1
Lagrangian kinematics

1.1 Conservation of particle identity

The essence of Lagrangian fluid dynamics is fluid particle identity acting as an independent variable. The identifier or *label* may be the particle position at some time, but could for example be a triple of the thermodyamic properties of the particle at some time. Time after labeling is the other independent variable. The fluid particle may not actually have been released into the flow at the time of labeling, but merely labeled with position or with some other properties at that time. Nevertheless, "time of release" will be used interchangeably with "labeling time." The subsequent position of the particle is a dependent variable, even though it may coincide with the independently chosen position of an Eulerian observer at the subsequent time. The Eulerian observer also employs time, after some convenient initial instant, as the other independent variable. Of course, a particle path can be calculated in the Eulerian framework by integrating velocity on the path, with respect to time. Indeed, the suppression or implicitness of this detailed path information is the basis of the relative simplicity of the Eulerian formulation. On the other hand, fluid velocity is readily calculated from the particle position in the Lagrangian framework by the local operation of particle differentiation with respect to time after labeling.

Conservation of particle identity is not an immediately compelling consideration in the Eulerian framework, but is fundamental in the Lagrangian. Bretherton (1970) correctly remarks that, since fluid particles having the same mass, momentum and energy can be interchanged without affecting the dynamics of the fluid, the particle identities are of no dynamical consequence. Yet kinematic information is the basis for the conceptualization of flow. Quantification of the kinematic principle of conservation of particle identity yields a striking identity which resembles but is entirely distinct from conservation laws for mechanical and thermodynamic properties. A first integral of the identity provides an exact formula for a generalized Stokes drift in laminar flow, and in each realization of a turbulent flow. The suitability of, for example, thermodynamic variables as particle identifiers does not require that they be conserved; it is their instantaneous values at the labeling time which are conserved for an individual particle.

The relationship between the Lagrangian and Eulerian formulations must be established with great pedantry, in order to establish the soundness of both. Consider, therefore, the fluid particle having the identifier or label a_i, $(i=1,2,3)$, such as its three-dimensional Cartesian coordinates, at some time s. At some later time t a Lagrangian observer, that is, an observer who moves with the particle, and who adopts a notation similar to that of Kraichnan (1965), records the position of the particle as $X_i(a_j, s|t)$. An Eulerian observer located at the position x_i at time t detects the particle if and only if

$$x_i = X_i(a_j, s|t). \tag{1.1}$$

See Figure 1.1.

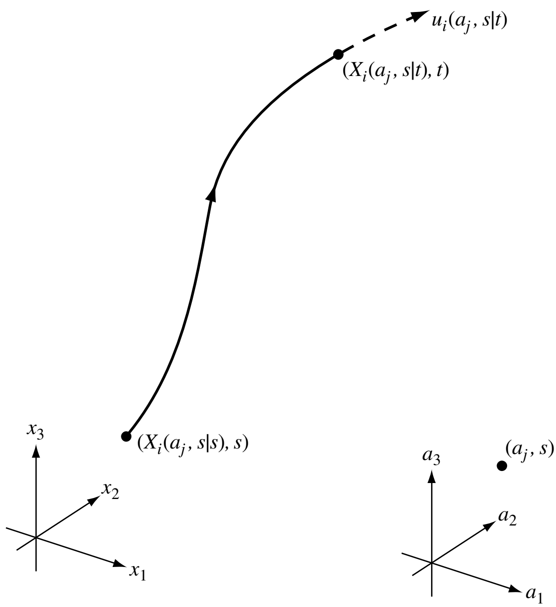

Figure 1.1 A fluid particle is given the label a_j at time s. Its position and velocity at time t are, respectively, $X_i(a_j, s|t)$ and $u_i(a_j, s|t)$. The label a_j is not necessarily the labeling position $X_i(a_j, s|s)$.

