The Analogy Between Electrodynamics and Fluid Mechanics

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Quick review of Classical Electrodynamics

$$\nabla \cdot \mathbf{B} = 0 \qquad \mathbf{B}_t + \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{E} = \sum_i q_i \, \delta\big(\mathbf{x} - \mathbf{x}_i(t)\big) \equiv \rho \qquad c^2 \nabla \times \mathbf{B} - \mathbf{E}_t = \sum_i q_i \, \dot{\mathbf{x}}_i \, \delta\big(\mathbf{x} - \mathbf{x}_i(t)\big) \equiv \mathbf{j}$$

$$m_i \ddot{\mathbf{x}}_i = q_i \, (\mathbf{E} + \dot{\mathbf{x}}_i \times \mathbf{B})$$

Potential representation: $\mathbf{E} = -\nabla\phi - \mathbf{A}_t$ $\qquad$ $\mathbf{B} = \nabla \times \mathbf{A}$

Gauge arbitrariness: $\phi \to \phi - \lambda_t$ $\qquad$ $\mathbf{A} \to \mathbf{A} + \nabla\lambda$

In this talk: $\mathbf{E}=(E_1, E_2, 0)$ $\qquad$ $\mathbf{B} = (0,0,B_3)$ $\qquad$ $\mathbf{x}_i = (x_i, y_i, 0)$ $\qquad$ $\mathbf{A} = (A, B, 0)$

Lagrangian for Classical Electrodynamics

$$L = L_1 + L_2$$

$$L_1[\phi, \mathbf{A}] = \frac{1}{2}\int dt \iiint d\mathbf{x}\left(\mathbf{E}\cdot\mathbf{E} - c^2\mathbf{B}\cdot\mathbf{B}\right) = \frac{1}{2}\int dt \iiint d\mathbf{x}\left((\nabla\phi + \mathbf{A}_t)^2 - c^2(\nabla\times\mathbf{A})^2\right)$$

$$L_2[\phi, \mathbf{A}, \mathbf{x}_i] = -\frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i + \sum_i q_i \int dt \iiint d\mathbf{x}\left(-\phi + \mathbf{A}\cdot\dot{\mathbf{x}}_i\right)\delta\left(\mathbf{x} - \mathbf{x}_i(t)\right)$$

In two dimensions:

$$L_1[\phi, A, B] = \frac{1}{2}\int dt \iint dx\, dy\left[\left(A_t + \phi_x\right)^2 + \left(B_t + \phi_y\right)^2 - c^2\left(B_x - A_y\right)^2\right]$$

$$L_2[\phi, A, B, x_i, y_i] = -\frac{1}{2}\sum_i m_i(\dot{x}_i^2 + \dot{y}_i^2) + \sum_i q_i \int dt \iint dx\, dy\, \left(-\phi + A\dot{x}_i + B\dot{y}_i\right)\, \delta\left(\mathbf{x} - \mathbf{x}_i(t)\right)$$

Fluid dynamics

(1) Regard $\mathbf{x}_i(t)$ as the location of a point vortex with vorticity q_i

(2) Set $m_i = 0$, i.e. delete the kinetic energy of the charged particles

(3) Attach new physical meanings to the potentials:

$$\hat{h} \equiv \frac{h}{h_0} = B_x - A_y$$

$$\hat{h}\, u = -\phi_y - B_t$$

$$\hat{h}\, v = \phi_x + A_t$$

(4) Add denominators to two terms in L_1

Lagrangian for Fluid Dynamics

$$L = L_1 + L_2$$

$$L_1[\phi, A, B] = \frac{1}{2}\int dt \iint dx\,dy \left[\frac{\left(A_t + \phi_x\right)^2}{\left(B_x - A_y\right)} + \frac{\left(B_t + \phi_y\right)^2}{\left(B_x - A_y\right)} - c^2\left(B_x - A_y\right)^2 \right]$$

$$L_2[\phi, A, B, x_i, y_i] = \sum_i q_i \int dt \iint dx\,dy \;\left(-\phi + A\dot{x}_i + B\dot{y}_i\right)\; \delta\left(\mathbf{x} - \mathbf{x}_i(t)\right)$$

