Lagrangian Methods in Fluid Mechanics

Rick Salmon
Scripps Institution of Oceanography
University of California, San Diego

Tel Aviv University
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Eulerian formulation (conventional)

velocity	$\mathbf{v}(x,y,z,t)$	$\dfrac{\partial \mathbf{v}}{\partial t} = \cdots$
mass density	$\rho(x,y,z,t)$	$\dfrac{\partial \rho}{\partial t} = \cdots$
pressure	$p(x,y,z,t)$	$p = F(\rho)$

Lagrangian formulation

fluid particle locations $\mathbf{x}(a,b,c,\tau)$ $\dfrac{\partial^2 \mathbf{x}}{\partial \tau^2} = \cdots$

(a,b,c) are particle labels; $\dfrac{\partial}{\partial \tau} \equiv \dfrac{D}{Dt}$

Solve for $\mathbf{x}(a,b,c,\tau)$; then determine $\mathbf{v}(a,b,c,\tau) = \dfrac{\partial}{\partial \tau}\mathbf{x}(a,b,c,\tau)$

However, it may be difficult to determine $\mathbf{v}(x,y,z,t)$.
If $\mathbf{v}(x,y,z,t)$ cannot be determined, you still have a parametric solution (e.g. Gerstner's wave) that could not have been found by seeking $\mathbf{v}(x,y,z,t)$.

Some key points....

1. The two descriptions are physically equivalent and mathematically closed.
2. The Eulerian description is dominant today, but this was not always the case.
3. Each has its advantages, but those of the Lagrangian approach are insufficiently appreciated.
4. A chief criticism of the L. approach is that it asks for too much.
5. However, it is possible to find L. solutions that would be impossible to find in the E. approach.

A major advantage of the Lagrangian approach:
the existence of variational principles

Analogy with Electrodynamics:

"Eulerian variables" **E** and **B** obey Maxwell's equations,
but no variational principle exists.

To obtain a variational principle, one must introduce
the potentials ϕ and **A**: $\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$

A and ϕ are analogous to (a,b,c)

A variational principle exists in terms of **A** and ϕ

Gauge symmetry of electrodynamics $\Leftrightarrow$
particle relabeling symmetry of fluid mechanics

This turns out to be more than an analogy!

Best known form of the variational principle for a fluid

$$A = \int L\, d\tau = \int (T - V)\, d\tau = \int d\tau \iiint da\, db\, dc \left[\frac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(x,y,z)}{\partial(a,b,c)} \right) \right]$$

Labels are assigned such that $da\, db\, dc = d(\text{mass})$

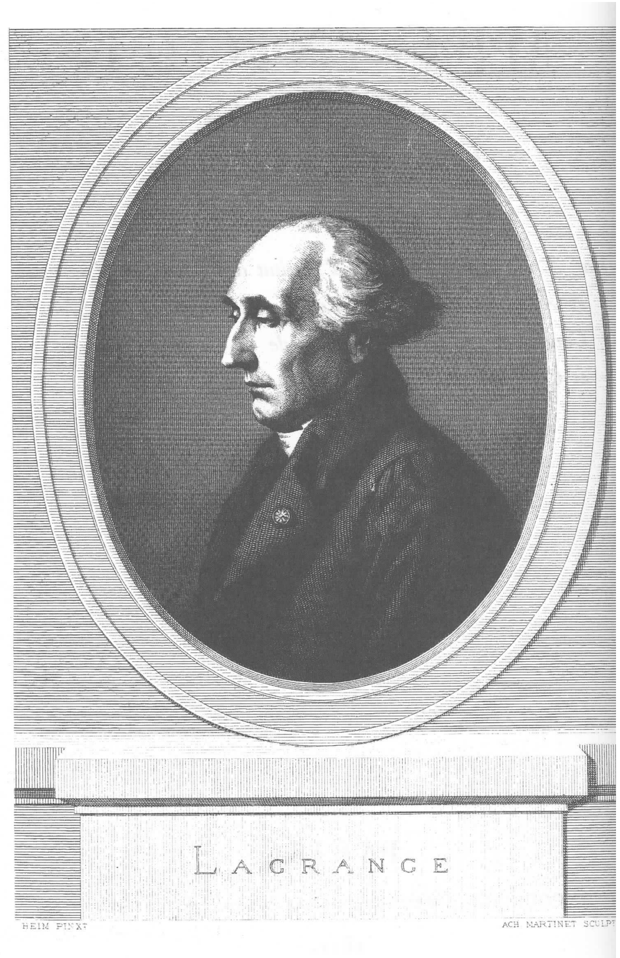
then $\rho = \dfrac{\partial(a,b,c)}{\partial(x,y,z)}$ and mass conservation is implicit