The Lagrangian velocity $u_i(a_j, s|t)$ is the particle velocity:

$$u_i(a_j, s|t) \equiv \frac{\partial}{\partial t} X_i(a_j, s|t). \tag{1.2}$$

Note that the partial derivative with respect to t is taken at fixed values for a_j and s, that is, the derivative is the Lagrangian partial in time. In the interest of notational simplicity, the same operator symbol $(\partial/\partial t)$ will be used subsequently for the Eulerian partial derivative in time, and the interpretation of the symbol will be made clear in the accompanying text. Subscripts will be used to distinguish thermodynamic partial derivatives of state variables, in the rare instances where such derivatives occur.

The labeling theorem Let q be any quantity associated with a fluid particle, such as density ρ, temperature T, or a velocity component u_i. The value of q at time t is denoted $q(a_j, s|t)$. *Assume that the label a_j is the particle position at time s.* Then, for any increment Δs in the labeling time s (see Figure 1.2),

$$q(X_i(a_j, s|s+\Delta s), s+\Delta s|t) = q(a_j, s|t), \tag{1.3}$$

since the labels are on the same path and they refer to the same particle. Expanding (1.3) and applying the definition (1.2) for the Lagrangian velocity yields (Kraichnan, 1965)

$$\frac{\partial}{\partial s} q(a_j, s|t) + u_k(a_j, s|s) \frac{\partial}{\partial a_k} q(a_j, s|t) = 0. \tag{1.4}$$

Note that there is an implied summation over the repeated index k in (1.4). The equation expresses that q is conserved along the characteristic direction

$$\frac{\partial a_k}{\partial s} = u_k(a_j, s|s)$$

in the (a_j, s) labeling space-time. This is the law of conservation of particle identity, or *labeling theorem.* ■

For example, choosing the quantity q to be any component u_i of the particle velocity,

$$\frac{\partial}{\partial s} u_i(a_j, s|t) + u_k(a_j, s|s) \frac{\partial}{\partial a_k} u_i(a_j, s|t) = 0, \tag{1.5}$$

and hence

$$u_i(a_j, t|t) = u_i(a_j, s|t) - \int_s^t u_k(a_j, r|r) \frac{\partial}{\partial a_k} u_i(a_j, r|t) dr. \tag{1.6}$$

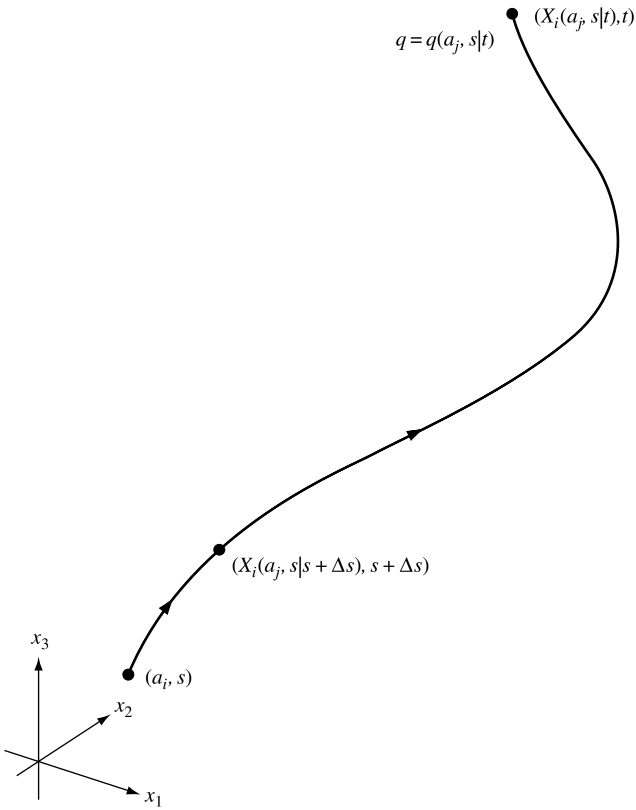

Figure 1.2 If a fluid particle is labeled by its position a_i at time s, then it could equally well be labeled by its position $a_i + u_i[a_j, s]\Delta s$ at time $s + \Delta s$. In particular, the value q for any state variable is the same for these two choices of labels.

