac{1}{\rho}\frac{\partial\tau}{\partial z}$$

$\overline{u}$-momentum balance: driven by convergence of eddy momentum flux or friction

$$\overline{w} = \frac{1}{N^2}\left[Q_b - \frac{1}{\cos\vartheta}\frac{\partial(\overline{v'b'}\cos\vartheta)}{\partial y}\right]$$

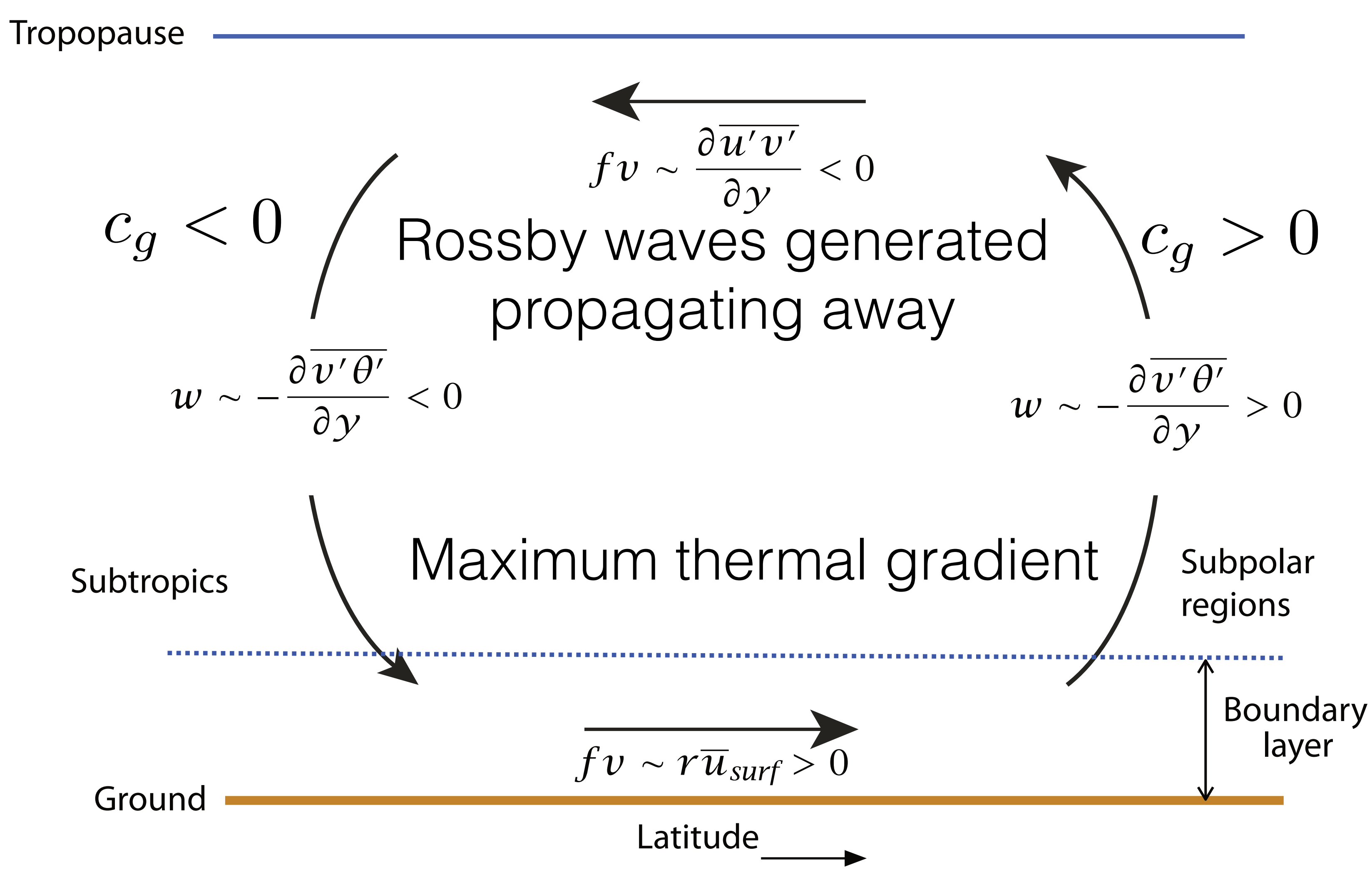
Buoyancy balance: driven by convergence of eddy buoyancy flux or diabatic terms



Momentum and buoyancy fluxes from baroclinic eddies drive the Eulerian overturning

$$\frac{\partial\overline{v}}{\partial y} + \frac{\partial\overline{w}}{\partial z} = 0$$

Mechanism of momentum flux convergence



from Vallis (2017)

$\psi = \mathrm{Re}\, C e^{\mathrm{i}(kx+ly-\omega t)}$ Rossby wave with dispersion relation $\omega = ck = \overline{u}k - \frac{\beta k}{k^2+l^2}$

Meridional group velocity $c_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2+l^2)^2}$

Horizontal velocity $u' = -\mathrm{Re}\, C\, \mathrm{i} l e^{\mathrm{i}(kx+ly-\omega t)}, \qquad v' = \mathrm{Re}\, C\, \mathrm{i} k e^{\mathrm{i}(kx+ly-\omega t)}.$

$\overline{u'v'} = -\frac{1}{2}C^2 kl = -\mu^2 c_g^y$ goes from negative to positive, i.e. $\frac{\partial}{\partial y}\overline{u'v'} < 0$

Is the net heat transport equatorward or poleward?

$$\bar{w}N^2 + \frac{1}{\cos\theta}\frac{\partial(\overline{v'b'}\cos\theta)}{\partial y} = Q_b$$

A small residual of two large terms almost balancing

Transport by the mean

Transport by eddies

$$\frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

Requires

$$\frac{\partial}{\partial y}\left(f\frac{\partial}{\partial z}\frac{\overline{v'b'}}{N^2} - \frac{\partial\overline{v'u'}}{\partial y}\right) = f\frac{\partial}{\partial z}\frac{Q_b}{N^2} + \frac{1}{\rho}\frac{\partial}{\partial y}\frac{\partial \tau}{\partial z}$$

$$\frac{\partial}{\partial y}\overline{v'q'}$$

Eddy PV flux divergence

We need to look at the PV fluxes

Look at the QGPV

In QG we have two variables, linearly related

$$\psi \text{ and } q \equiv \nabla^2\psi + f^2\frac{\partial}{\partial z}\left(\frac{\partial_z\psi}{N^2}\right)$$

We want to know the average, large-scale, slow-time evolution of $\bar{\psi}$ and $\bar{q}$

$$\partial_t\bar{q} + J(\bar{\psi},\bar{q}) = -\nabla\cdot\overline{\mathbf{u}'q'} + curl\bar{F}$$

+ boundary conditions:

$$f\left[\partial_t\bar{\psi}_z + J(\bar{\psi},\bar{\psi}_z)\right] + N^2\bar{w} = -\nabla\cdot\overline{\mathbf{u}'\psi'_z} + \bar{S} \quad \text{at } z = 0, H$$

We need to know $\bar{\psi}$, $\overline{\mathbf{u}'q'}$ and $\overline{\mathbf{u}'b'}$ on boundaries

In general

$$v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{1}{2}\frac{\partial}{\partial x}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right).$$

For a zonal average

$$\overline{v'q'} = -\frac{\partial}{\partial y}\overline{u'v'} + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}\overline{v'b'}\right).$$

Eliassen-Palm fluxes $\boldsymbol{\mathcal{F}}$

We can write: $\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}$, with: $\boldsymbol{\mathcal{F}} \equiv -\overline{u'v'}\,\mathbf{j} + \dfrac{f_0}{N^2}\overline{v'b'}\,\mathbf{k}$ $\quad \nabla_x \cdot \equiv (\partial/\partial y, \partial/\partial z)\cdot$

$$f_0 \frac{\partial \overline{u}}{\partial z} = -\frac{\partial \overline{b}}{\partial y},$$

In QG, momentum and buoyancy are:

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v} - \frac{\partial}{\partial y}\overline{u'v'} + \overline{F},$$

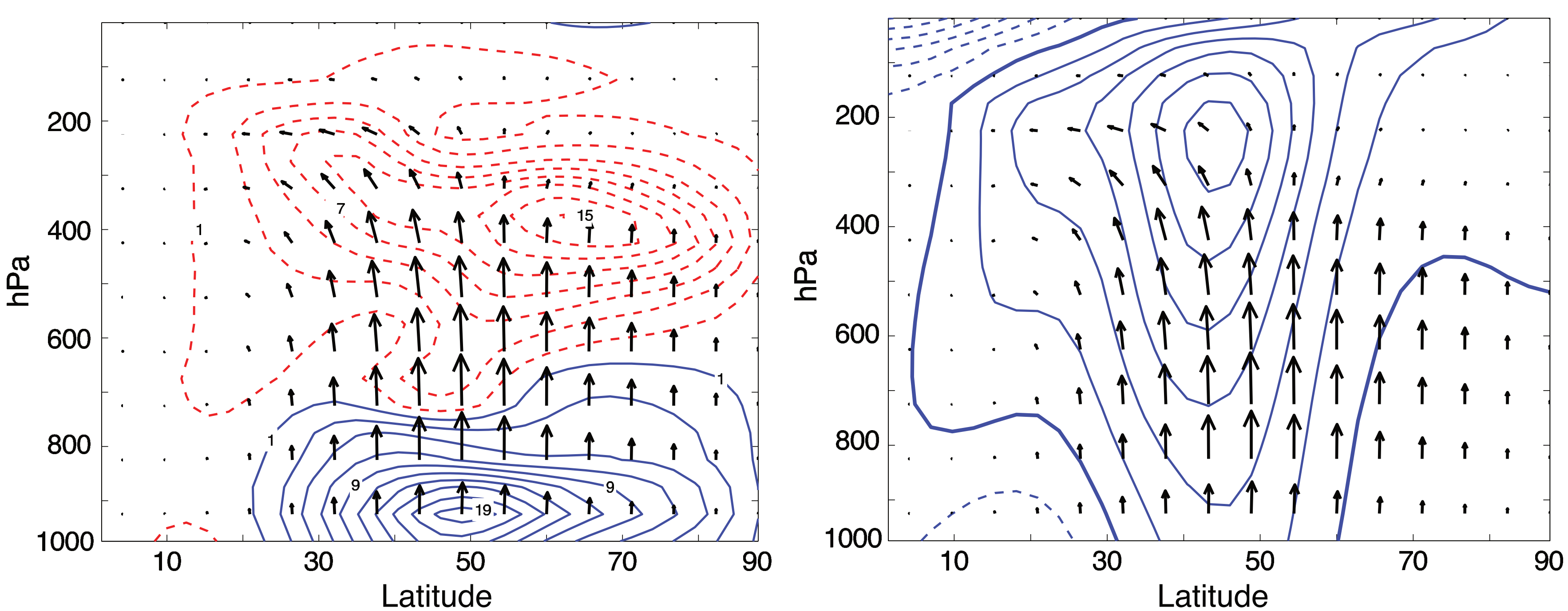
$$\frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w} - \frac{\partial}{\partial y}\overline{v'b'} + \overline{S}.$$

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v}^* + \overline{v'q'} + \overline{F},$$

$$\frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w}^* + \overline{S},$$

Where: $\overline{v}^* = \overline{v} - \dfrac{\partial}{\partial z}\left(\dfrac{1}{N^2}\overline{v'b'}\right)$, $\quad \overline{w}^* = \overline{w} + \dfrac{\partial}{\partial y}\left(\dfrac{1}{N^2}\overline{v'b'}\right)$ is an incompressible velocity

Propagation and breaking of EP fluxes with momentum deposition and jet acceleration



$$\boldsymbol{\mathcal{F}} = -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k},$$

dashed colors: $\nabla \cdot \boldsymbol{\mathcal{F}} = \overline{v'q'}$

solid contours: $\overline{u}$

from Vallis (2017)

The transformed eulerian mean - TEM (QG)

Remember: $\overline{v'q'} = \nabla_x \cdot \boldsymbol{\mathcal{F}}$, with: $\boldsymbol{\mathcal{F}} \equiv -\overline{u'v'}\,\mathbf{j} + \frac{f_0}{N^2}\overline{v'b'}\,\mathbf{k}$ $\quad \nabla_x \cdot \equiv (\partial/\partial y, \partial/\partial z)\cdot$

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v}^* + \overline{v'q'} + \overline{F},$$

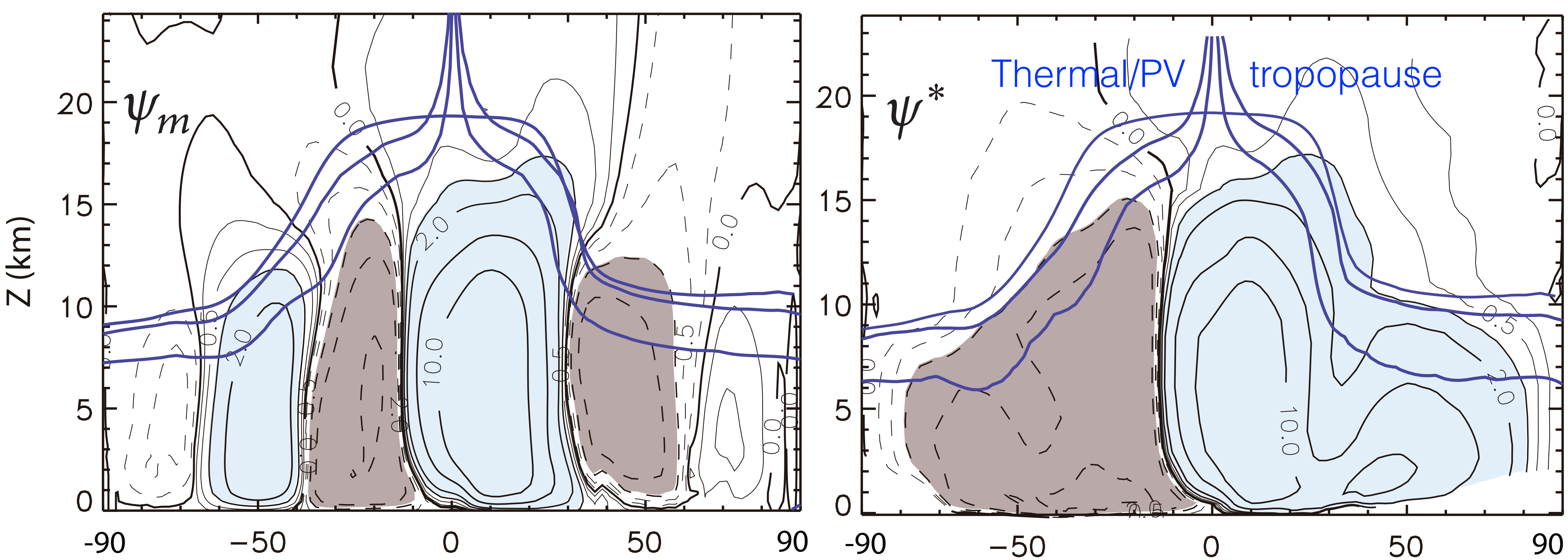
$$\frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w}^* + \overline{S},$$

$$\overline{v}^* = \overline{v} - \frac{\partial}{\partial z}\left(\frac{1}{N^2}\overline{v'b'}\right),$$

$$\overline{w}^* = \overline{w} + \frac{\partial}{\partial y}\left(\frac{1}{N^2}\overline{v'b'}\right).$$

$$(\overline{v}, \overline{w}) = \left(-\frac{\partial \psi_m}{\partial z}, \frac{\partial \psi_m}{\partial y}\right)$$

Residual streamfunction: $\psi^* \equiv \psi_m + \frac{1}{N^2}\overline{v'b'}$, $\quad (\overline{v}^*, \overline{w}^*) = \left(-\frac{\partial \psi^*}{\partial z}, \frac{\partial \psi^*}{\partial y}\right)$



Ferrel cell is reversed

from Vallis (2017)

Use thermal wind to eliminate u and b: $$f_0^2 \frac{\partial^2 \psi^*}{\partial z^2} + N^2 \frac{\partial^2 \psi^*}{\partial y^2} = f_0 \frac{\partial}{\partial z}\overline{v'q'} + f_0 \frac{\partial \overline{F}}{\partial z} + \frac{\partial \overline{S}}{\partial y}.$$

The residual circulation - TEM (QG)