Resulting equations:

$$\delta\phi: \qquad v_x - u_y = q \equiv \sum_i q_i \;\delta(\mathbf{x} - \mathbf{x}_i(t))$$

$$\delta A, \delta B: \qquad \mathbf{u}_t + \nabla\left(c^2\hat{h} + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = \sum_i q_i(\dot{y}_i, -\dot{x}_i)\;\delta(\mathbf{x} - \mathbf{x}_i(t)) \rightarrow q(v, -u)$$

$$\delta\mathbf{x}_i: \qquad \dot{\mathbf{x}}_i = \mathbf{u}(\mathbf{x}_i, t)$$

Mass conservation, $\hat{h}_t + \nabla\cdot(\hat{h}\mathbf{u}) = 0,$ is automatically satisfied.

How was this variational principle discovered?

development of a numerical algorithm similar to the Lattice Boltzmann Method

$$\left(\partial_t, \partial_x, \partial_y\right) \cdot \left(\hat{h}, \hat{h}u, \hat{h}v\right) = 0 \qquad \Rightarrow \qquad \left(\hat{h}, \hat{h}u, \hat{h}v\right) = \left(\partial_t, \partial_x, \partial_y\right) \times \left(-\phi, A, B\right)$$

Classical electrodynamics also has two variational principles.

See: Wheeler & Feynman, RMP 1949,

"Classical electrodynamics in terms of direct interparticle action."

Generalizations required for geophysical fluid dynamics

(1) continuous vorticity (requires labeling fields for the vorticity)

(2) three-dimensional Boussnesq dynamics (must remain hydrostatic)

(3) coordinate-system rotation

(4) adopt the Coulomb gauge $A_x + B_y = 0 \quad \Rightarrow \quad (A,B) = (\gamma_y, -\gamma_x)$

Theory of wave/mean interactions:

$$L = L_1 + L_2 = L_{QG} + L_{IG} + \varepsilon \, L_{coupling}$$

Wave packet propagating into a quiescent region (Bretherton flow)

$\Leftrightarrow$ electrodynamics in the absence of charge

$$L_2 = 0 \quad \Rightarrow \quad L = L_1$$

but since L_1 is non-quadratic (because of the "denominator terms"), the dynamics is nontrivial.

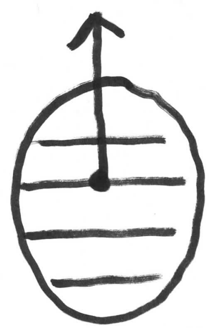
Apply Whitham's "averaged Lagrangian" method to obtain

$$\nabla^2 \overline{\psi} + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2} \frac{\partial \overline{\psi}}{\partial z} \right) = \nabla \times \mathbf{p}$$

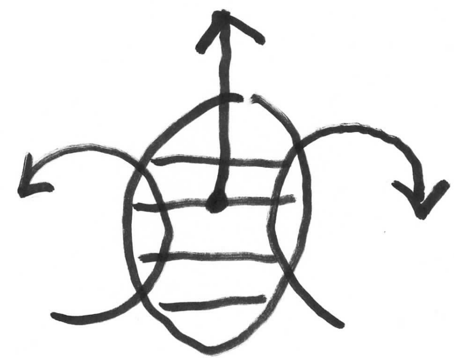
QG potential vorticity = curl of wave pseudomomentum