When the label a_i is the particle position at the labeling time, as is the case here, it is convenient to introduce a special notation for the Lagrangian velocity at the labeling time:

$$u_i[a_j, r] \equiv u_i(a_j, r|r), \tag{1.7}$$

which is obviously the velocity recorded by an Eulerian observer at (a_j, r); this assertion will be carefully confirmed later. Introducing the Eulerian notation (1.7) into (1.6) yields

$$u_i(a_j, s|t) - u_i[a_j, t] = \int_s^t u_k[a_j, r] \frac{\partial}{\partial a_k} u_i(a_j, r|t) dr. \tag{1.8}$$

The relation (1.8) is an explicit expression for a *generalized* Stokes drift at $X_i(a_j, s|t)$ since, in general,

$$X_i(a_j, s|t) \neq a_i, \tag{1.9}$$

and thus the drift is the difference of Lagrangian and Eulerian velocities at different points on the one-particle path.

If the Eulerian velocity is solenoidal:

$$\frac{\partial}{\partial x_k} u_k[x_j, t] = 0, \tag{1.10}$$

then the drift is the spatial gradient of a mixed Eulerian–Lagrangian "prediffusivity:"

$$u_i(a_j, s|t) - u_i[a_j, t] = \frac{\partial}{\partial a_k} K_{ik}(a_j, s|t), \tag{1.11}$$

where

$$K_{ik}(a_j, s|t) = \int_s^t u_k[a_j, r] u_i(a_j, r|t) dr. \tag{1.12}$$

Notes

(i) The above formulae hold for a laminar flow, and for individual realizations of a turbulent flow; in particular the "prediffusivity" K_{ik} has not been averaged over an ensemble.
(ii) The product in the integrand involves total velocities, rather than departures from ensemble means.
(iii) The prediffusivity is asymmetric: $K_{ik} \neq K_{ki}$.
(iv) Equation (1.12) is hardly surprising: if the velocities in the integrand are known, then so is the drift (1.11). Nevertheless, it is instructive to assess the data needed to evaluate K_{ij}: a current meter (to use oceanographic terminology) must be deployed at a_i for $s < r < t$, and floats must be released at a_i at each time r in that interval: see Figure 1.3.

Exercise 1.1 Consider labeling by the particle position at the labeling time. Show that for any particle property q,

$$q(a_i, s|t) = q\big[X_i(a_j, s|t), t\big]. \tag{1.13}$$

Hint: let $q\big[X_i(a_j, s|t), t\big] \equiv q\big(X_i(a_j, s|t), t|t\big) = Q(a_i, s|t)$, say. Verify that $Q(a_i, s|t)$, like $q(a_i, s|t)$, satisfies the labeling theorem (1.4), and note that $Q(a_j, t|t) = q(a_i, t|t)$. This exercise establishes that the Lagrangian value of q at time t is the Eulerian value at the particle position at that time. Thus $q[x_i, t]$ is aptly named the Eulerian value. □