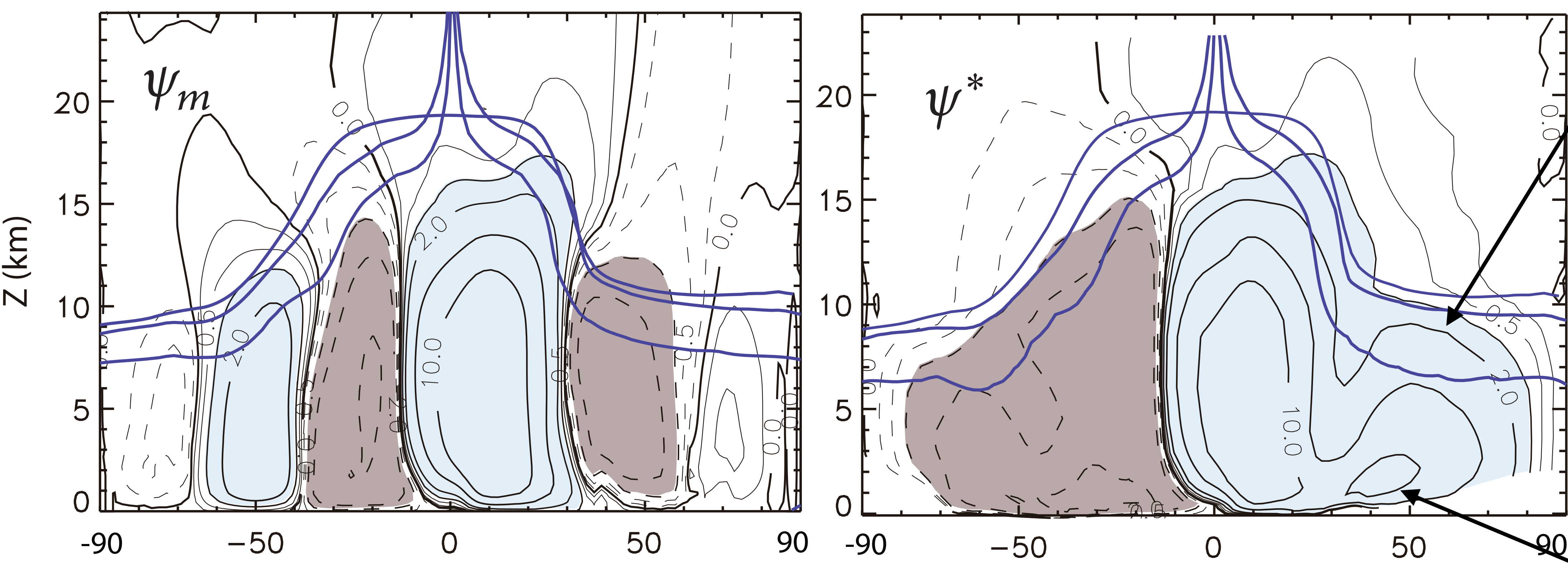
$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v}^* + \overline{v'q'} + \overline{F},$$

$$\frac{\partial \overline{b}}{\partial t} = -N^2 \overline{w}^* + \overline{S},$$

$$(\overline{v}^*, \overline{w}^*) = \left(-\frac{\partial \psi^*}{\partial z}, \frac{\partial \psi^*}{\partial y}\right)$$

$$\psi^* \equiv \psi_m + \frac{1}{N^2}\overline{v'b'},$$

In the upper branch: $f_0 \overline{v}^* \approx -\overline{v'q'}$



from Vallis (2017)

In the lower branch: $f_0 \overline{v}^* \approx -\overline{F}$

The residual circulation is more representative of tracer transport than Eulerian flow

Summary so far

The apparent equatorward heat transport by the Ferrel cells is resolved by including eddy-transport of (potential) temperature.

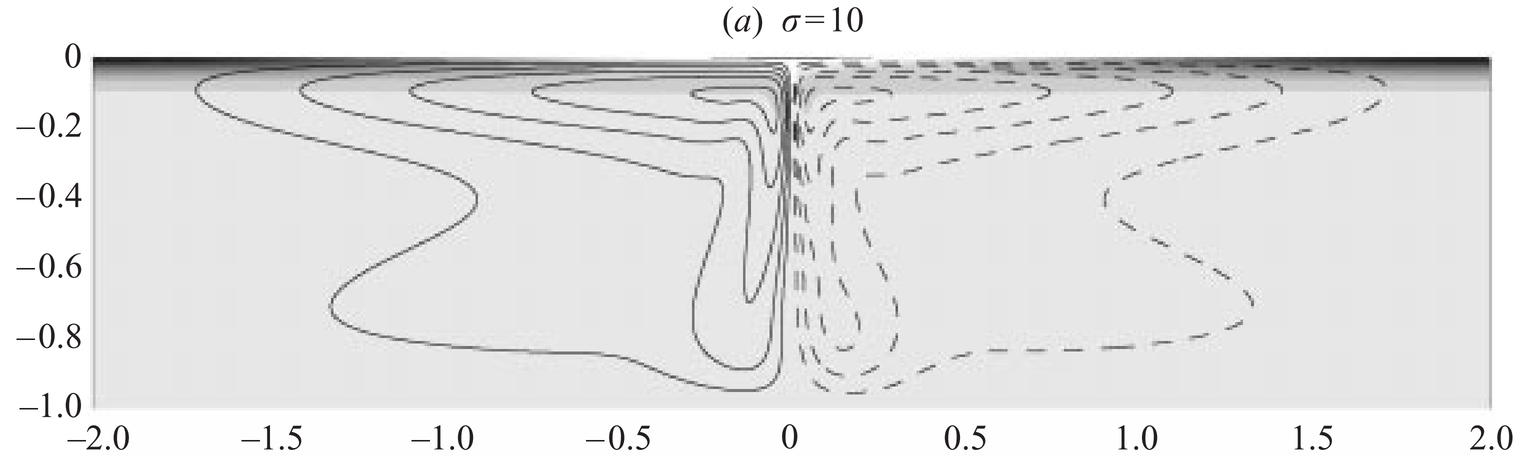
The momentum eddy-transport and form-stress maintain the Eulerian Ferrel cells.

TEM accounts for these processes in a simple QG framework: a breakthrough.

In QG, isentropes (potential density or potential temperature) are horizontal, so there is little difference between diabatic and vertical transport.

In general isentropes are not horizontal, so TEM needs to be generalized for usage in the primitive equations.

Example 1: 2D Steady Flow



Steady 2D nonhydrostatic convection—Paparella & Young (2002)
Reyleigh # = 10^8

Buoyancy equation:

$$b_t + \boldsymbol{u} \cdot \nabla b = \mathcal{D}$$

For 2D steady flow:

$$v b_y + w b_z = \mathcal{D}$$

$$v_y + w_z = 0$$

$$\psi_z = -v \qquad \psi_y = w$$

b is advected by ψ and dissipated by $\mathcal{D}$: $\qquad J(\psi, b) = \mathcal{D}$

where $$J(\psi, b) = \frac{\partial(\psi, b)}{\partial(y, z)} = \psi_y b_z - \psi_z b_y$$

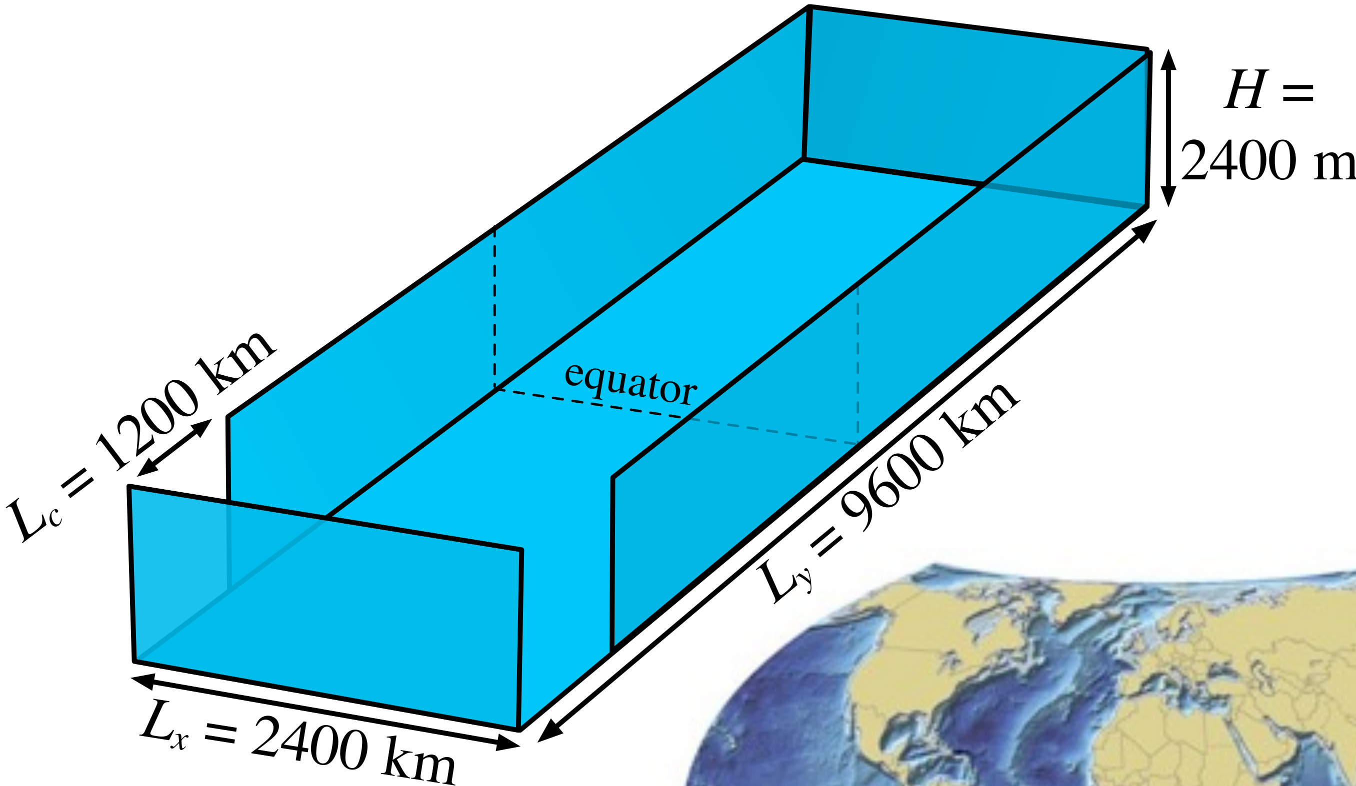
Use buoyancy's vertical coordinate:

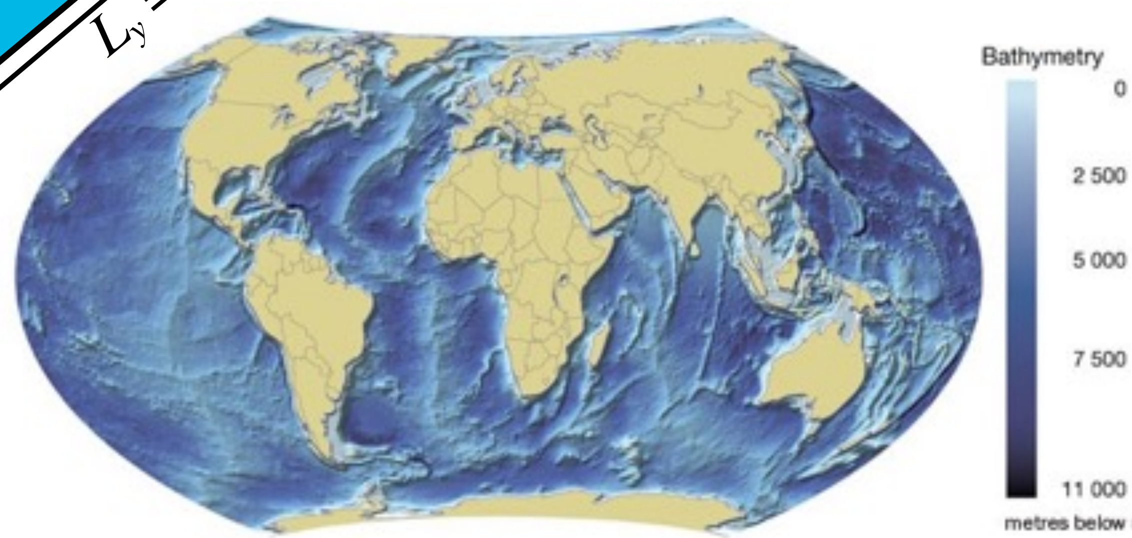
$$\frac{\partial(\psi, b)}{\partial(y, z)} = \frac{\partial(\psi, b)}{\partial(\tilde{y}, \tilde{b})} \frac{\partial(\tilde{y}, \tilde{b})}{\partial(y, z)} = b_z \psi_{\tilde{y}} \qquad \tilde{y} \to \text{at constant } b$$

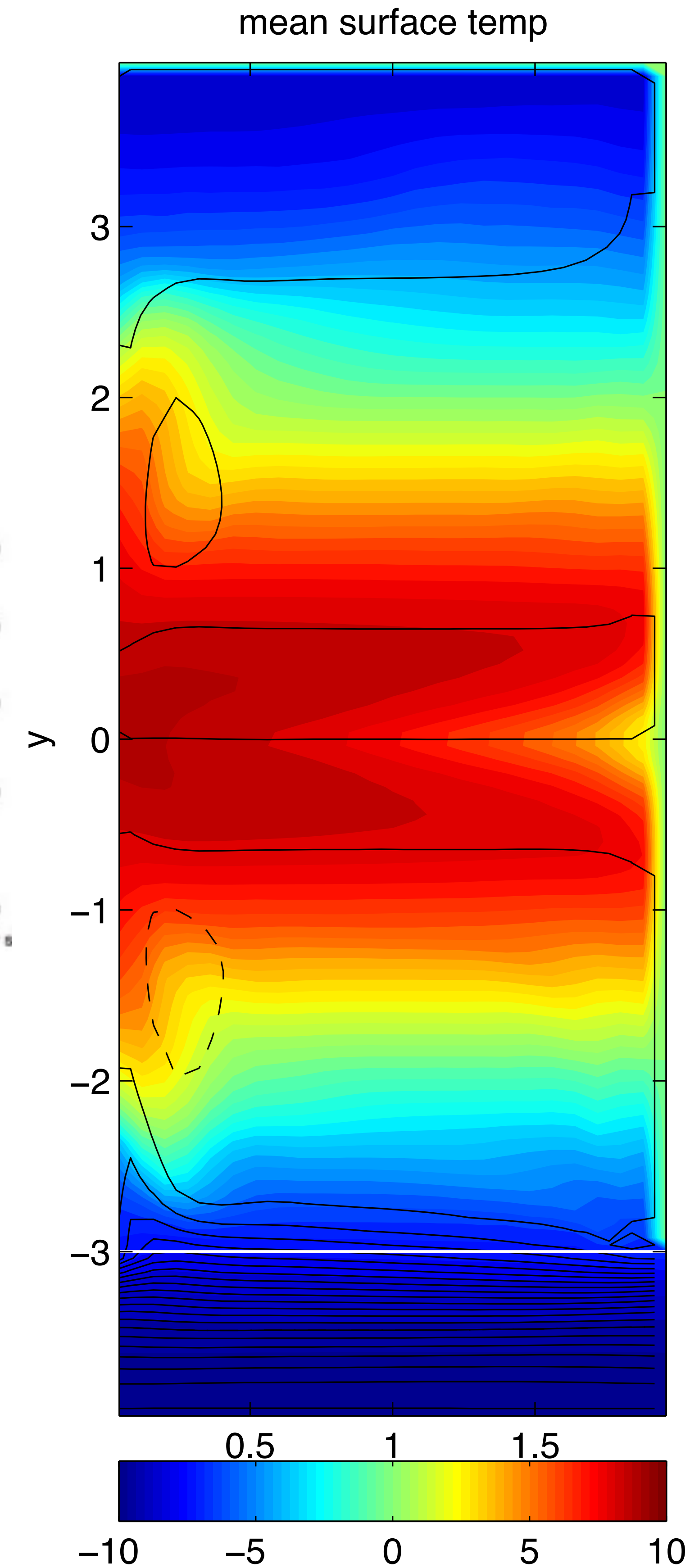
$$\boxed{\psi_{\tilde{y}} = \sigma \mathcal{D} \qquad \sigma = b_z^{-1}}$$

The mean flow advects the mean buoyancy and the diapycnal velocity balances diabatic sources and sinks