$\delta x, \delta y, \delta z :$ momentum equations for the fluid

Incompressible limit:

$$A = \int d\tau \iiint da\, db\, dc \left[\frac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} + \lambda \left(\frac{\partial(x,y,z)}{\partial(a,b,c)} - \frac{1}{\rho_0} \right) \right]$$

$\lambda =$ Lagrange multiplier (pressure)



No figures will be found in this work.

The methods I present require neither constructions nor geometrical or mechanical arguments, but solely algebraic operations subject to a regular and uniform procedure.

Mecanique Analytique

Nouvelle Edition (1815)

Tome Second

$$S\left[\left(\frac{d^2x}{dt^2}+X\right)\delta x+\left(\frac{d^2y}{dt^2}+Y\right)\delta y+\left(\frac{d^2z}{dt^2}+Z\right)\delta z\right]Dm+S\lambda\,\delta L=0$$

$$L = Dx\ Dy\ Dz = \text{constant}$$

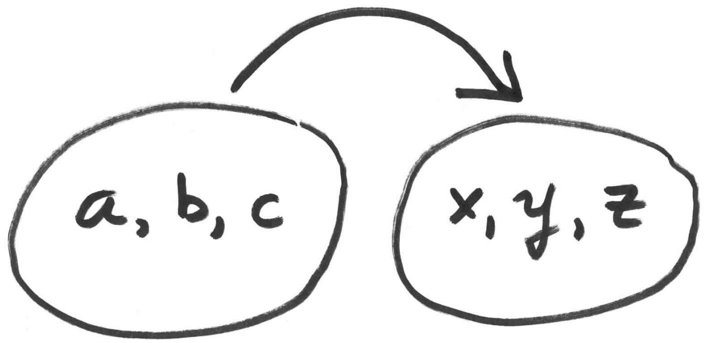
Emmy Noether (1882-1935)

"What will our soldiers think when they return to the university and find that they are required to learn at the feet of a woman?"

David Hilbert: "We are a university, not a bath house."

Particle-relabeling symmetry property

$$\delta \int d\tau \iiint da\,db\,dc \left[\frac{1}{2}\frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(x,y,z)}{\partial(a,b,c)}\right)\right] = 0 \quad \text{for arbitrary} \quad \delta x(a,b,c), \delta y(a,b,c), \delta z(a,b,c)$$



$$\delta \int d\tau \iiint dx\,dy\,dz \; \frac{\partial(a,b,c)}{\partial(x,y,z)} \left[\frac{1}{2}\frac{\partial \mathbf{x}}{\partial \tau}\cdot\frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(x,y,z)}{\partial(a,b,c)}\right)\right] = 0$$