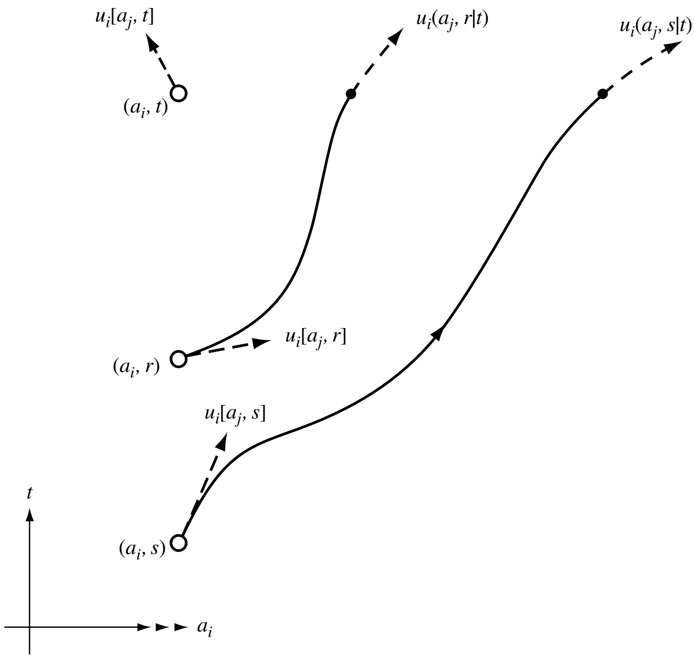

Figure 1.3 Evaluation of the generalized drift (1.11) requires that a current meter be deployed at position a_i for $s \le r \le t$, and that labeled fluid particles be released at a_i throughout the same time interval.

Exercise 1.2 (Lin, 1963) The notation of the labeling theorem, like that the path function $X_i(a_j, s|t)$, can be reversed for further illumination. Let a_i be the label, at time s, of a particle observed at position x_j at time t; that is, $a_i = A_i(x_j, t|s)$. Show that the "total" or "material" derivative of the labeling function A_i vanishes identically:

$$\frac{\partial}{\partial t} A_i(x_j, t|s) + u_k[x_j, t] \frac{\partial}{\partial x_k} A_i(x_j, t|s) = 0. \tag{1.14}$$

Note that, unlike Kraichnan's equation (1.4), Lin's equation (1.14) holds not only for labeling by position at time s, but for arbitrary labeling at that time.

□

Exercise 1.3 Extend the labeling theorem to labels other than the particle position at the labeling time, according to the following principle: for a fluid particle at position x_i at time t, the value of any particle property q is independent of the time s at which the particle is assigned the arbitrary label a_j. Verify that the original theorem (1.4) does obtain when the label is in fact the particle position at the time of release. Alternatively, express any label as a function of the release position and invoke the original labeling theorem. Reconcile these extensions. Finally, given Lagrangian kinematics labeled by a_j at time s, relabel by b_j at time r. □

Exercise 1.4 Consider a Lagrangian flow formulation having arbitrary labels a_j, that is, labels other than the particle position $X_j(a_k, s|t)$ at the release time $t = s$. Express the Eulerian velocity in terms of the Lagrangian kinematics. Establish the aptness of the construction of Eulerian fields from Lagrangian fields having arbitrary labels. □

Exercise 1.5 Assume that a particle path of the form $X_i = X_i(a_j|t)$ is known to be a solution of the Lagrangian equations of fluid dynamics, for some label a_i. Is $X_i = X_i(a_j|t-s)$ also a solution, for some time s? Show that the labeling theorem may be used to extend the known solution to a family of solutions of the form $X_i = X_i(a_j, s|t)$. □

1.2 Streaklines, streamlines and steady flow

Fluid flow tends to be time dependent, and is most naturally made visible with streaklines. These are neither particle paths nor streamlines, except for steady flow in which all three are identical.

Exercise 1.6 A streakline is the locus, at one time t, of fluid particles released at the position x_i at previous times r in some interval $s \le r \le t$. Express streaklines with Lagrangian notation. A streamline is a path everywhere tangential to the local fluid velocity, at one time t. Express streamlines with Lagrangian notation. Illustrate planar particle paths, streakline and streamlines with a single perspective sketch in the (x_1, x_2, t) space-time. □

Flow is defined to be "steady" if Lagrangian values are invariant under time translation:

$$q(a_i, s|t) = q(a_i, s-T|t-T), \tag{1.15}$$

for some time shift T. The left-hand side of (1.15) can depend on s and t only in the combination $t-s$. We may then define

$$q(a_i|t-s) \equiv q(a_i, s|t). \tag{1.16}$$

The "streamline" $X_i(a_j|t-s)$ is the sole particle path through $X_i(a_j, s|s)$:

$$X_i(a_j|t-s) = X_i(a_j, s|t). \tag{1.17}$$

Exercise 1.7 Assuming that particles are labeled by their positions a_j at time s, show that on a streamline in steady flow,

$$u_i(a_j|t-s) = \left(\frac{\partial}{\partial a_k} X_i(a_j|t-s) \right) u_k(a_j|0). \tag{1.18}$$