Example 11: 3D Steady Flow in oceanic box

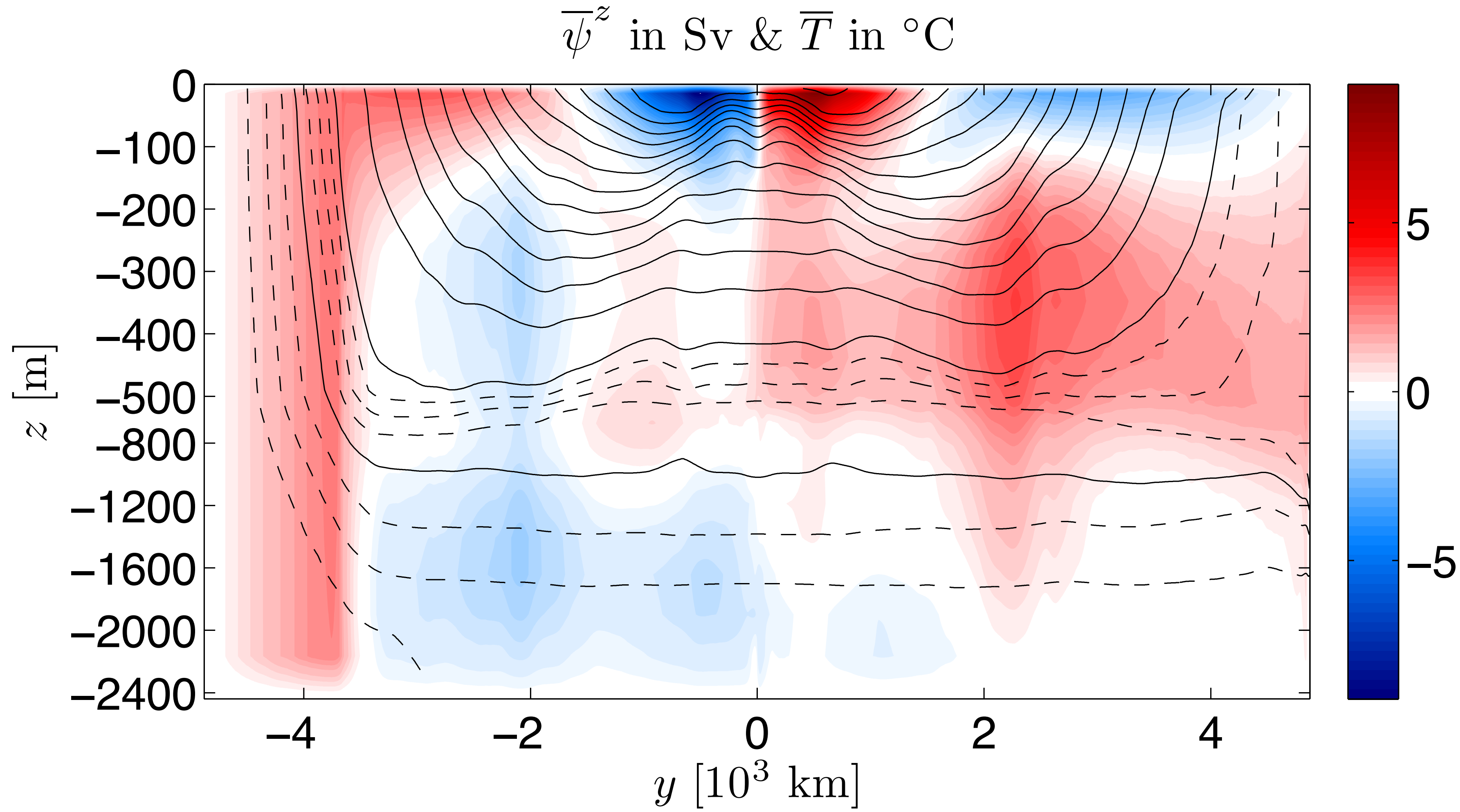






- Idealized single basin +ACC, forced at surface
- Half-sized basin in a notched-box
- Coarse resolution (100 km), hydrostatic MITgcm
- No salt: buoyancy linearly related to temperature
- GM eddy parameterization
- Explicit mixing only in surface layer ~50 m deep

3D Steady Flow



$\overline{(\cdot)}^z \rightarrow$ zonal mean (at constant z)

$$J\left(\overline{\psi}^z, \overline{b}^z\right) + \overline{(v'b')}^z_y + \overline{(w'b')}^z_z = \overline{\mathcal{D}}^z$$

The mean flow does not advect the mean buoyancy:

$$\overline{\psi_{\tilde{y}}}^z \neq \overline{\sigma}^z \overline{\mathcal{D}}^z \qquad \tilde{y} \rightarrow \text{at constant } \overline{b}$$

A thermally indirect cell in the periodic portion of the domain: Deacon cell equivalent to Ferrell cell in atmosphere.

Residual Streamfunction (3D steady)

Begin in buoyancy coordinates: 3-D variation:

$$(\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\sigma \mathcal{D})_{\tilde{b}} = 0 \qquad \sigma = b_z^{-1}$$

Zonally average at constant b:

$$\overline{(\sigma v)}_{\tilde{y}} + \overline{(\sigma \mathcal{D})}_{\tilde{b}} = 0$$

Define the residual streamfunction:

$$\psi^\dagger_{\tilde{b}} = -\overline{(\sigma v)} = -\bar{\sigma}\hat{v} \qquad \psi^\dagger_{\tilde{y}} = \overline{(\sigma \mathcal{D})} = \bar{\sigma}\hat{\varpi}$$

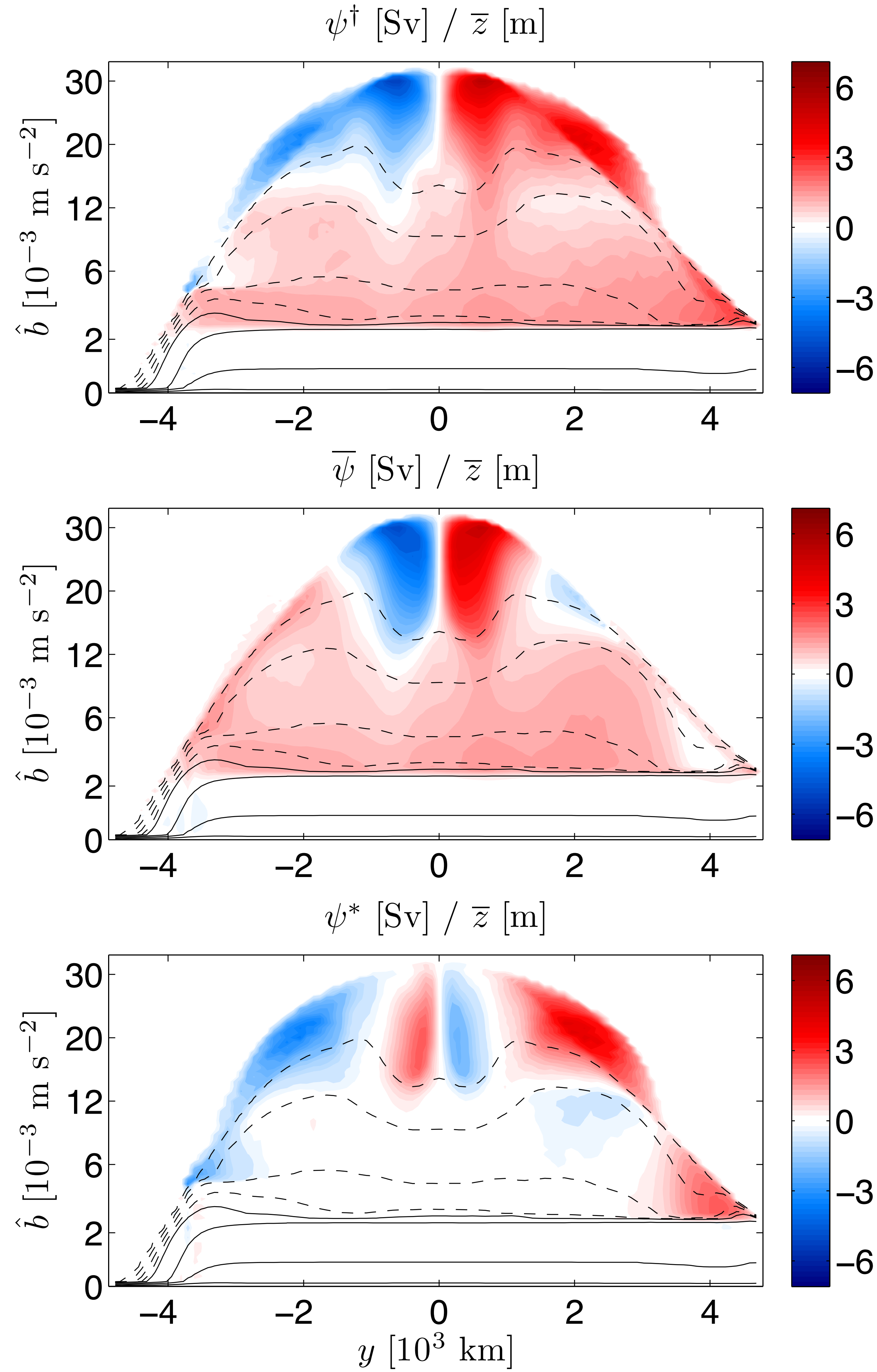
Decompose into mean and eddy components:

$$\psi^\dagger = \overline{\psi} + \psi^*$$

$$\overline{\psi}_{\tilde{b}} = -\bar{\sigma}\bar{v} \qquad \psi^*_{\tilde{b}} = -\overline{\sigma' v'}$$

Mean isopycnal height:

$$\overline{z}_{\tilde{b}} = \bar{\sigma}$$



Calculation in level coordinates (3D steady)

Easier to calculate in level coordinates:

$$\psi^\dagger(\tilde{y},\tilde{b}) = -\int_{-\infty}^{\tilde{b}} \overline{\sigma v}\,\mathrm{d}b = -\overline{\int_{-H}^{\zeta} v\,\mathrm{d}z} = -\int_{-H}^{0} \overline{v\,\mathcal{H}\left[\tilde{b} - b(x,y,z)\right]}\,\mathrm{d}z$$

Heaviside function

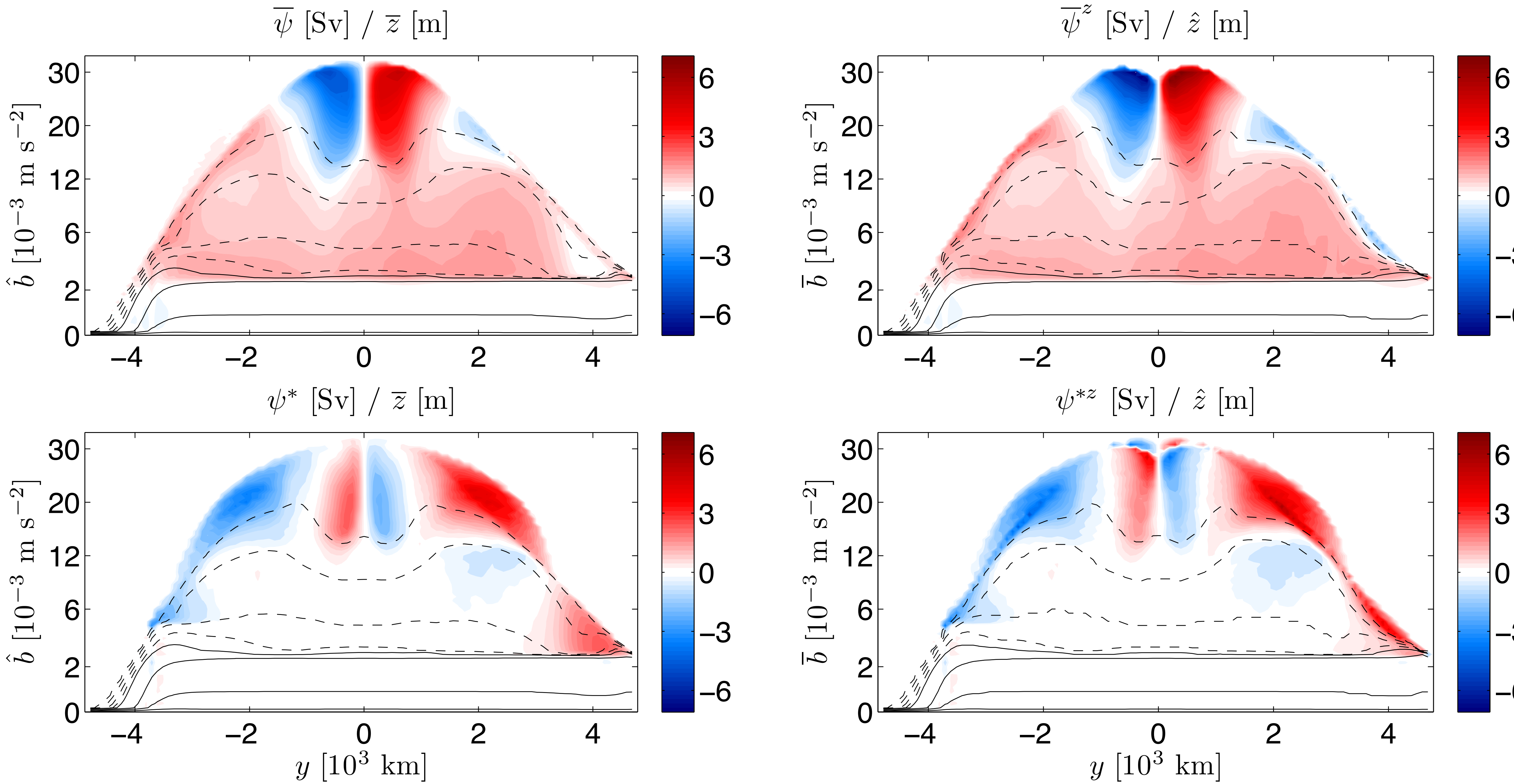
where ζ satisfies $\tilde{b} = b\left(x, y, \zeta(\tilde{x},\tilde{y},\tilde{b})\right)$

Do the same with the mean streamfunction:

$$\bar{\psi}(\tilde{y},\tilde{b}) = -\int_{-\infty}^{\tilde{b}} \bar{\sigma}\bar{v}\,\mathrm{d}b = -\int_{-H}^{0} \bar{v}\,\overline{\mathcal{H}\left[\tilde{b} - b(x,y,z)\right]}\,\mathrm{d}z$$

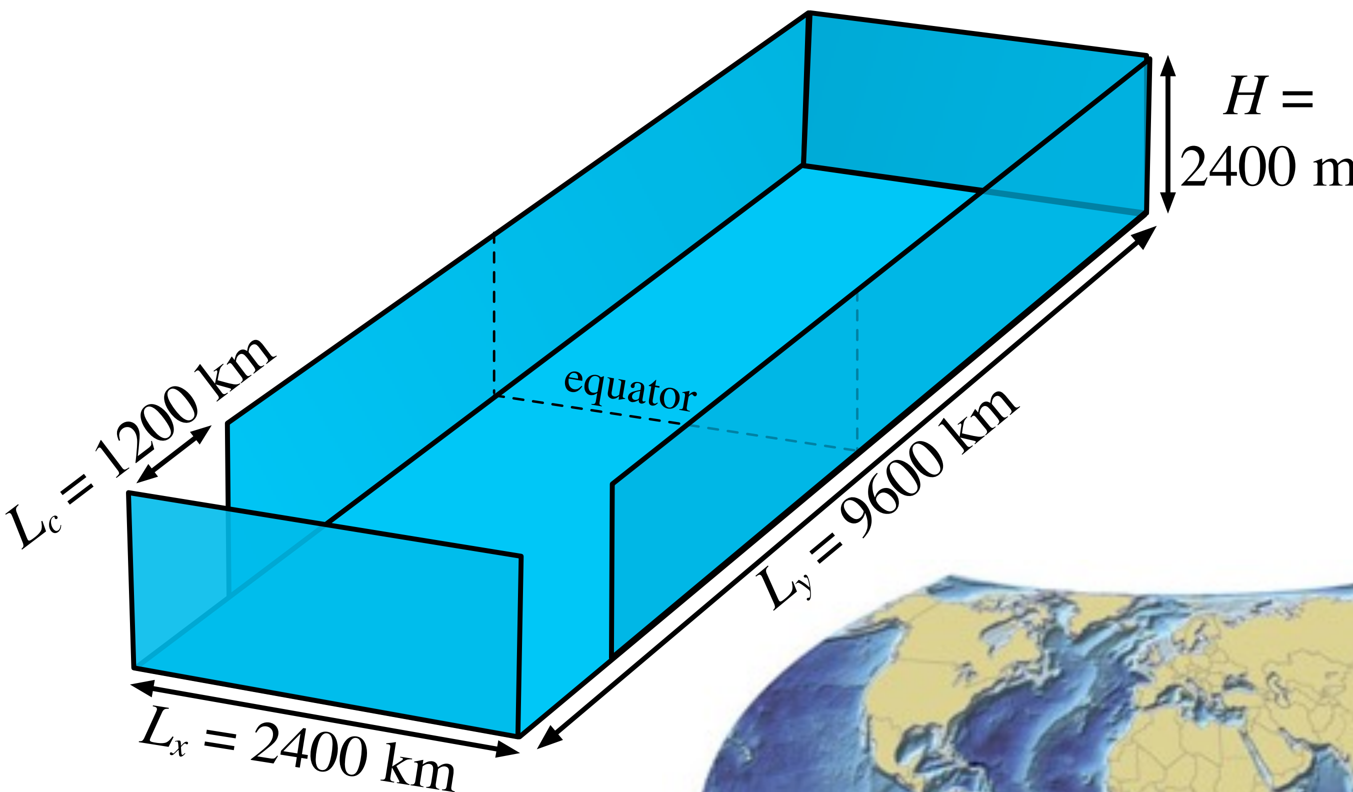
Note: $\bar{\sigma} = \overline{b_z^{-1}} \neq \bar{b}_z^{-1}$ so $\bar{\psi}^z(\tilde{y},\tilde{b}) = -\int_{-H}^{0} \bar{v}\,\mathcal{H}\left[\tilde{b} - \bar{b}\right]\mathrm{d}z \neq \bar{\psi}(\tilde{y},\tilde{b})$