$$\mathbf{p} = \frac{E\mathbf{k}}{\omega}$$

Wave packet propagating in the direction of **k**



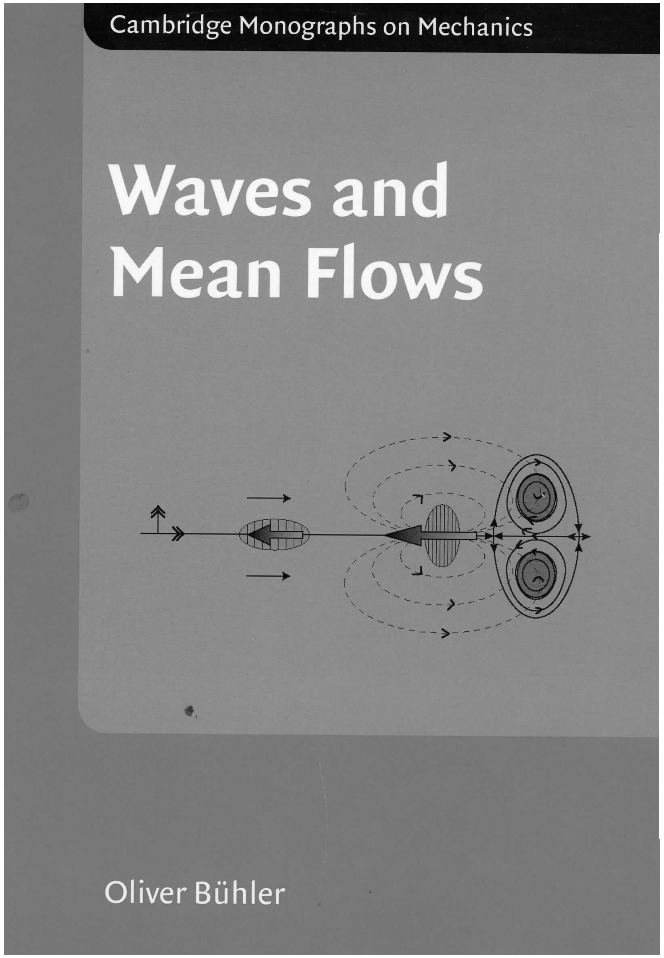
$$\mathbf{p} = \frac{E\mathbf{k}}{\omega}$$

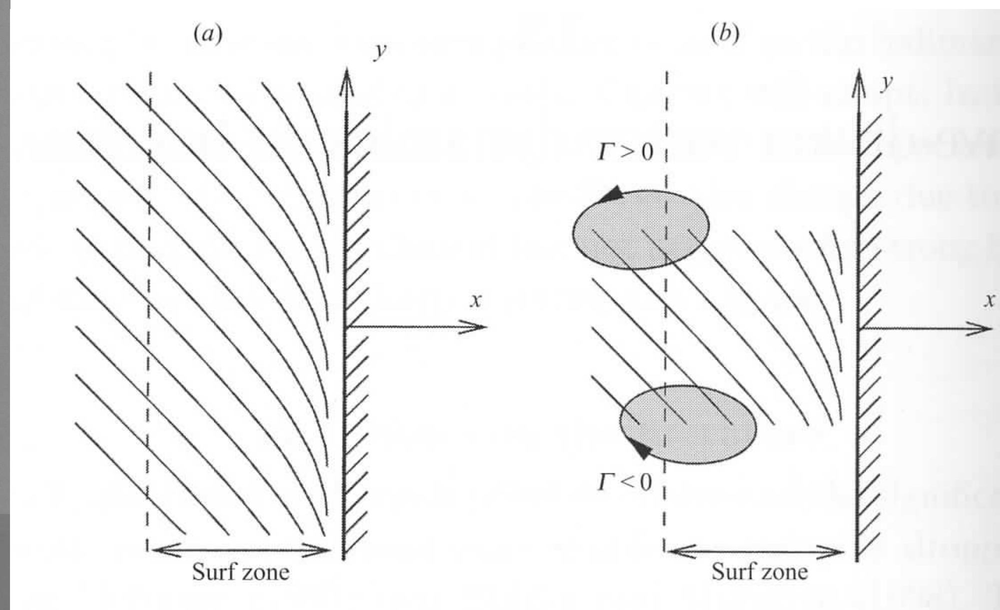


$$\nabla^2 \overline{\psi} = \nabla \times \mathbf{p}$$

electrodynamic analogy: pair production from a vacuum

However, fluid viscosity allows the vortices to detach.