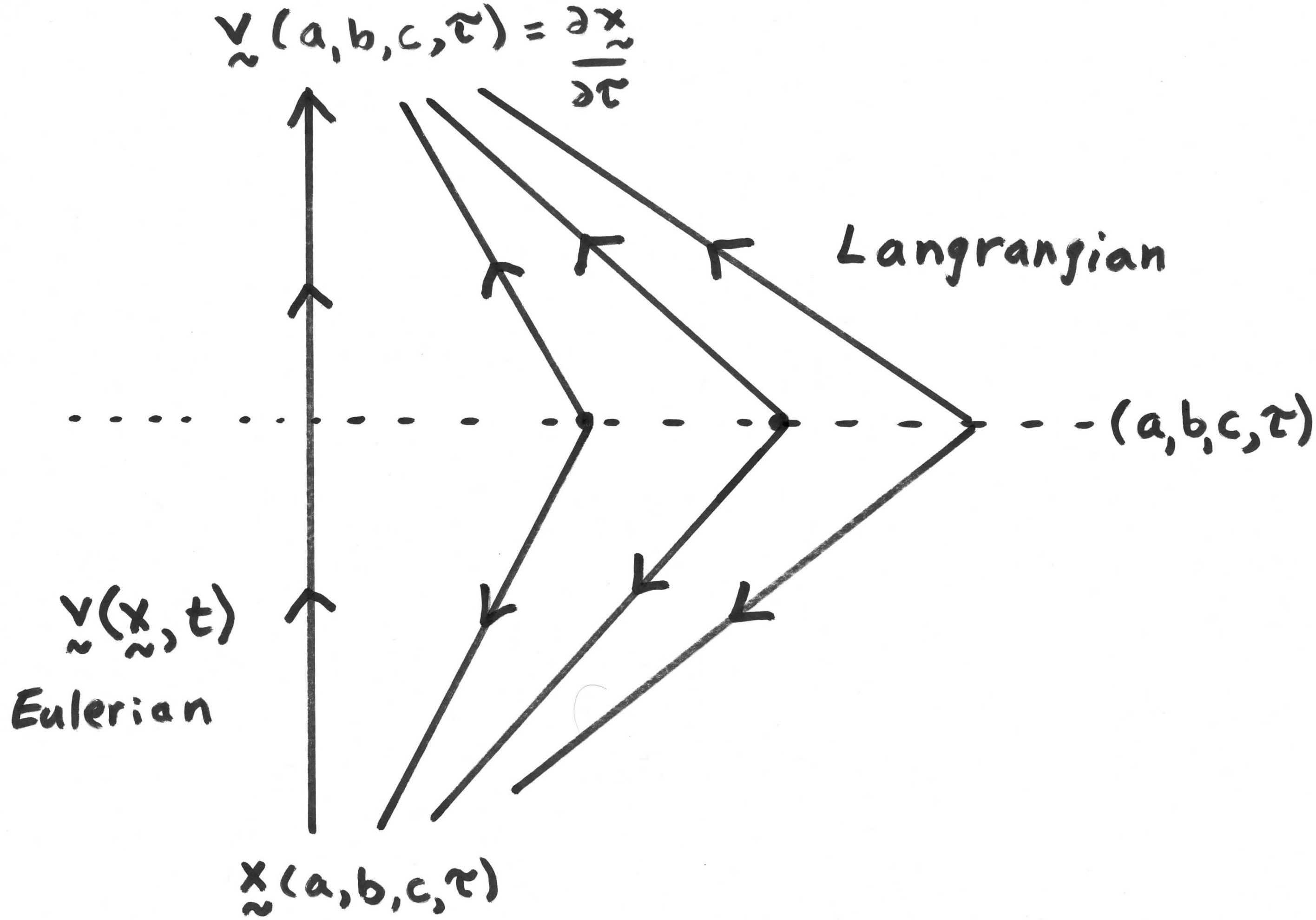
for arbitrary $\delta a(x,y,z), \delta b(x,y,z), \delta c(x,y,z)$

SYMMETRY: Choose $\delta a, \delta b, \delta c$ such that $\delta \dfrac{\partial(a,b,c)}{\partial(x,y,z)} = 0$