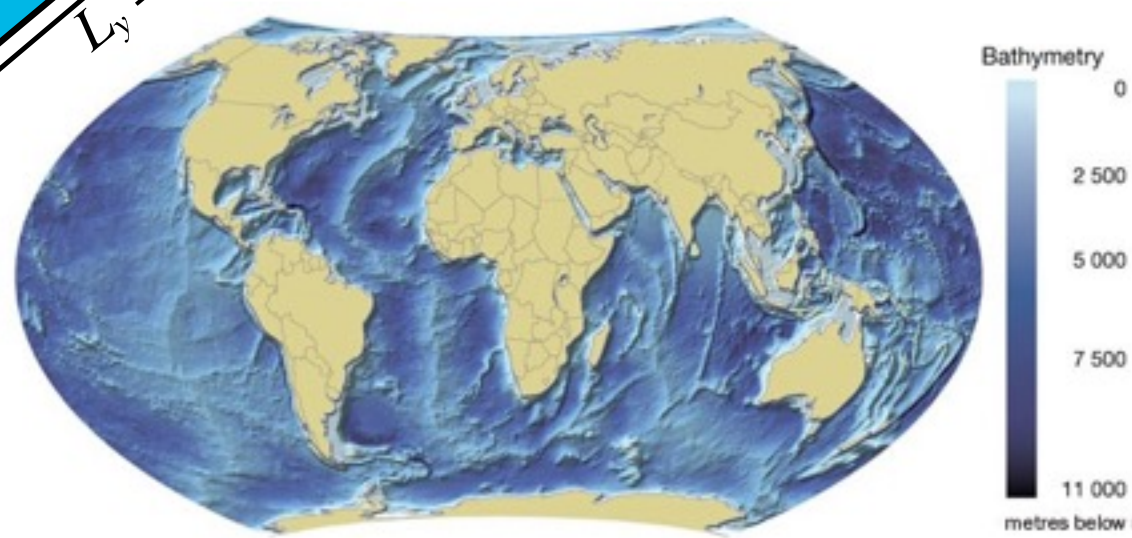
Definition of the Mean (3D steady)

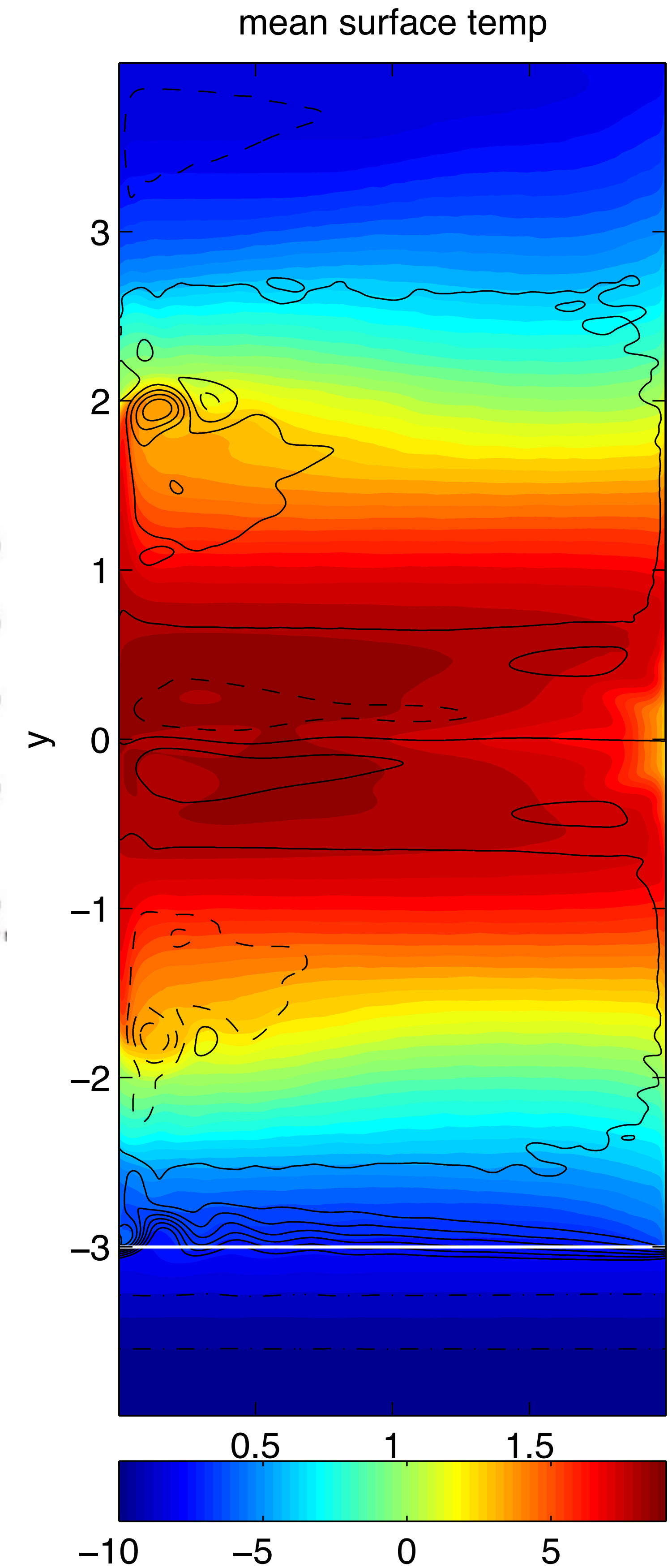


Thus, $\overline{\psi}$ is not simply a remapping of $\overline{\psi}^z$

Example III: 3D unsteady flow







- Idealized single basin +ACC forced at surface
- Half-sized basin in a notched-box
- High resolution (5.4 km), hydrostatic MITgcm
- No salt: buoyancy linearly related to temperature
- No eddy or mixed layer parameterizations
- $\kappa = 1.2 \times 10^{-5}$ m^2 s^{-1}

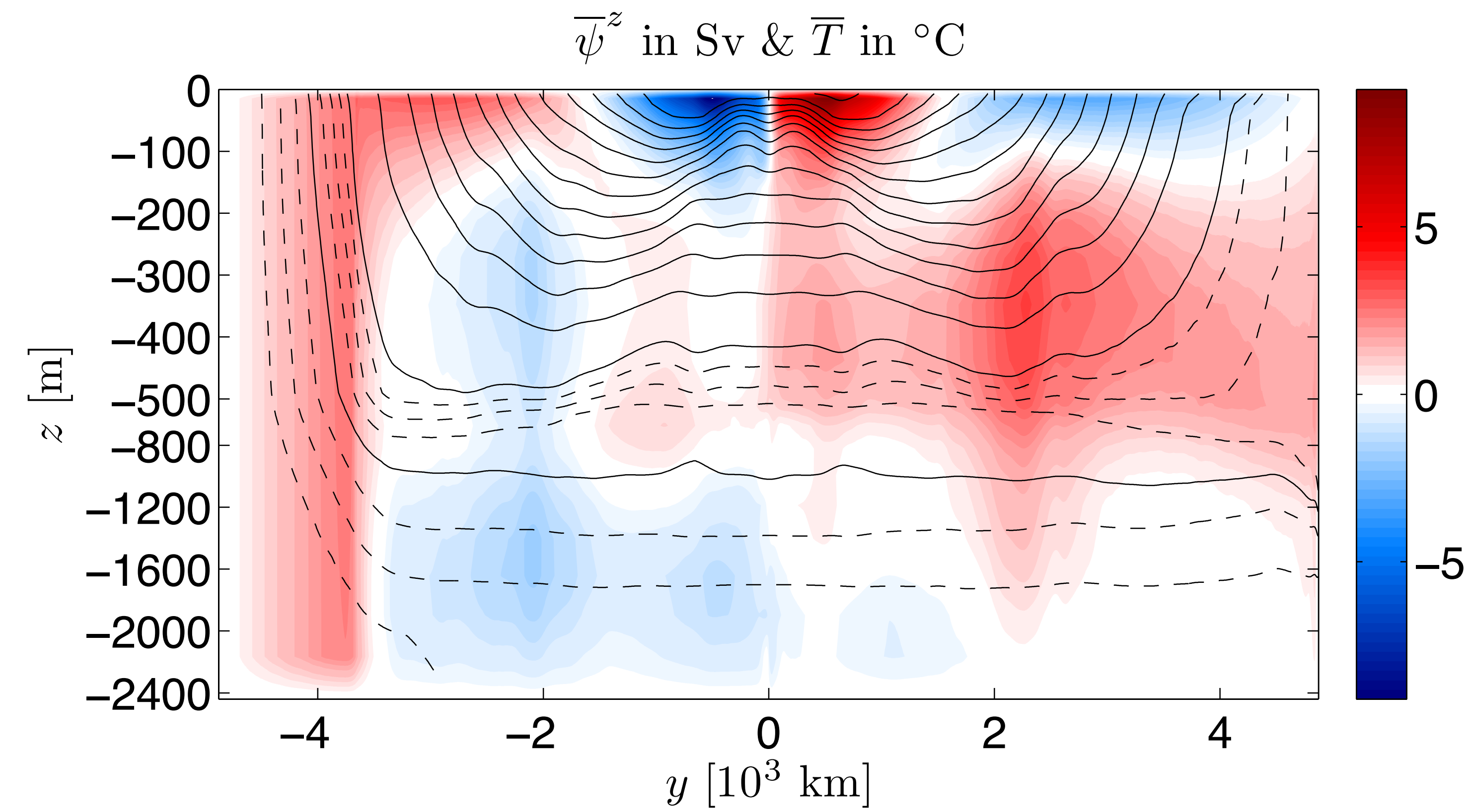
3D unsteady flow

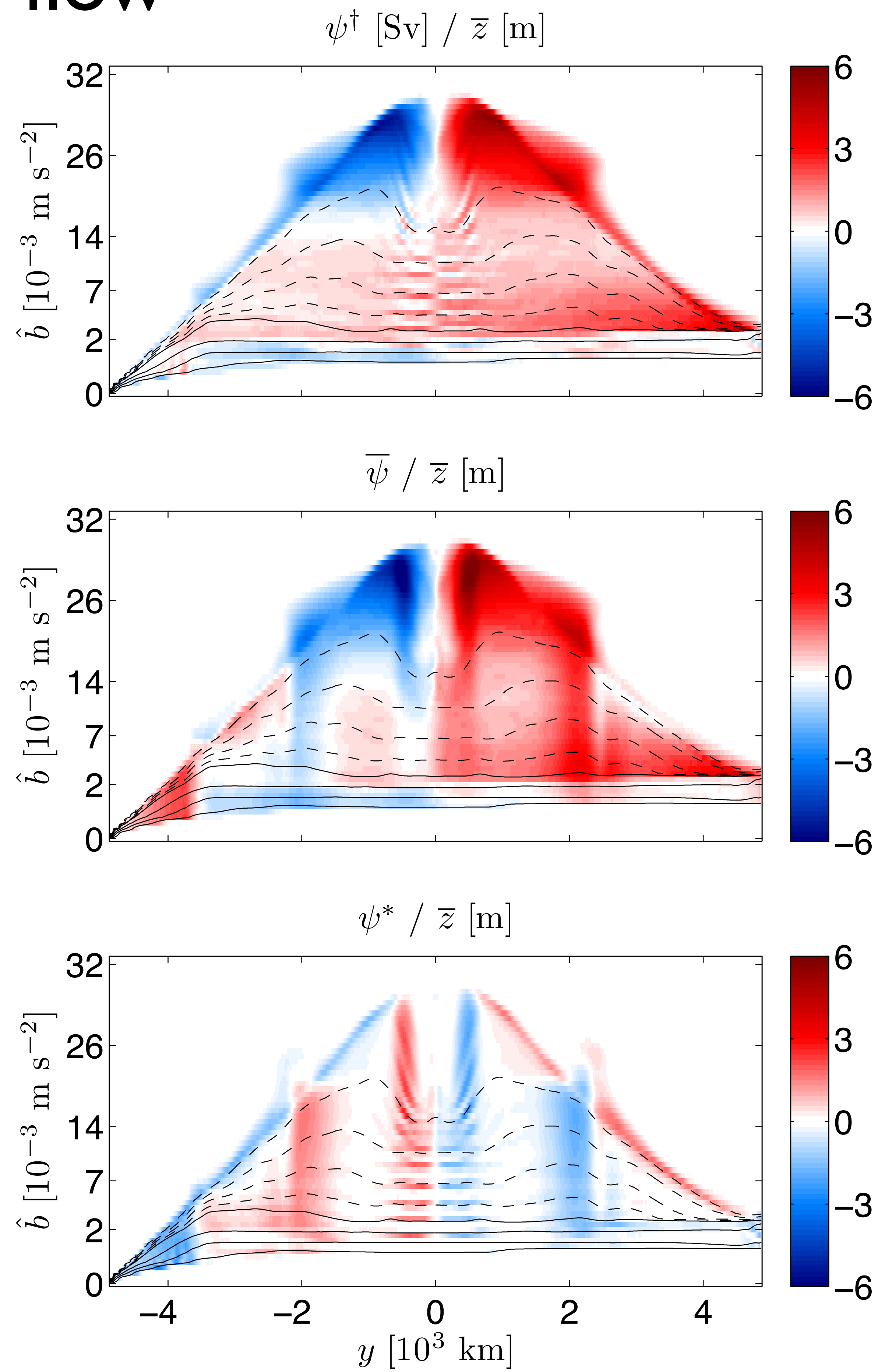
Now $\overline{(\cdot)} \rightarrow$ time mean

$$\psi^{\dagger}(\tilde{y},\tilde{b}) = -\int_{x_w}^{x_e}\int_{-H}^{0}\overline{v\,\mathcal{H}\left[\tilde{b}-b(x,y,z,t)\right]}\,\mathrm{d}z\,\mathrm{d}x$$

$$\bar{\psi}(\tilde{y},\tilde{b}) = -\int_{x_w}^{x_e}\int_{-H}^{0}\bar{v}\,\overline{\mathcal{H}\left[\tilde{b}-b(x,y,z,t)\right]}\,\mathrm{d}z\,\mathrm{d}x$$

