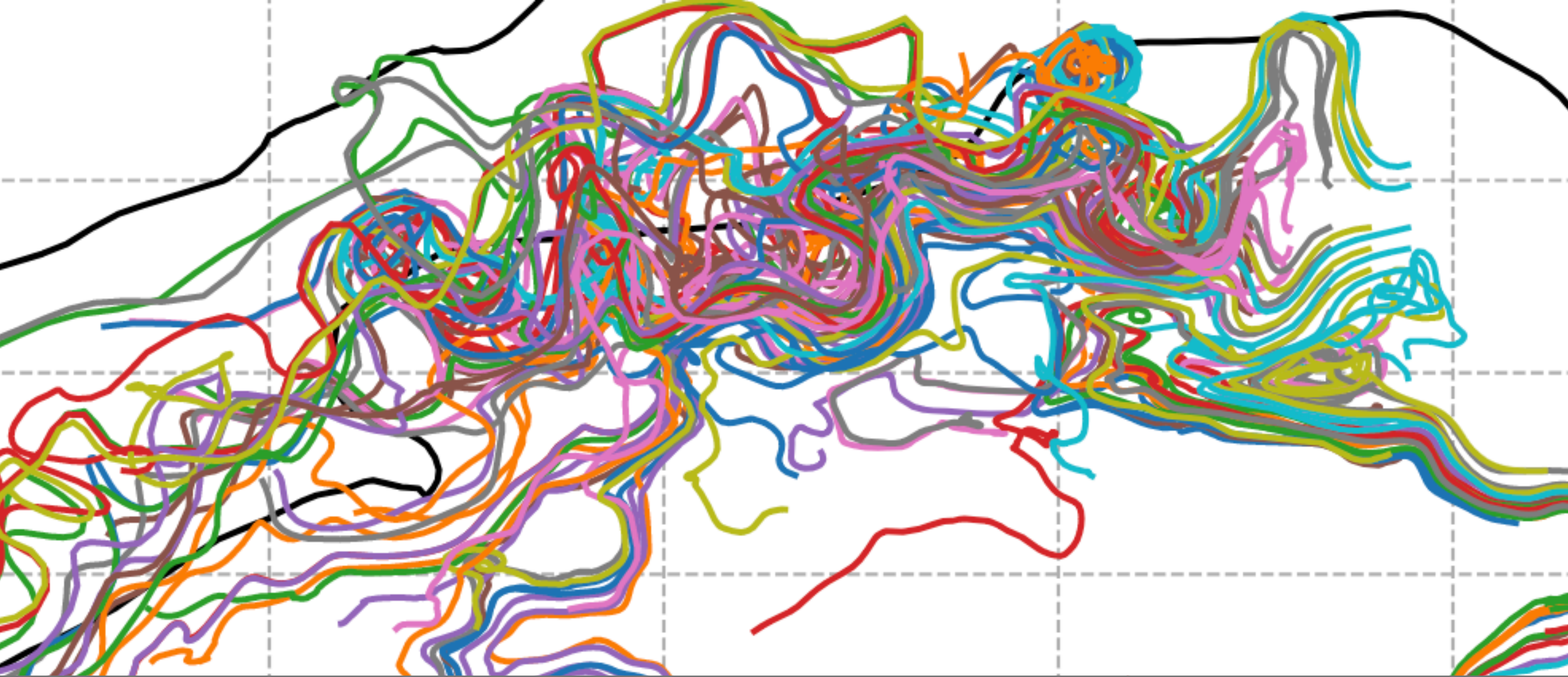