Cambridge Monographs on Mechanics
Waves and Mean Flows
Oliver Bühler


(a)
y
x
Surf zone
(b)
y
Γ > 0
Γ < 0
x
Surf zone

Pair production in fluid dynamics & electrodynamics

Fluid Lagrangian with no vorticity present:

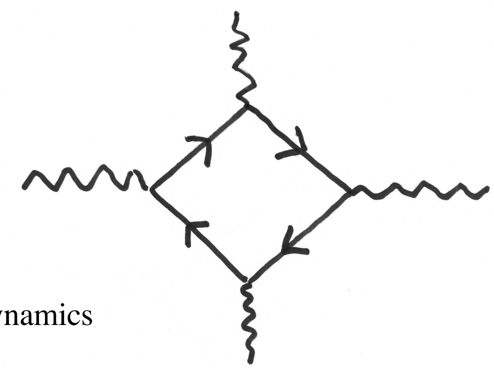
$$L = \frac{1}{2}\frac{\left(A_t - \phi_x\right)^2}{\left(1 + B_x - A_y\right)} + \frac{1}{2}\frac{\left(B_t - \phi_y\right)^2}{\left(1 + B_x - A_y\right)} - \frac{1}{2}c^2\left(1 + B_x - A_y\right)^2$$

$$\sim \frac{1}{2}\frac{E_1^2}{(1 + B_3)} + \frac{1}{2}\frac{E_2^2}{(1 + B_3)} - \frac{1}{2}c^2 B_3^2$$

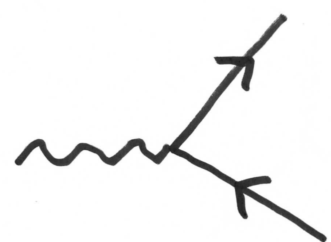
$$\approx L_0 + \frac{1}{2}\left(E_1^2 + E_2^2\right)B_3$$

Electrodynamics Lagrangian with no charge present (Heisenberg-Euler):

$$L \sim L_0 + \alpha_1\left(\mathbf{E}\cdot\mathbf{E} - \mathbf{B}\cdot\mathbf{B}\right)^2 + \alpha_2\left(\mathbf{E}\cdot\mathbf{B}\right)^2$$