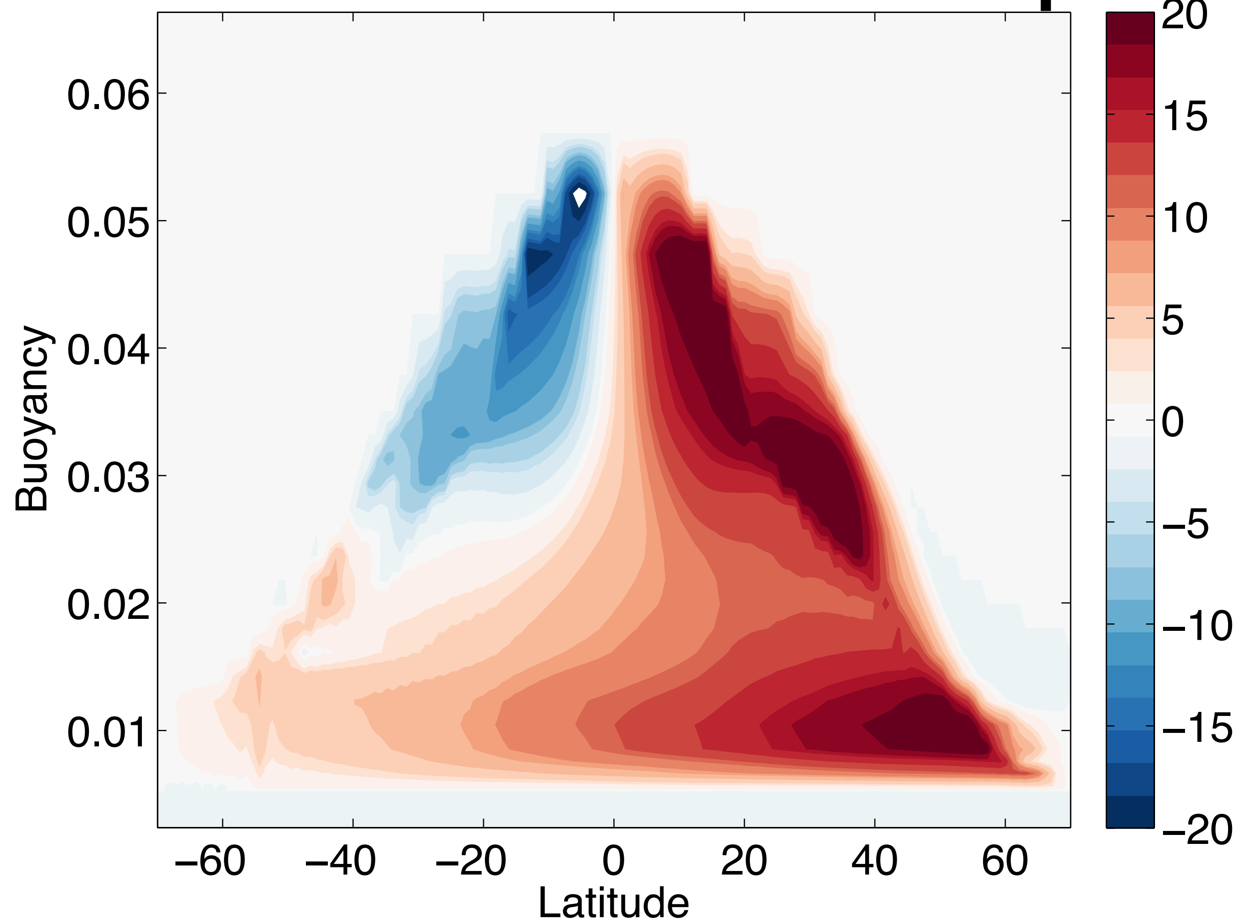
$$\bar{z}(\tilde{y},\tilde{b}) = -H + \int_{x_w}^{x_e}\frac{1}{x_e-x_w}\int_{-H}^{0}\overline{\mathcal{H}\left[\tilde{b}-b(x,y,z,t)\right]}\,\mathrm{d}z\,\mathrm{d}x$$



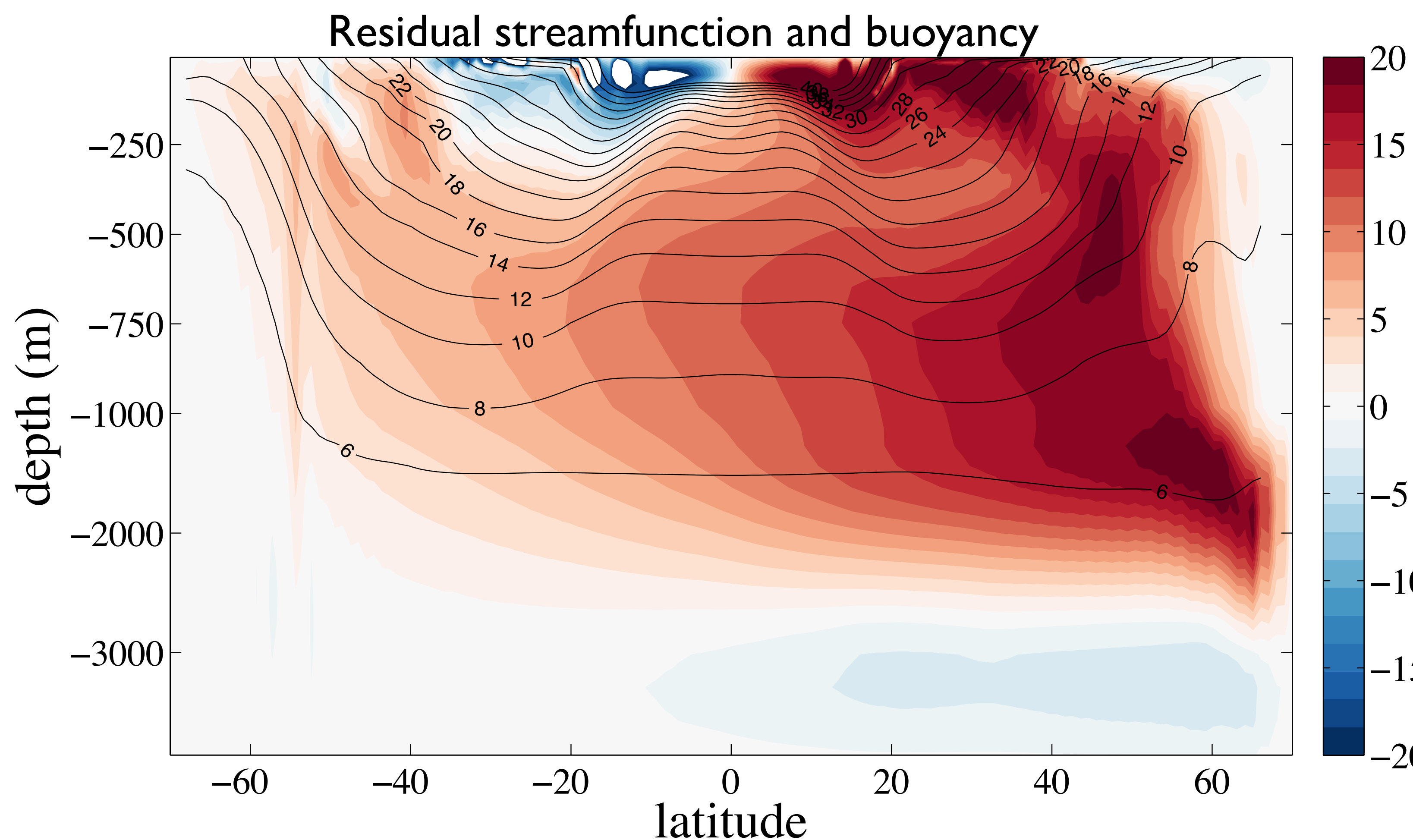


Visualization for human consumption

We have $\psi^{\dagger}(\tilde{y}, \tilde{b}) \quad \bar{z}(\tilde{y}, \tilde{b})$



We can express $\psi^{\dagger}(\tilde{y}, \bar{z})$

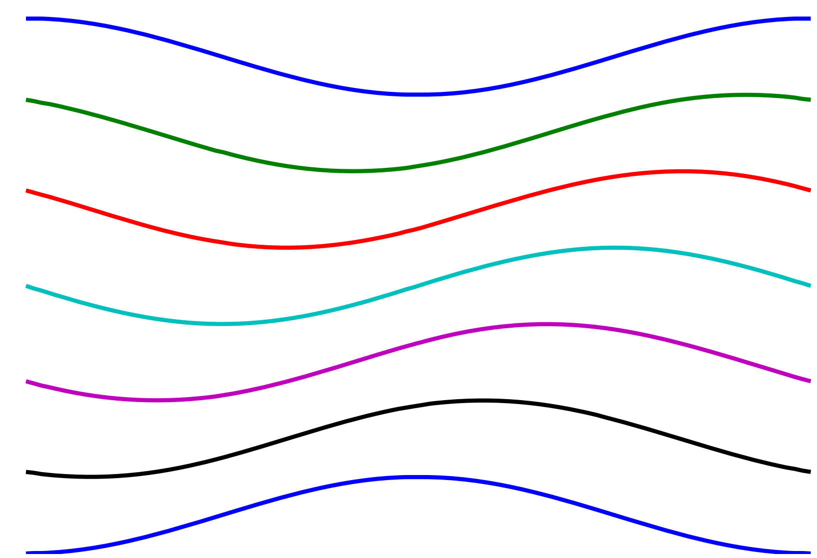


TEM beyond QG: Thickness Weighted Average

The stacked primitive equations:

average in buoyancy coordinates
over quasi-adiabatic eddies

Stacked:
$b_z \neq 0$



The TWA equations:

can be presented in any coordinate
system (buoyancy or spatial)

The “thickness” is: $$\sigma = \frac{1}{b_z}$$

The transport velocity a.k.a.
the residual velocity is: $$\hat{u} = \frac{\overline{\sigma u}}{\overline{\sigma}}$$

The (unweighted)
mean velocity is: $$\bar{u}$$

Young, W.R., 2012. An exact thickness-weighted average formulation of the Boussinesq equations. Journal of Physical Oceanography, 42(5), pp.692-707.

The TWA equations in buoyancy coordinates

EP vectors

(1) Only the residual velocity appears.

(2) All tracers - including momentum - are advected by the residual velocity.

(3) Eddy effects are confined to the momentum equations, and appear in EP vectors.

(4) EP vectors are quadratic in eddy amplitude.

(5) EP divergences are expressed in terms of the eddy-flux of PV. There is a fully 3D and nonlinear generalization of Taylor's identity connection EP to PV.

(6) This is not the most general formulation, but it is probably the most useful because buoyancy is the best stacked tracer.

$$\sigma \overset{\text{def}}{=} \zeta_{\tilde{b}} \qquad \sigma = \frac{1}{b_z} \quad \text{Thickness}$$

$$\hat{u}_{\tilde{t}} + \hat{\varpi}\hat{u}_{\tilde{b}} - \overline{\sigma}\hat{v}\Pi^{\sharp} + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{x}} = \hat{\mathcal{X}} - \nabla \cdot \mathbf{E}^{u}$$

$$\hat{v}_{\tilde{t}} + \hat{\varpi}\hat{v}_{\tilde{b}} - \overline{\sigma}\hat{u}\Pi^{\sharp} + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{y}} = \hat{\mathcal{Y}} - \nabla \cdot \mathbf{E}^{v}$$

$$\overline{\zeta} + \overline{m}_{\tilde{b}} = 0$$

$$\overline{\sigma}_{\tilde{t}} + (\overline{\sigma u})_{\tilde{x}} + (\overline{\sigma v})_{\tilde{y}} + (\overline{\varpi\sigma})_{\tilde{b}} = 0,$$

diabatic effects

$$\Pi^{\sharp} \overset{\text{def}}{=} \frac{f + \hat{v}_{\tilde{x}} - \hat{u}_{\tilde{y}}}{\overline{\sigma}} \qquad \text{Ertel's PV}$$

The residual velocity is: $\hat{u} = \dfrac{\overline{\sigma u}}{\overline{\sigma}}$

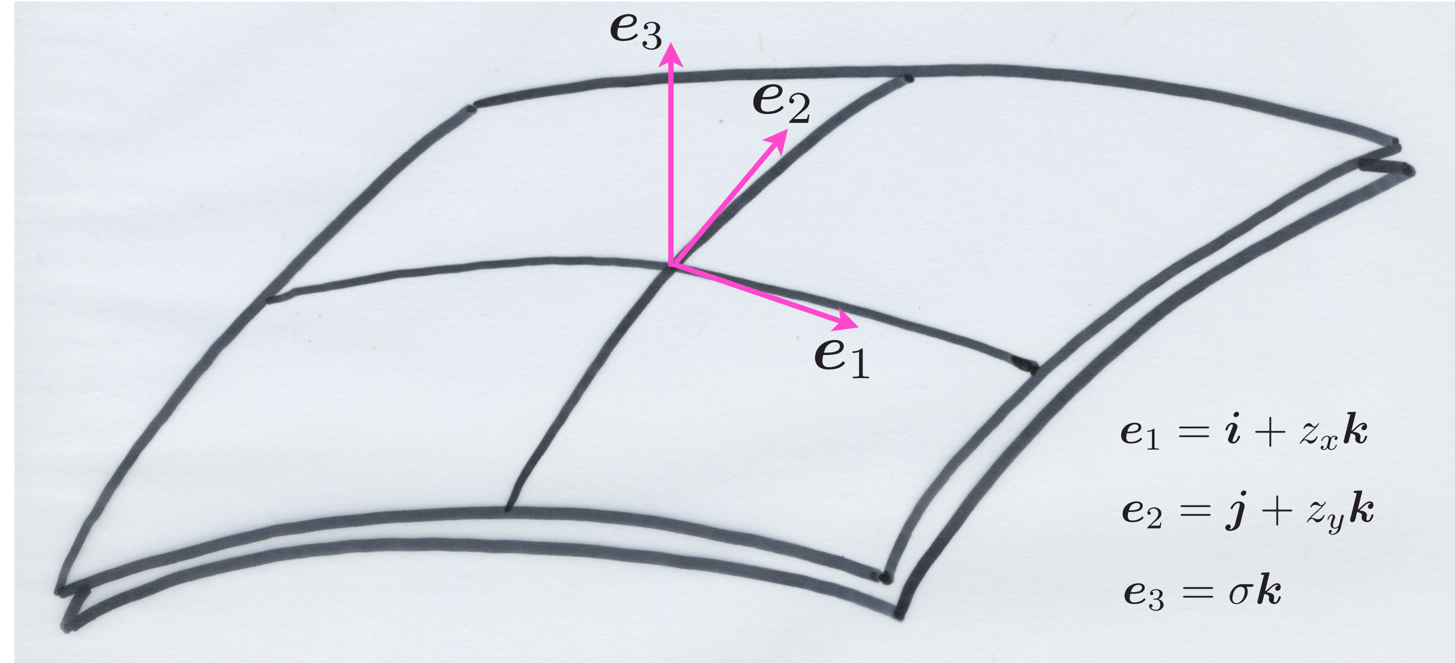
$$\mathbf{E}^{u} = \widehat{u''u''}\,\overline{\mathbf{e}}_1 + \widehat{u''v''}\,\overline{\mathbf{e}}_2 + \overline{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\,\overline{\mathbf{e}}_1 + \overline{\zeta' m'_{\tilde{x}}}\,\overline{\mathbf{e}}_3\right)$$

$$u = \hat{u} + u'' \qquad \zeta = \overline{\zeta} + \zeta'$$

Div, grad, curl

$$\boldsymbol{q} = q^1\boldsymbol{e}_1 + q^2\boldsymbol{e}_2 + q^3\boldsymbol{e}_3$$

$$\sigma\boldsymbol{\nabla}\cdot\boldsymbol{q} = \left(\sigma q^1\right)_{\tilde{x}} + \left(\sigma q^2\right)_{\tilde{y}} + \left(\sigma q^3\right)_{\tilde{b}}$$



$$\nabla f = f_{\tilde{x}}\mathbf{e}^1 + f_{\tilde{y}}\mathbf{e}^2 + f_{\tilde{b}}\mathbf{e}^3$$

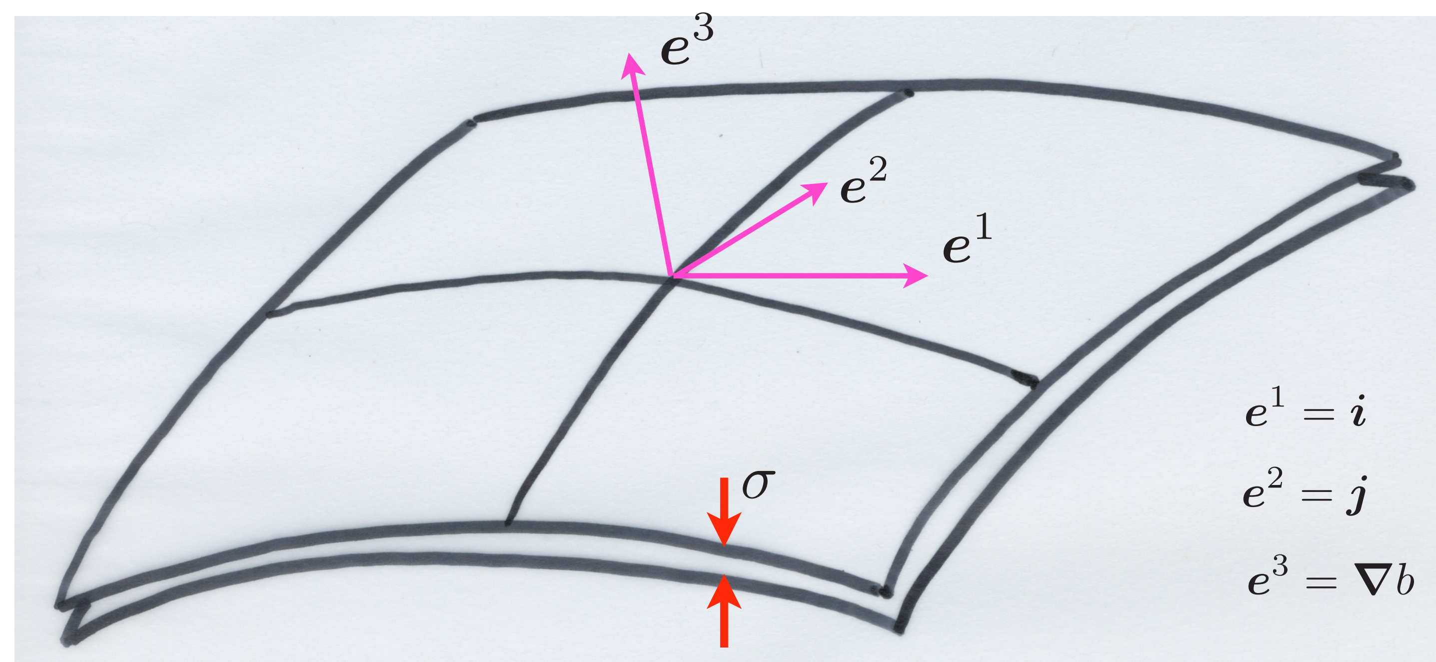
Dual basis vectors

Buoyancy coordinates are so close to cartesian coordinates that one is tempted to wing it. But buoyancy coordinates are not orthogonal...