$\Rightarrow$ conservation of potential vorticity

Practical Applications

1. Analytical approximations that maintain conservation laws
2. Analytical approximations that add conservation laws
3. Numerical algorithms that maintain conservation laws
4. Exact solutions of the fluid equations



$\underset{\sim}{v}(a,b,c,\tau) = \frac{\partial \underset{\sim}{x}}{\partial \tau}$
Langrangian
(a,b,c,τ)
$\underset{\sim}{v}(\underset{\sim}{x},t)$
Eulerian
$\underset{\sim}{x}(a,b,c,\tau)$

Work of A. A. Abrashkin & E. I. Yakubovich (1984)

Lagrangian equations for two-dimensional incompressible flow with arbitrary labels

$$J(a,b)\frac{\partial^2 x}{\partial \tau^2} = -\frac{\partial(p,y)}{\partial(a,b)}$$

$$J(a,b)\frac{\partial^2 y}{\partial \tau^2} = -\frac{\partial(x,p)}{\partial(a,b)} - J(a,b)\, g$$

$$\frac{\partial(x,y)}{\partial(a,b)} = J(a,b)$$ $J(a,b)$ is an arbitrary time-independent function that cannot change sign

Select the solution $x(a,b,\tau)$, $y(a,b,\tau)$ arbitrarily. It is a solution if $p(a,b,\tau)$ can be found.

Solve for $\frac{\partial p}{\partial a}$ and $\frac{\partial p}{\partial b}$ from the first two equations. Require $\frac{\partial}{\partial b}\frac{\partial p}{\partial a} = \frac{\partial}{\partial a}\frac{\partial p}{\partial b}$.

This leads to

$$\frac{\partial(x_\tau, x)}{\partial(a,b)} + \frac{\partial(y_\tau, y)}{\partial(a,b)} = J(a,b)\,\zeta(a,b)$$ where $\zeta(a,b)$ is another arbitrary time-independent function.

Result of this procedure:

Let $z = x + iy$ and $s = a + ib$

Then $z = s + f(s^*)e^{i\omega\tau}$ is a solution for *any* analytic $f(\ )$

where

$$J(a,b) = 1 - |f'(s)|^2 \qquad \text{and} \qquad \zeta(a,b) = \frac{-2\omega|f'(s)|^2}{1-|f'(s)|^2}$$

$f(s) = Cs \quad \Rightarrow \quad$ Kirchhoff's vortex (1876)

$f(s) = e^{Cs} \quad \Rightarrow \quad$ Gerstner's wave (1809)

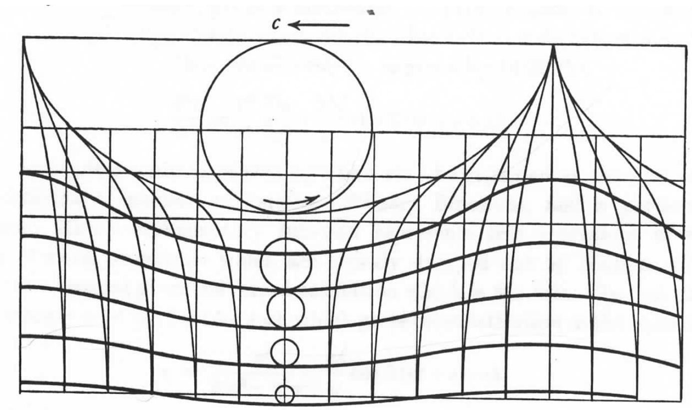


Fig. 14·81 (ii).