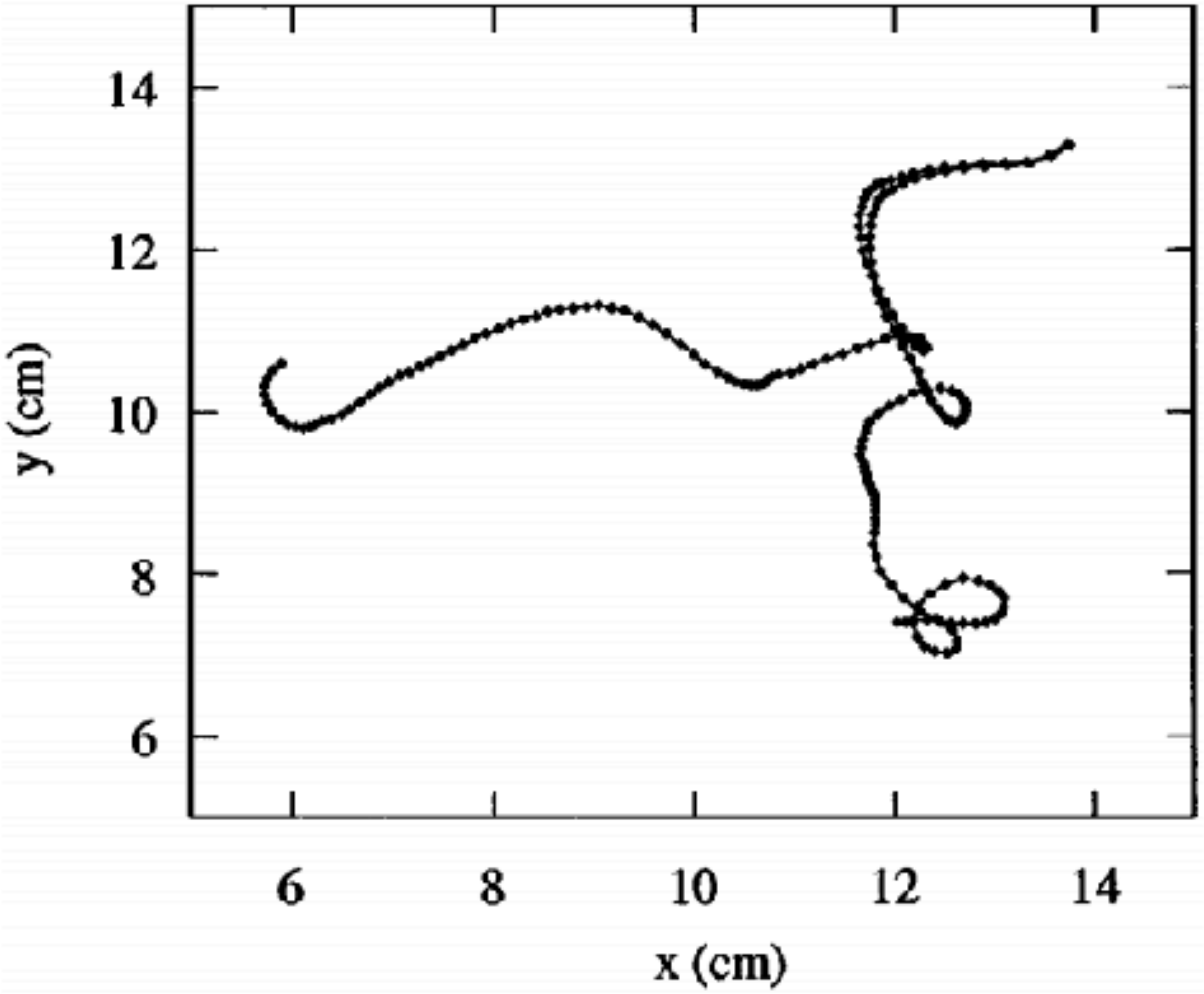