$$\boldsymbol{e}_i\boldsymbol{e}^j = \delta_{ij}$$

$$\boldsymbol{q} = q_1\boldsymbol{e}^1 + q_2\boldsymbol{e}^2 + q_3\boldsymbol{e}^3$$

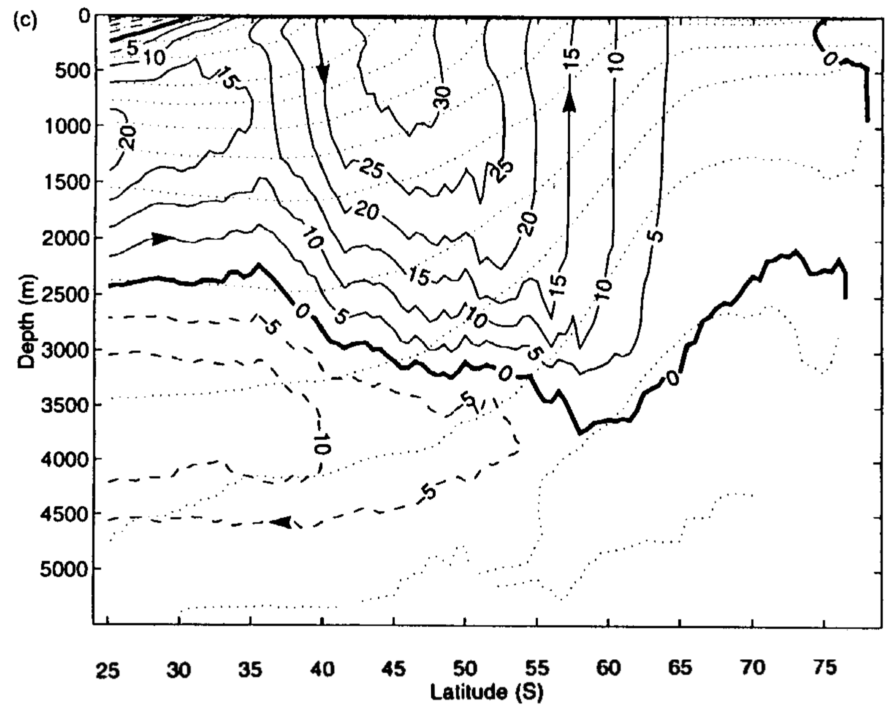
$$\begin{aligned}\sigma\boldsymbol{\nabla}\times\boldsymbol{q} =& (q_{3\tilde{y}} - q_{2\tilde{b}})\boldsymbol{e}_1\\ &+ (q_{1\tilde{b}} - q_{3\tilde{x}})\boldsymbol{e}_2\\ &+ (q_{2\tilde{x}} - q_{1\tilde{y}})\boldsymbol{e}_3\end{aligned}$$



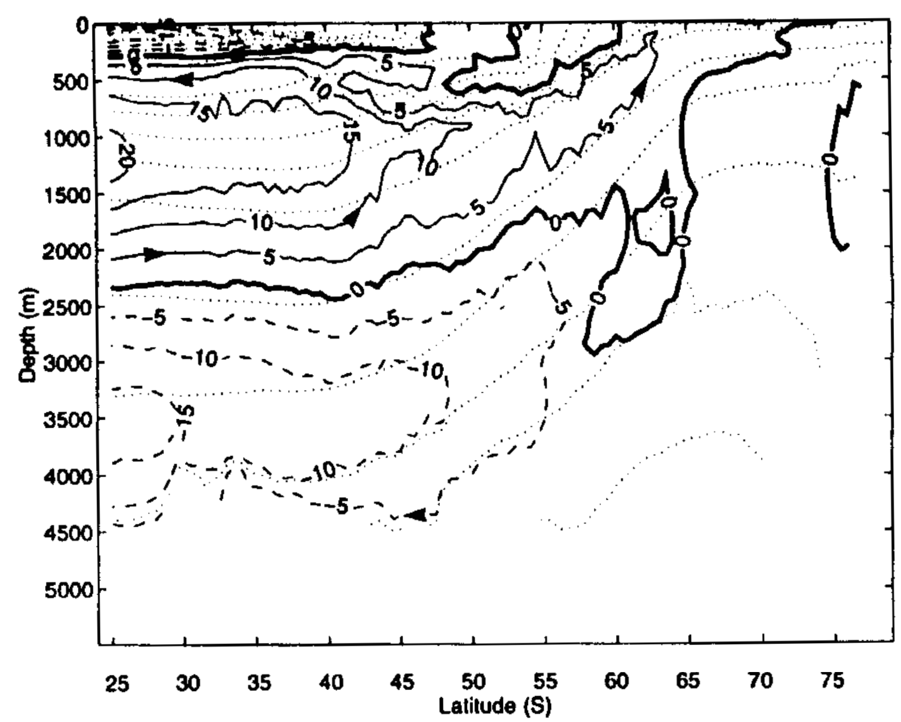
Overturning circulations in vertical and density coordinates

Southern-ocean

Eulerian zonally averaged meridional mass transport



Zonal average in density coordinates,remapped in height



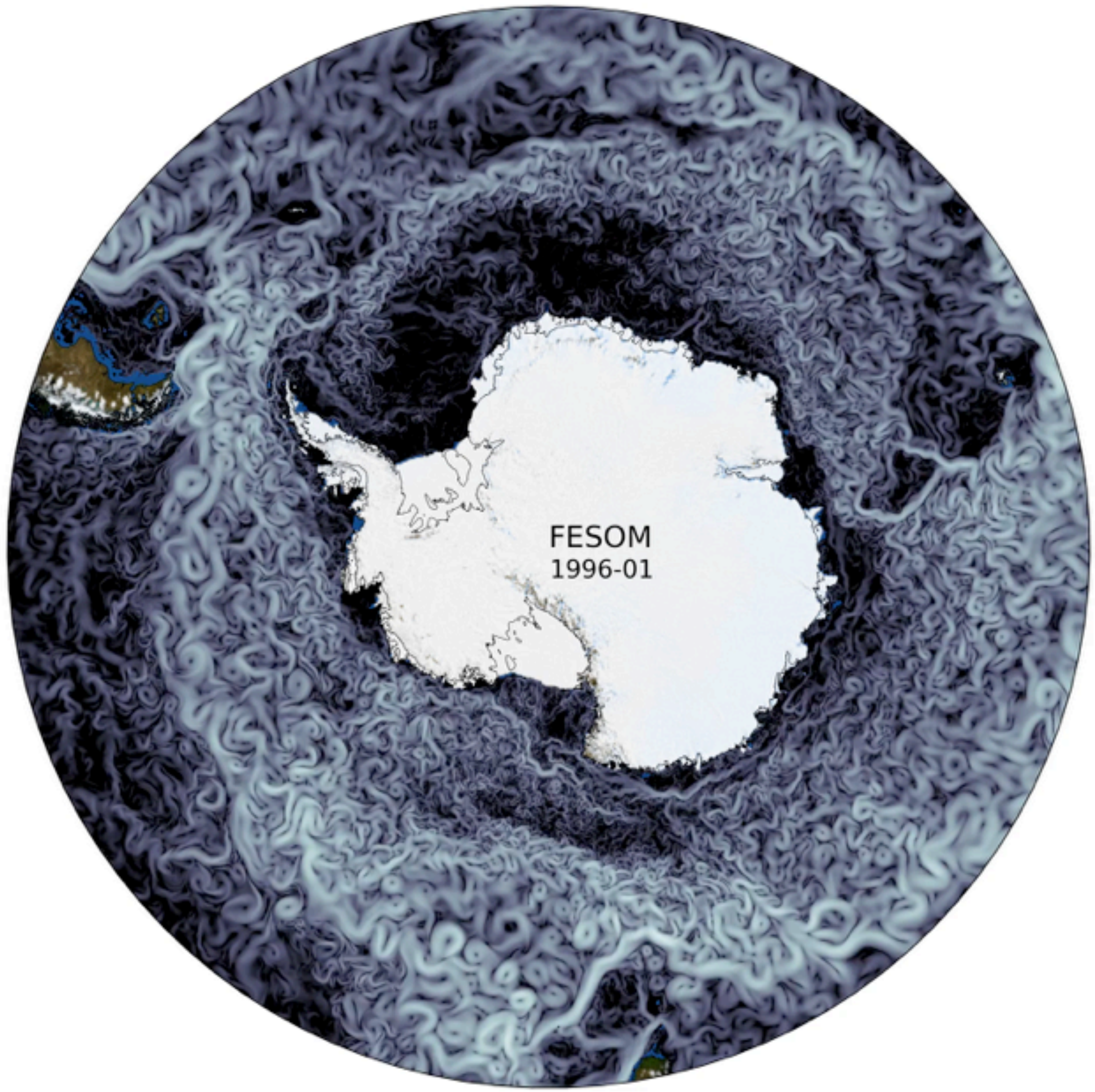
The thermally indirect cell (Deacon cell) disappear in density coordinates

Q. J. R. Meteorol. Soc. (1997), **123**, pp. 519–526

Similarities of the Deacon cell in the Southern Ocean and Ferrel cells in the atmosphere

By D. J. KAROLY[1]*, P. C. McINTOSH[2], P. BERRISFORD[3], T. J. McDOUGALL[2] and A. C. HIRST[2]

The ACC velocity (FESOM model)



A massive westward current with rich eddy-field

Application to TWA to idealized ACC

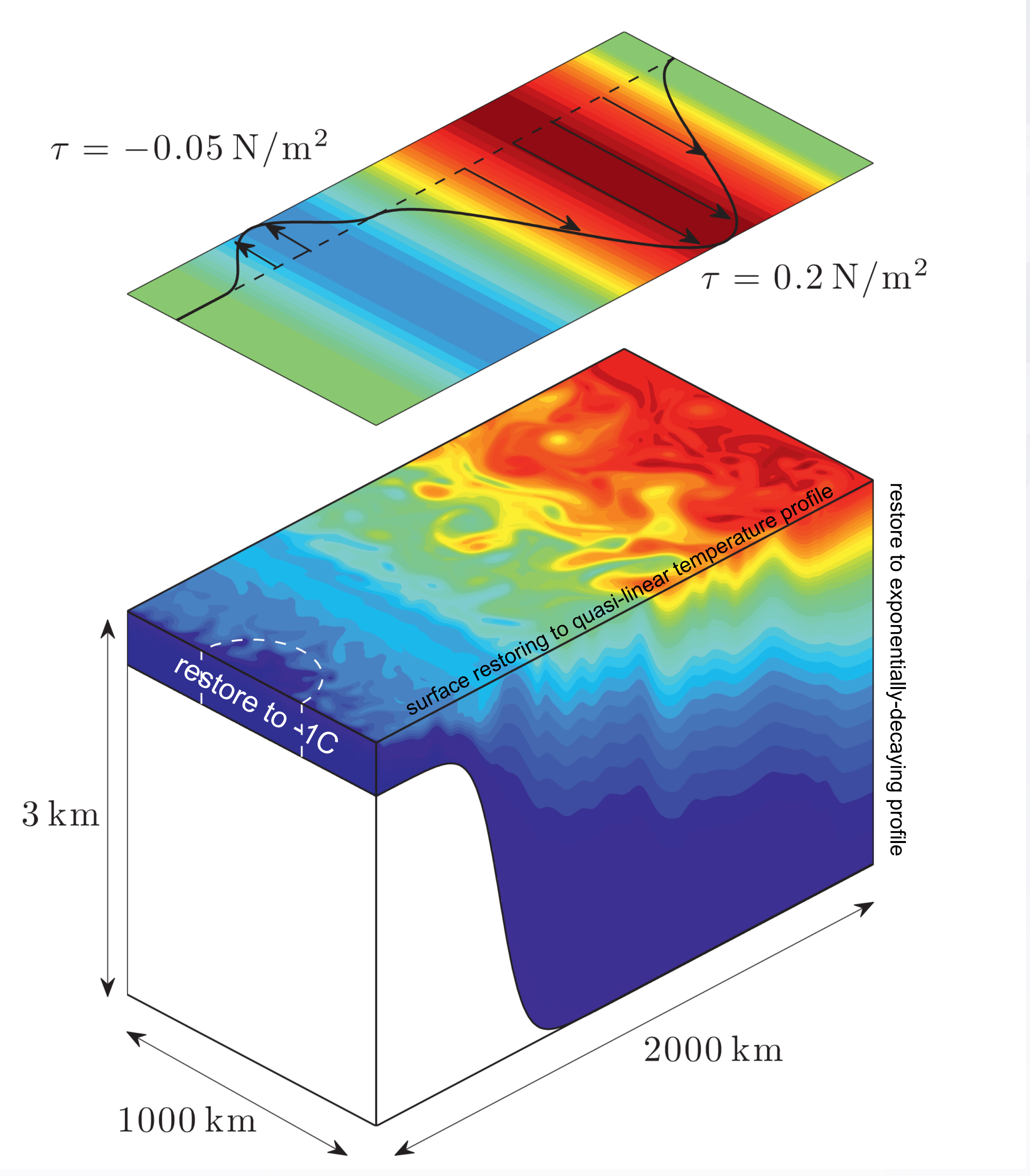
Idealized Southern Ocean

Spin up:
100 years at 20 km from rest
20 years at 10 km interpolated from 10 km
15 years at 5 km interpolated from 5 km

Simulation and Analysis:
20 years of simulation sampled every 3 day
ocean PDE solver uses 100 levels
TWA analysis uses 100 buoyancy levels

Forcing:
zonally-uniform wind stress as shown
linear restoring of surface temperature
linear restoring of interior temperature at boundaries.

Configuration:
1000 km x 2000 km x 2.5 km
includes continental shelf and shelf break.
zonally-periodic
linear EOS with uniform salinity
(surfaces of temperate == surfaces of buoyancy)



A Thickness-Weighted Average Perspective of Force Balance in an Idealized Circumpolar Current
Ringler et al.,JPO 2017 https://doi.org/10.1175/JPO-D-16-0096.1

Thickness-Weighted Averaged (TWA) equations:

$$\hat{u}_{\tilde{t}} + \hat{\varpi}\hat{u}_{\tilde{b}} - \overline{\sigma}\hat{v}\Pi^{\sharp} + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{x}} = \hat{\mathcal{X}} - \boxed{\nabla \cdot \mathbf{E}^u}$$

$$\hat{v}_{\tilde{t}} + \hat{\varpi}\hat{v}_{\tilde{b}} + \overline{\sigma}\hat{u}\Pi^{\sharp} + \left(\overline{m} + \frac{1}{2}\hat{u}^2 + \frac{1}{2}\hat{v}^2\right)_{\tilde{y}} = \hat{\mathcal{Y}} - \boxed{\nabla \cdot \mathbf{E}^v}$$

$$\overline{\zeta} + \overline{m}_{\tilde{b}} = 0$$

$$\overline{\sigma}_{\tilde{t}} + (\overline{\sigma u})_{\tilde{x}} + (\overline{\sigma v})_{\tilde{y}} + (\overline{\sigma \varpi})_{\tilde{b}} = 0$$

The advecting velocity is now the thickness-weighted velocity (aka residual mean velocity).

The TWA machinery leaves the structure of the equations intact and isolates the action of the eddies into a single term.

$$u_{\tilde{t}} + \varpi u_{\tilde{b}} - \sigma v\Pi + \left(m + \frac{1}{2}u^2 + \frac{1}{2}v^2\right)_{\tilde{x}} = \mathcal{X}$$

$$v_{\tilde{t}} + \varpi v_{\tilde{b}} + \sigma u\Pi + \left(m + \frac{1}{2}u^2 + \frac{1}{2}v^2\right)_{\tilde{y}} = \mathcal{Y}$$

$$\zeta + m_{\tilde{b}} = 0$$

$$\sigma_{\tilde{t}} + (\sigma u)_{\tilde{x}} + (\sigma v)_{\tilde{y}} + (\sigma \varpi)_{\tilde{b}} = 0$$

$(\mathcal{X}, \mathcal{Y})$: non-conservative forcing

m : Montgomery potential

$\Pi = \dfrac{f + u_x - v_y}{\sigma}$: potential vorticity

$\varpi = \dfrac{D\tilde{b}}{D\tilde{t}}$: diabatic velocity

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TWA at out-cropped buoyancy layers …. what to do?