That is, the velocity on the streamline is the "strained initial value". Hint: use the labeling theorem. Is (1.18) a linear relationship? □

In general, the matrix of "Lagrangian strains"

$$J_{ij}(a_k, s|t) \equiv \frac{\partial}{\partial a_j} X_i(a_k, s|t) \tag{1.19}$$

is the Jacobi matrix for the transformation $a_j \to X_i$. The Lagrangian formulation is useful only so long as the determinant of this transformation, or Jacobi determinant, does not vanish.

Recall that for labeling by release position, the Eulerian velocity is

$$u_i[x_j, t] \equiv u_i(x_j, t|t). \tag{1.20}$$

If the flow is steady, then

$$u_i(x_j, t|t) = u_i(x_j|0), \tag{1.21}$$

and the Eulerian velocity is independent of time:

$$u_i[x_j, t] = u_i[x_j]; \tag{1.22}$$

thus it suffices to find the Eulerian velocity at time $t = s$. The Eulerian and Lagrangian velocities coincide at that time:

$$u_i[x_j] \equiv u_i(x_j|0). \tag{1.23}$$

Exercise 1.8 Show that in steady flow, particle paths are also streaklines and streamlines. □

Now consider an ideally conserved quantity such as entropy η. That is,

$$\frac{\partial \eta}{\partial t} = 0. \tag{1.24}$$

If the flow is steady: $\eta(a_j, s|t) = \eta(a_j|t-s)$, then by the labeling theorem

$$u_k[a_j]\frac{\partial}{\partial a_k}\eta(a_j|t-s) = 0. \tag{1.25}$$

This startling conclusion may be reconciled to the Eulerian expression of steady convection:

$$\begin{aligned} u_k[a_j]\frac{\partial}{\partial a_k}\eta(a_j|t-s) &= u_k[a_j]\frac{\partial}{\partial x_m}\eta[X_i]\frac{\partial}{\partial a_k}X_m(a_j|t-s) \\ &= -\frac{\partial}{\partial s}X_m(a_j|t-s)\frac{\partial}{\partial x_m}\eta[X_i] \\ &= \frac{\partial}{\partial t}X_m(a_j|t-s)\frac{\partial}{\partial x_m}\eta[X_i] \\ &= u_m[X_i]\frac{\partial}{\partial x_m}\eta[X_i] = 0. \end{aligned} \tag{1.26}$$

Note:

(i) Relations (1.14) and (1.23) have been applied to η, and the labeling theorem has been applied to the steady particle path $X_m(a_j|t-s)$.
(ii) There is seemingly more information in (1.25) than in the rightmost equality of (1.26), since the former refers to the Lagrangian gradient of η at times other than the labeling time. However, since η is conserved and since steady flow is assumed, $\eta = \eta(a_j|t-s) = \eta(a_j|0) = \eta[a_j]$.

1.3 Local kinematics

The definition (1.2) for the Lagrangian velocity $u_i(a_j, s|t)$, the definition (1.7) for Eulerian velocity $u_i[x_k, t]$, and the identity (1.13) lead to the well-known relation

$$\frac{\partial}{\partial t}X_i(a_j, s|t) = u_i[X_k(a_j, s|t), t]. \tag{1.27}$$

Assuming that $X_i(a_j, s|s) = a_i$, that is, the particle is labeled by its position at time s, assuming smoothness of the Eulerian velocity field, and expanding in a Taylor series about the local reference point a_i^* for small $t-s$ yields

$$\begin{aligned} \frac{\partial}{\partial t}X_i(a_j, s|t) =& u_i[a_j^*, s] + \frac{\partial}{\partial x_k}u_i[x_j, s]\bigg|_{x_j=a_j^*}\Big(X_k(a_j, s|t) - a_k^*\Big) \\ &+ O(|a_j - a_j^*|^2) + O(t-s). \end{aligned} \tag{1.28}$$