# MULTIPLE PARTICLE STATISTICS

LaCasce (2008), as told by Channing Prend

# MOTIVATION

**Fundamental Problem:** 2 particles deployed simultaneously at slightly different locations follow very different paths.

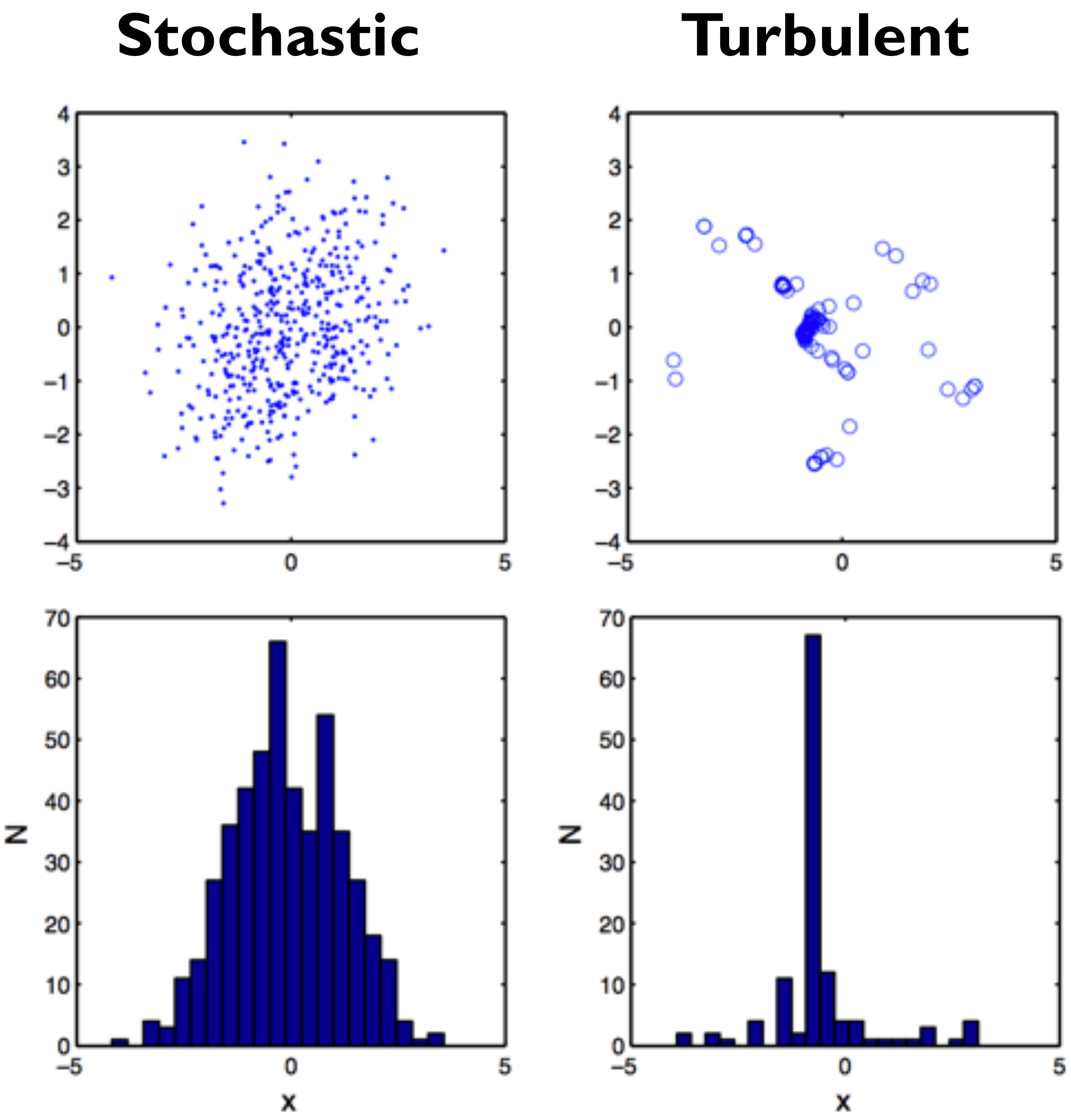


Jullien *et al.* (1999)

Multiple particle statistics deal with the relative dispersion of particle pairs (or larger groups of particles).

At small and large scales, relative dispersion behaves like absolute dispersion. At intermediate scales, relative dispersion reflects aspects of the flow.

# MOTIVATION



LaCasce (2008)

Variance of particle displacements:

$$D_x(t) = \frac{1}{N-1}\sum_{i=1}^{N}[x_i(t) - x_i(0) - M_x(t)]^2$$

$$\downarrow$$

$$D_x(t) = \frac{1}{2N(N-1)}\sum_{i \neq j}[x_i(t) - x_j(t)]^2$$

Relative dispersion is useful to consider because it’s proportional to cloud dispersion and depends on the flow at intermediate scales.

# DEFINING RELATIVE DISPERSION

**Joint displacement PDF:** $P(x,y,t) = P(x_0,y_0,t_0)\mathcal{Q}(x,y,t|x_0,y_0,t_0)\mathrm{d}x_0\mathrm{d}y_0.$

**PDF of particle separations:** $q(y,t|y_0,t_0) = \int \mathcal{Q}(x,y,t|x_0,y_0,t_0)\mathrm{d}x_0.$

$p(y,t) = \int p(y_0,t_0)q(y,t|y_0,t_0)\mathrm{d}y_0.$ (probability of a given separation)

**Relative dispersion**

$$\overline{y^2}(t) = \int y^2 p(y,t)\mathrm{d}y.$$

**Relative diffusivity**

$$K \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{y^2} = \overline{yv} = \overline{y_0 v} + \int_{t_0}^{t} \overline{v(t)v(\tau)}\mathrm{d}\tau,$$

"memory" of initial state

two particle velocity cross-correlation

# SCALE DEPENDENCE OF RELATIVE DIFFUSIVITY

$$K \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{y^2} = \overline{yv} = \overline{y_0 v} + \boxed{\int_{t_0}^{t} \overline{v(t)v(\tau)}\mathrm{d}\tau,}$$

**Mean square separation velocity:** $\overline{v^2}(t) = \overline{(u_i(t) - u_j(t))^2} = 2v^2 - 2\overline{u_i u_j}$

If pair velocities are uncorrelated $\overline{v^2}(t) = \overline{(u_i(t) - u_j(t))^2} = 2v^2 - 2\cancelto{0}{\overline{u_i u_j}}$

Relative diffusivity is 2x the absolute diffusivity

Relative dispersion behaves like absolute dispersion at small and large scales. At intermediate scales, pair velocities are correlated and relative dispersion depends on the flow.