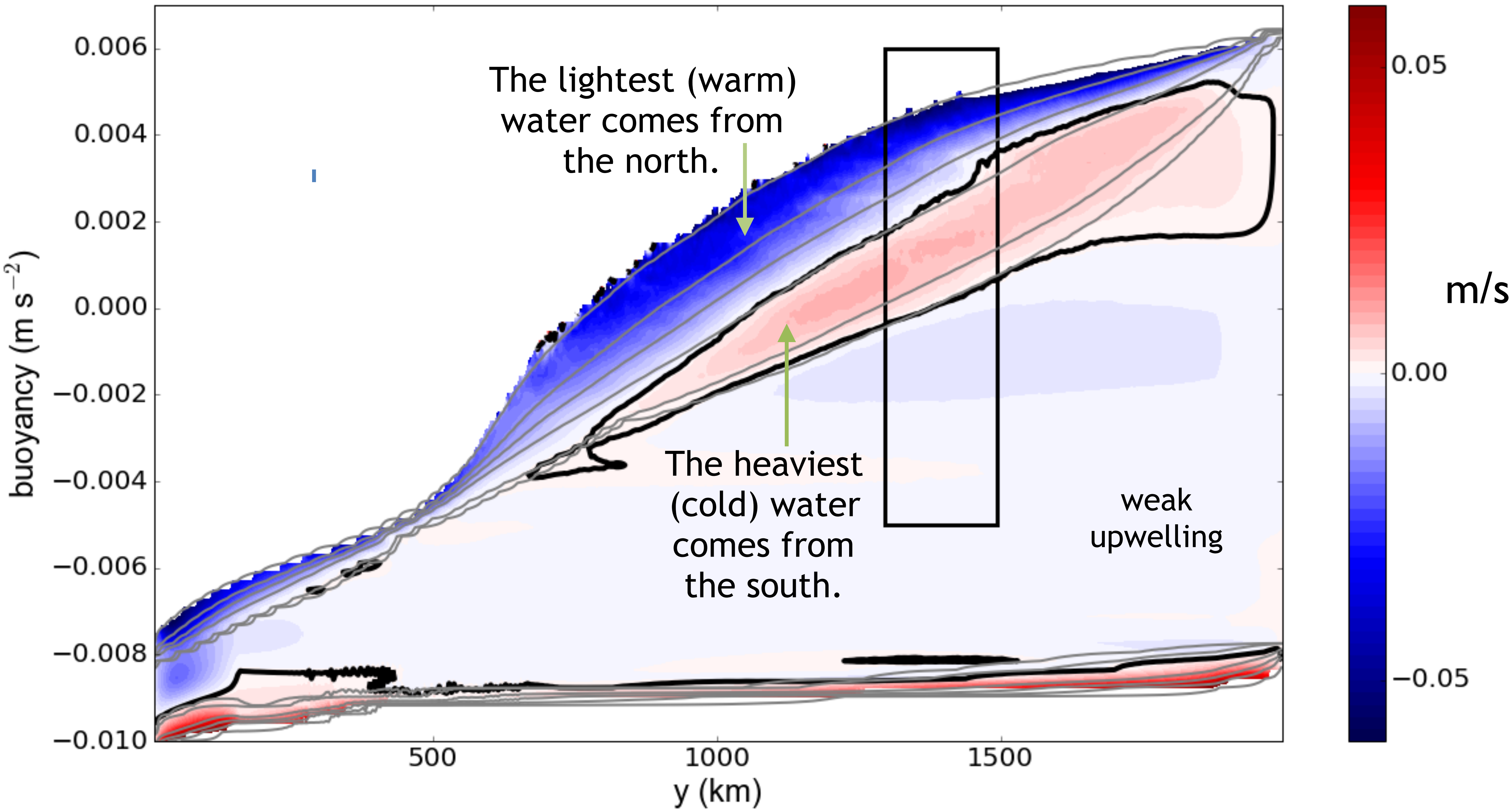
For each sample,

$\sigma = \zeta_{\tilde{b}}$, is given a value of zero for out-cropped layers.

$\sigma \mathbf{u}$, is given a value of zero for out-cropped layers.

$$\overline{\sigma} = \frac{1}{M} \sum_{m=1}^{M} \sigma \,, \quad \overline{\sigma \mathbf{u}} = \frac{1}{M} \sum_{m=1}^{M} \sigma \mathbf{u} \,, \quad \hat{\mathbf{u}} = \frac{\overline{\sigma \mathbf{u}}}{\overline{\sigma}}$$

TWA meridional velocity time and zonally averaged

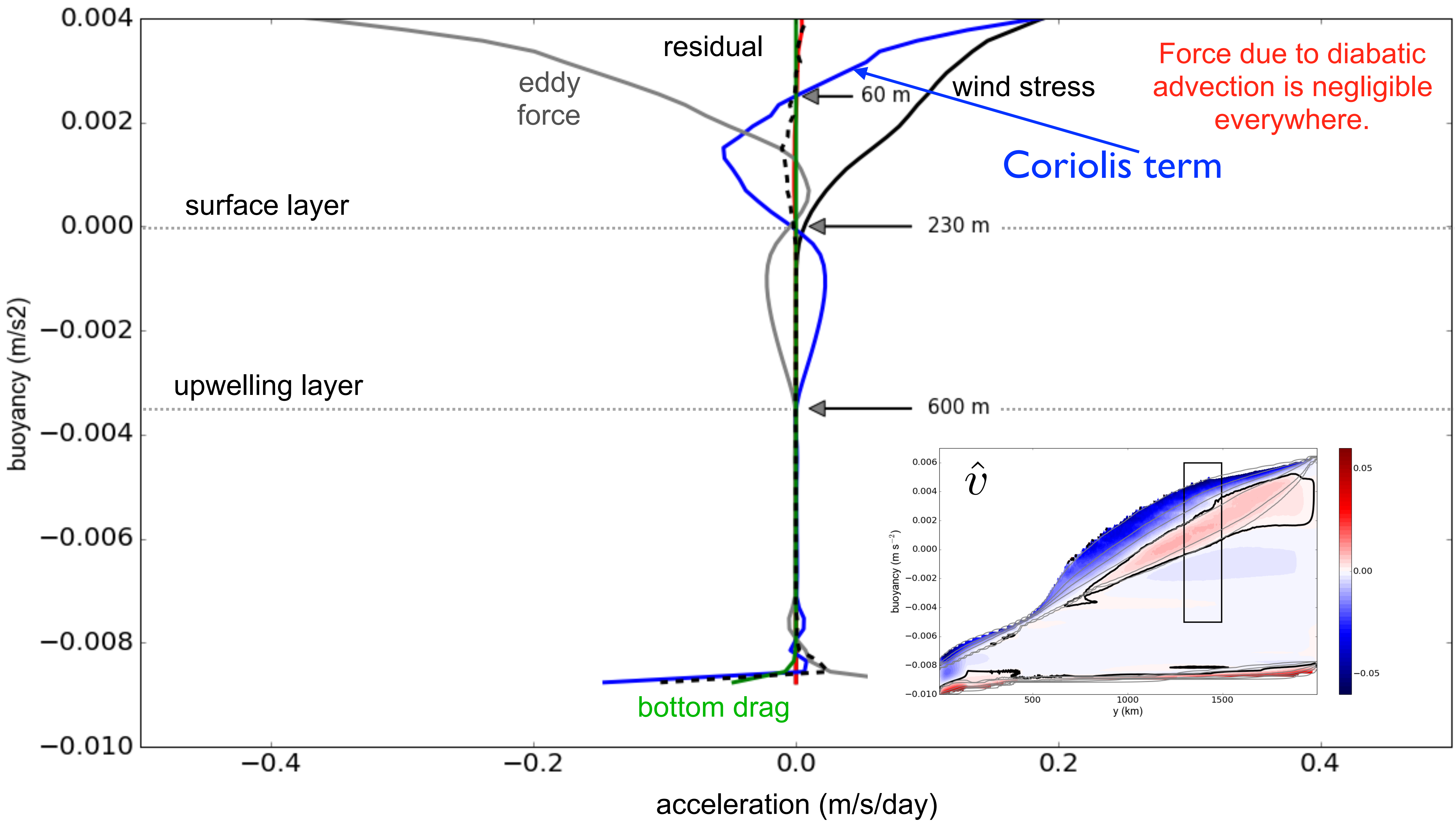


The TWA meridional velocity is in the surface diabatic layer and at the bottom

Almost no TWA meridional velocity in the interior

Momentum balance in the box

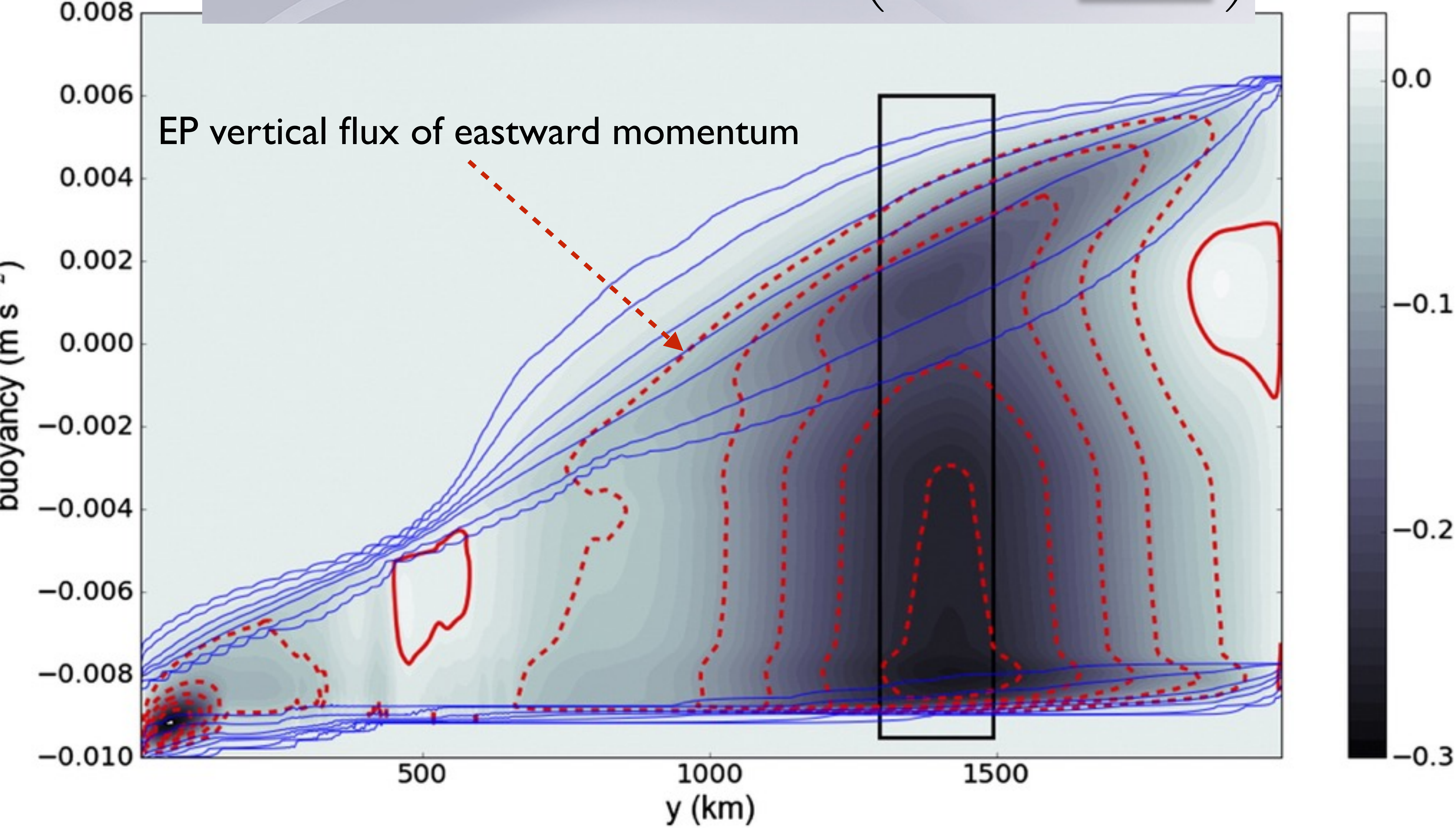
$$[\hat{u}_{\tilde{t}}] = -\left[\hat{\varpi}\hat{u}_{\tilde{b}}\right] + \left[\overline{\sigma}\hat{v}\Pi^{\sharp}\right] + \left[\hat{\mathcal{X}}\right] - \left[\nabla \cdot \mathbf{E}^{u}\right]$$



Nothing below 600m until the bottom boundary layer near 3000m.

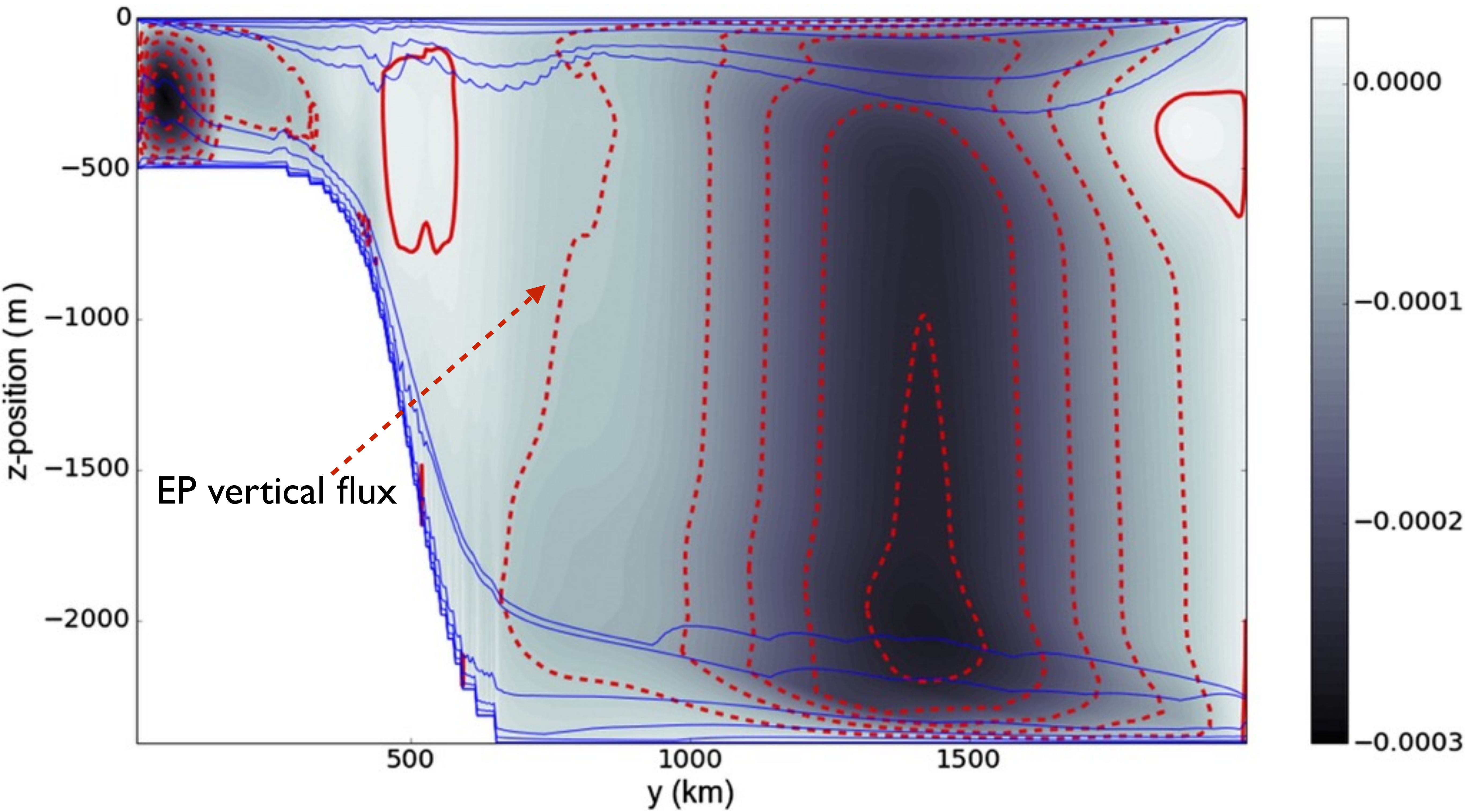
The EP fluxes: vertical flux of eastward momentum

$$\mathbf{E}^u = \widehat{u''u''}\,\overline{\mathbf{e}}_1 + \widehat{u''v''}\,\overline{\mathbf{e}}_2 + \bar{\sigma}^{-1}\left(\frac{1}{2}\overline{\zeta'^2}\,\overline{\mathbf{e}}_1 + \underline{\overline{\zeta' m'_{\tilde{x}}}\,\overline{\mathbf{e}}_3}\right)$$