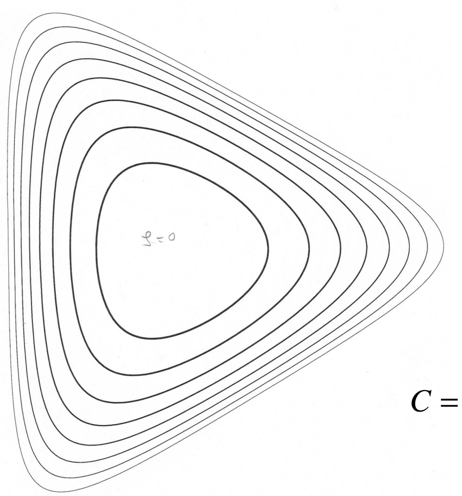
$$f(s) = Cs^2$$

$$x + iy = a + ib + C\left(a - ib\right)^2 e^{i\omega\tau}$$

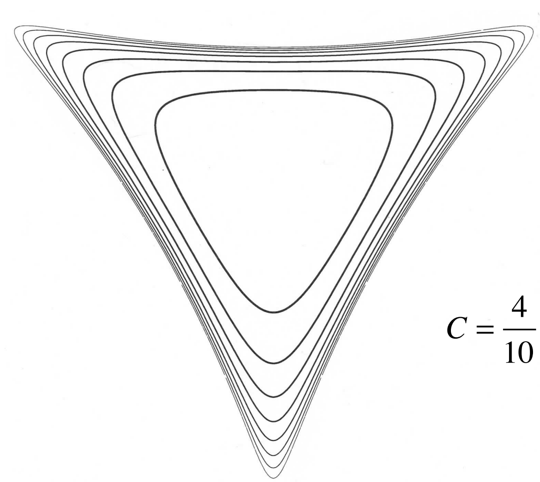
$$J \equiv \frac{\partial\left(x,y\right)}{\partial\left(a,b\right)} = 1 - 4C^2\left(a^2 + b^2\right)$$

If $a^2 + b^2 < 1$ then $C < \dfrac{1}{2}$

$$\zeta = \frac{-2\omega\left[4C^2\left(a^2 + b^2\right)\right]}{1 - 4C^2\left(a^2 + b^2\right)}$$



$C = \frac{1}{4}$



$C = \frac{4}{10}$

$f(s) = Cs^n \quad$ (requires $\quad C < \dfrac{1}{n}$)

vortex boundary: $\quad r(\theta,t) = 1 + C^2 + 2C\cos\big((n+1)\,\theta - \omega\, t\big)$

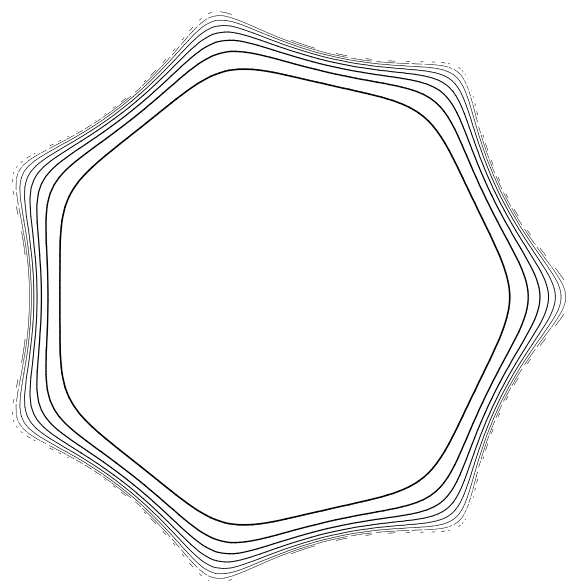
There are $\quad n+1 \quad$ "arms"

The structure rotates at angular speed $\quad \dfrac{\omega}{n+1}$

The fluid particles circulate at angular speed ω

$n = 6$

$C = \dfrac{1}{2}\cdot\dfrac{1}{6}$



"Ptolemaic solutions"

work with Nick Pizzo

Summary points

1. "Lagrangian thinking" is very old but has long been neglected.
2. A chief advantage is the existence of a variational principle
   with particle-relabeling symmetry.
3. The arbitrariness of the labels adds flexibility to the search for solutions.
4. But the concept of "particle labels" seems old-fashioned.
   The more modern viewpoint would be gauge freedom.