electrodynamics



fluid dynamics

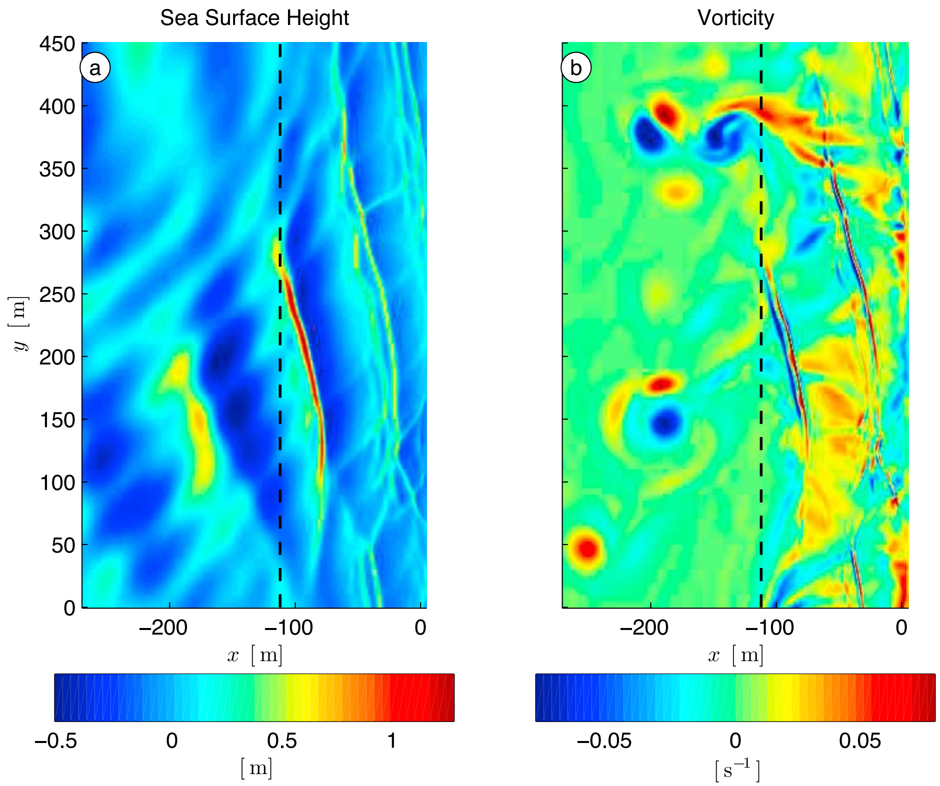


Figure 3. Snapshot in time of modeled (a) sea surface elevation η and (b) vorticity ζ versus x and y for R3, 2700 s into the model run. The shoreline is located at $x = 0$ m and the black dashed line is the approximate outer limit of the surf zone. Only a subset of the model domain is shown. Note the broad range of vorticity length scales within the surf zone.

Surface wave packet propogating to the right

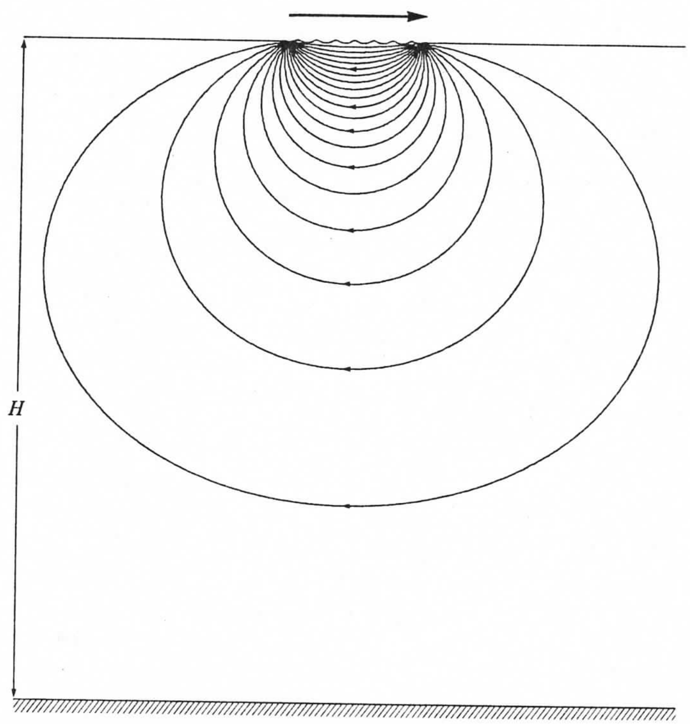


FIGURE 2. The irrotational, $O(a^2)$ return flow underneath a packet of surface gravity waves propagating to the right. (The streamlines, plotted at equal intervals, are quantitatively correct for a two-dimensional wave packet whose amplitude is constant except near its ends.)

McIntyre (1981)

Advantages of the electrodynamic analogy:

1. Lagrangian in which vorticity plays a prominent role.
2. Sharp distinction between virtual vorticity (L_1) and actual vorticity (L_2).
3. Wave/vorticity replaces wave/mean.
4. Vorticity is the true slow variable.