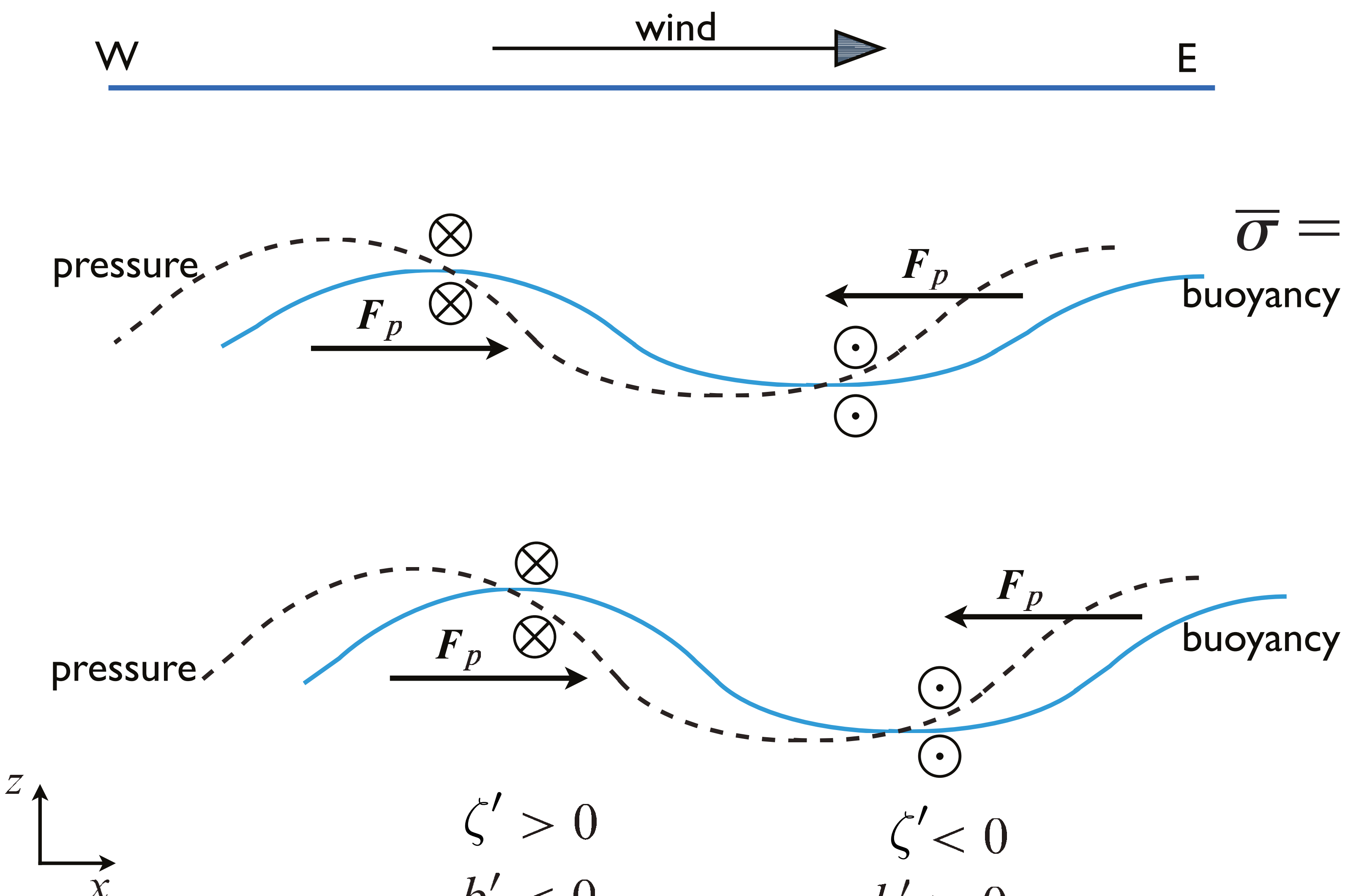
The vertical component of eastward momentum flux is dominated by $\overline{\zeta' m'_x}$

$\overline{\zeta' m'_x}$ in z-coordinates



$\overline{\zeta' m'_x}$ is vertically uniform except in the top and bottom boundary layer: no divergence and no residual velocity except in top and bottom layers

$\overline{\zeta' m'_x}$: form-stress



$\zeta' > 0$ | $\zeta' < 0$
$b' < 0$ | $b' > 0$
$v' > 0$ | $v' < 0$

$$\overline{\sigma m_{\tilde{x}}} = \overline{\sigma}\,\overline{m}_{\tilde{x}} + (\overline{\zeta' m'_{\tilde{x}}})_{\tilde{b}} + \left(\frac{1}{2}\overline{\zeta'^2}\right)_{\tilde{x}}$$

$$\overline{\sigma} = \overline{\zeta}_{\tilde{b}} = -\overline{m}_{\tilde{b}\tilde{b}}.$$

In QG-TEM

$$f_0\, v' = p'_x$$

$$v'q' = -\frac{\partial}{\partial y}(u'v') + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}v'b'\right) + \frac{1}{2}\frac{\partial}{\partial x}\left((v'^2 - u'^2) - \frac{b'^2}{N^2}\right).$$

On a constant buoyancy surface $b = \bar{b}(z) + b'(x,y,z,t) \Longrightarrow \zeta' \approx -\frac{b'}{\bar{b}_z} = -\frac{b'}{N^2}$

Form-stress transfers momentum vertically (or across buoyancies)

Using residual velocities in a prognostic model: cartesian coordinates

$$\hat{u}_t + \hat{u}\hat{u}_x + \hat{v}\hat{u}_y + w^\sharp \hat{u}_z - f\hat{v} + p^\sharp_x + \nabla \cdot \mathbf{E}^u = 0,$$

$$\hat{v}_t + \hat{u}\hat{v}_x + \hat{v}\hat{v}_y + w^\sharp \hat{v}_z + f\hat{u} + p^\sharp_y + \nabla \cdot \mathbf{E}^v = 0,$$

$$p^\sharp_z = b^\sharp,$$

$$\hat{u}_x + \hat{v}_y + w^\sharp_z = 0,$$

$$b^\sharp_t + \mathbf{u}^\sharp \cdot \nabla b^\sharp = \hat{\varpi}.$$

diabatic effects

(1) Only residual velocity appears.

(2) All tracers are advected by the residual velocity.

(3) Eddy effects are confined to the momentum equations, and appear in EP vectors.

The residual velocity is:

$$\boldsymbol{u}^\sharp = \hat{u}\boldsymbol{i} + \hat{v}\boldsymbol{j} + \underbrace{(\bar{z}_t + \hat{u}\bar{z}_x + \hat{v}\bar{z}_y)}_{=w^\sharp}\boldsymbol{k}$$

$$\hat{u} = \frac{\overline{\sigma u}}{\overline{\sigma}} \qquad \sigma = \frac{1}{b_z}$$

An eulerian observer at (x,y,z,t) is at the mean depth z of some buoyancy surface. This defines

$$b^\sharp(x, y, z, t)$$

Parametrization of EP fluxes

A model in terms of the TWA fields requires parametrizing the EP fluxes

$$\overline{\zeta' m'_x} = -\mu \bar{\sigma} \hat{u}_z$$
$$\overline{\zeta' m'_y} = -\mu \bar{\sigma} \hat{v}_z$$

Vertical viscosity of horizontal momentum (Rhines and Young, 1982)

This is equivalent to adding extra velocities to the Coriolis terms such that

$$f v^* \equiv \left(\frac{\overline{\zeta' m'_x}}{\bar{\sigma}} \right)_z \qquad f u^* \equiv - \left(\frac{\overline{\zeta' m'_y}}{\bar{\sigma}} \right)_z$$

If $\hat{u}$ $\hat{v}$ are in geostrophic balance, then $(u^*, v^*) = \left(\kappa_a \frac{\nabla \rho^\sharp}{\rho^\sharp_z} \right)_z$

With $\kappa_a = \mu f^{-2} \bar{\sigma}^{-1} = \mu f^{-2} \bar{N}^2$

The parametrization for the extra velocity is equivalent
to Gent-McWilliams scheme with diffusivity κ_a

Prognostic Residual Mean Flow in an Ocean General Circulation Model and its Relation to Prognostic Eulerian Mean Flow
Saenz, J.A. et al. JPO 2017. https://doi.org/10.1175/JPO-D-15-0024.1

Comparison to a Eulerian model

A model in terms of the **Eulerian** fields requires parametrizing the eddy-fluxes: start with the buoyancy fluxes (no momentum fluxes)

$$\frac{D\overline{u}^z}{Dt} - f\overline{v}^z + \frac{\partial p}{\partial x} = \overline{R}_x,$$

$$\frac{D\overline{v}^z}{Dt} + f\overline{u}^z + \frac{\partial p}{\partial y} = \overline{R}_y,$$

$$\frac{\partial p}{\partial z} = b,$$

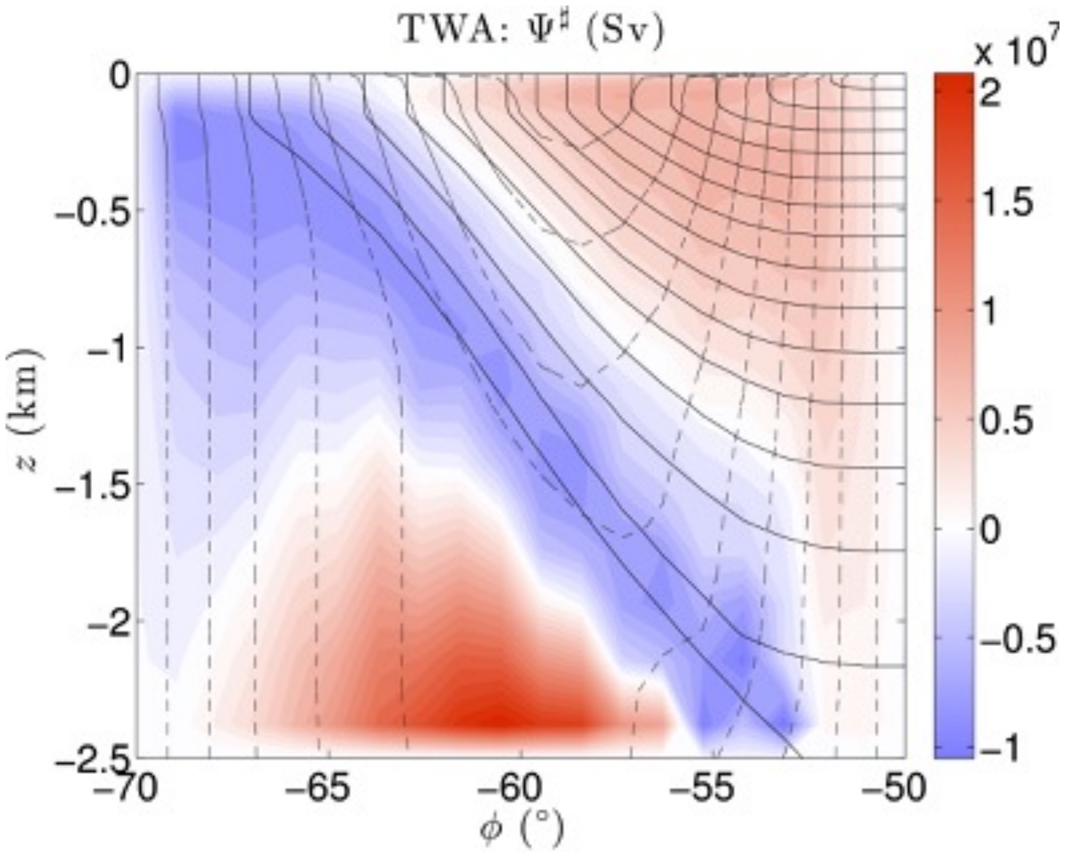
$$\frac{\partial(\overline{u}^z + u_*^z)}{\partial x} + \frac{\partial(\overline{v}^z + v_*^z)}{\partial y} + \frac{\partial(\overline{w}^z + w_*^z)}{\partial z} = 0,$$

$$\frac{Db}{Dt} + u_*^z\frac{\partial b}{\partial x} + v_*^z\frac{\partial b}{\partial y} + w_*^z\frac{\partial b}{\partial y} = 0,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{u}^z\frac{\partial}{\partial x} + \overline{v}^z\frac{\partial}{\partial y} + \overline{w}^z\frac{\partial}{\partial z},$$

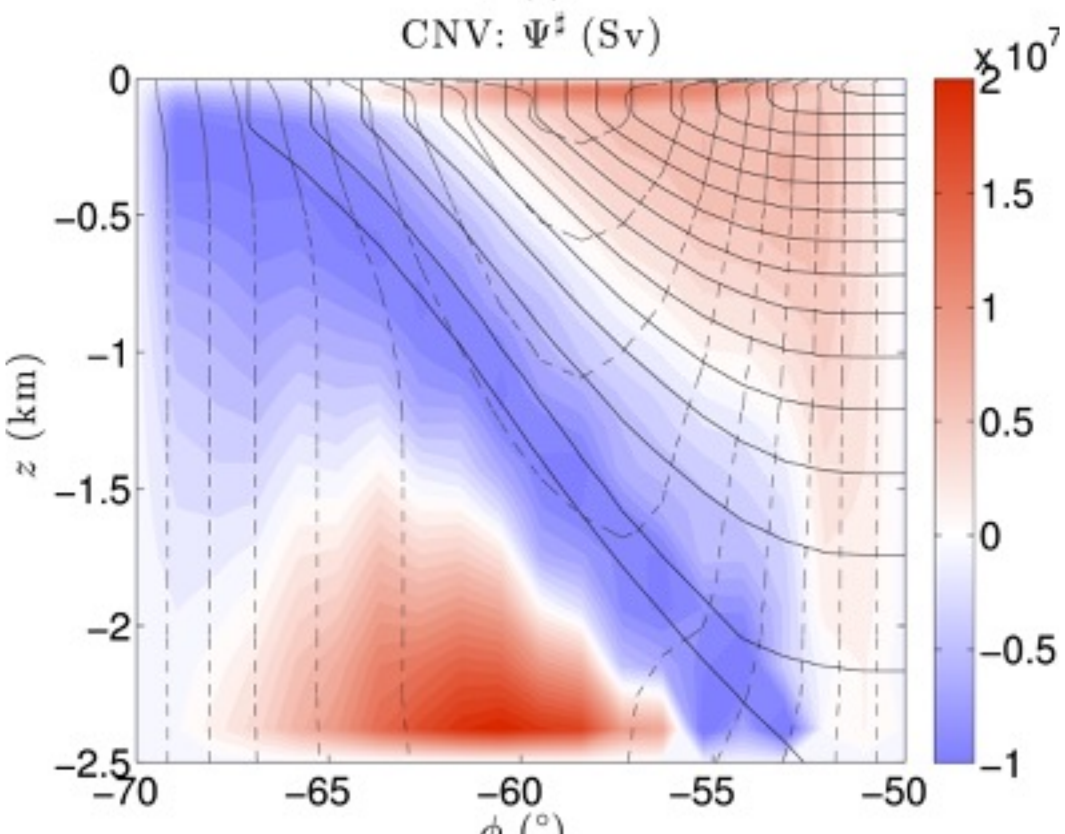
$$u_*^z = -\frac{\partial}{\partial z}\left(\kappa\frac{\overline{\rho}_x^z}{\overline{\rho}_z^z}\right), \quad \text{and} \quad v_*^z = -\frac{\partial}{\partial z}\left(\kappa\frac{\overline{\rho}_y^z}{\overline{\rho}_z^z}\right),$$

Implementation in a numerical model of the ACC



Residual overturning using TWA model EP fluxes parametrized as vertical viscosity

With $\kappa_a = \mu f^{-2}\bar{\sigma}^{-1} = \mu f^{-2}\bar{N}^2$



Residual overturning using conventional Eulerian mean, parametrized buoyancy fluxes, assuming $(\hat{u}, \hat{v}) = (\bar{u}, \bar{v}) + (u^*, v^*)$

Quantitative agreement, because eddy mom. flux is negligible.

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Saenz, J.A. et al. JPO 2017. https://doi.org/10.1175/JPO-D-15-0024.1

Summary

Residual mean formalism is very useful to capture the effect of eddy-fluxes on buoyancy transport (and possibly other tracers).

TWA places eddy-effect in EP flux divergence in the momentum equation using a single velocity. Not clear how to get the Eulerian flow (is it needed?)

Not widely implemented yet, but it can and has been done in the ACC setting.

Agrees with parametrization of eddy-fluxes, if confined to buoyancy fluxes.

Not clear how to parametrize all of the EP fluxes (momentum), which are important for jets formation and maintenance.