The Eulerian rate of strain tensor at $[a_i^*, s]$ may be decomposed into symmetric and skewsymmetric tensors:

$$\left(\frac{\partial u_i}{\partial x_k}\right)^* = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)^* + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)^*. \tag{1.29}$$

The skew tensor may be expressed in terms of a vector product:

$$\left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i}\right)^* = -\epsilon_{ikl}\omega_l^*, \tag{1.30}$$

where the alternating tensor $\epsilon_{ikl} = 1$ for $i = 1, k = 2, l = 3$, etc. (Jeffreys, 1931), and ω_l^* is the value at $[a_i^*, s]$ of the Eulerian vorticity ω_l:

$$\omega_l = \epsilon_{lmn}\frac{\partial u_n}{\partial x_m}. \tag{1.31}$$

Transforming to a new spatial variable ξ_i, according to

$$X_i = a_i^* + (t-s)\left(u_i^* - \frac{1}{2}\epsilon_{ikl}\omega_l^*\xi_k\right) + \xi_i, \tag{1.32}$$

the local relation 1.28 becomes

$$\frac{\partial \xi_i}{\partial t} = e_{ik}^*\xi_k + O(|a_j - a_j^*|^2) + O(t-s) \tag{1.33}$$

where e_{ik}^* is the local value of the symmetric rate of strain tensor

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right). \tag{1.34}$$

Note that $\xi_i = a_i - a_i^*$ at $t = s$. Again, all the Eulerian fields are evaluated at the reference point a_i^* and at time s. The transformation (1.32) consists of an infinitesimal translation with the local velocity u_i^*, plus an infinitesimal rotation with the local angular velocity $\omega_l^*/2$. A further transformation to the principal axes of the symmetric tensor e_{ik}^*, and (1.33) is diagonalized:

$$\frac{\partial \xi_i{}'}{\partial t} = \lambda_i \xi_i' + \dots, \tag{1.35}$$

where ξ_i' is the component of displacement in the i^{th} principal direction, and λ_i is the i^{th} principal moment. The trace of a matrix is an invariant, thus

$$\lambda_1 + \lambda_2 + \lambda_3 = e_{kk} = \frac{\partial u_k}{\partial x_k}. \tag{1.36}$$

It follows that for a solenoidal Eulerian velocity field or "incompressible flow," the sum of the eigenvalues λ_i vanishes. If all vanish, then the flow is stagnant in the transformed coordinates, and the particle motions in the original coordinates are circles superimposed on a uniform translation. Assume to the contrary that at least one eigenvalue is positive and one is negative. The corresponding principal axes are, respectively, a *dilatation* axis and a *compression* axis passing through the reference point a_i^*. According to (1.33), other particles released near a_i^* approach the axis of greatest dilatation, asymptotically for large $t-s$. Note that, consistent with the preceding approximations, the Taylor series expansion of the Jacobi matrix for small elapsed time $t-s$ is

$$J_{ij}(a_k, s|t) = \delta_{ij} + (t-s)\Big(e_{ij}[a_k, s] - \frac{1}{2}\epsilon_{ijl}\omega_l[a_k, s]\Big) + O(t-s)^2, \qquad (1.37)$$

to first order. Thus the time evolution to this order of accuracy is determined by the Eulerian symmetric rate of strain tensor and the Eulerian vorticity, both evaluated at the labeling position and time. Higher order terms are determined by the pressure field, that is, by the dynamics of the fluid.

The preceding local analysis of particle kinematics is traditional, but is used to great effect in the study of turbulent diffusion by Batchelor (1959), and is the basis for much topological investigation (e.g., Ottino, 1989). The analysis is essentially Eulerian, as the characteristics of the particle motion are all determined by the spatial gradients of the Eulerian velocity at the original labeling position.