# 2-D TURBULENT FLOW

**Lagrangian velocity difference:**

$$\overline{v(y)^2} = \overline{(u(x+y,t)-u(x,t))^2} = 2\int_0^\infty \underbrace{E(k)}_{E(k)\propto k^{-\alpha}}\underbrace{[1-J_0(ky)]}_{\begin{array}{l} 1-J_0(ky)\approx \frac{1}{4}k^2y^2, \quad ky \ll 1, \\ 1-J_0(ky)\approx 1+\mathrm{O}(ky)^{-1/2}, \quad ky \gg 1. \end{array}}\mathrm{d}k$$

$$\overline{v(y)^2} \approx 2\int_0^{1/y} k^{-\alpha}\left(\frac{1}{4}k^2y^2\right)\mathrm{d}k + 2\int_{1/y}^{\infty} k^{-\alpha}\mathrm{d}k$$

$$\Big\downarrow$$

$$\overline{v(y)^2} = \underbrace{\frac{1}{2}y^2\frac{1}{3-\alpha}k^{3-\alpha}|_0^{1/y}}_{\text{diverges if } \alpha \geq 3} + \underbrace{\frac{2}{1-\alpha}k^{1-\alpha}|_{1/y}^{\infty}}_{\text{diverges if } \alpha \leq 1}$$

# 2-D TURBULENT FLOW CONTINUED

$$\overline{v(y)^2} = \frac{1}{2}y^2 \frac{1}{3-\alpha} k^{3-\alpha}\Big|_0^{1/y} + \frac{2}{1-\alpha} k^{1-\alpha}\Big|_{1/y}^{\infty}$$

$1 < \alpha < 3$:

$$\overline{v(y)^2} \propto y^{\alpha-1} \longrightarrow K = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{y^2} \propto y^{(\alpha+1)/2}$$

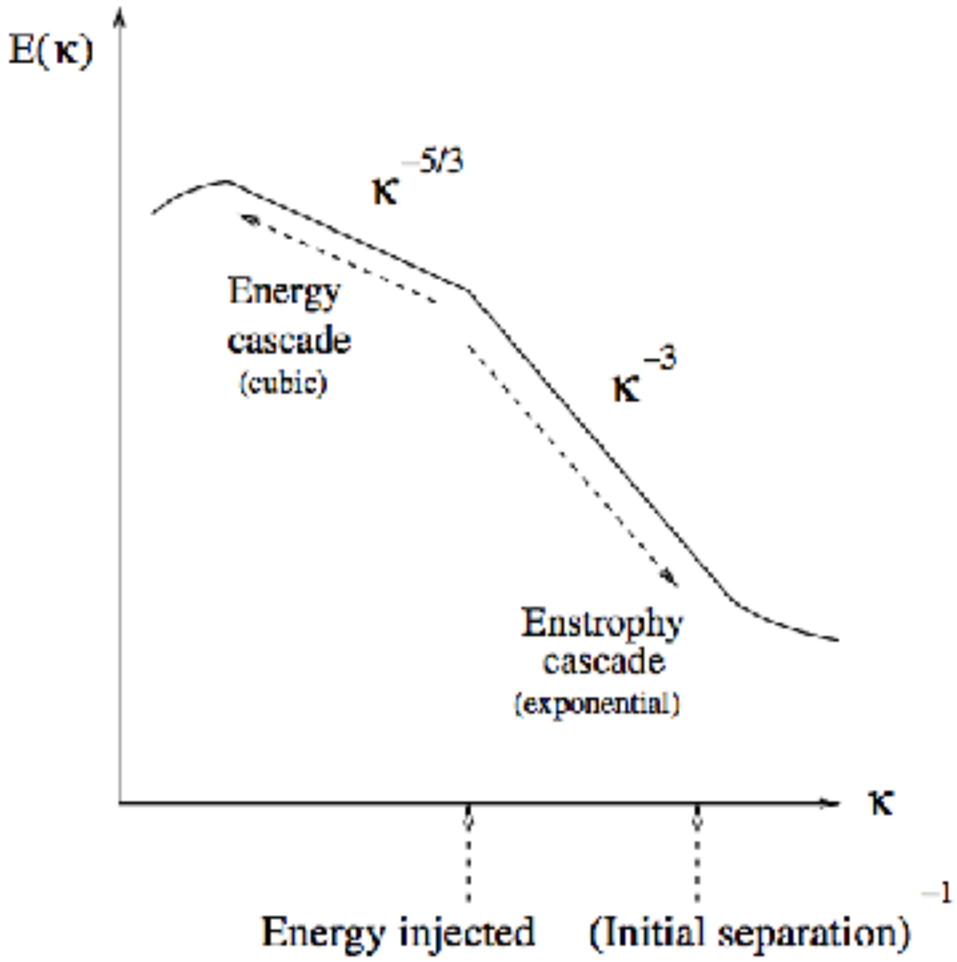
Relative diffusivity depends directly on separation distance (local dispersion).

$\alpha \geq 3$:

$$\overline{v(y)^2} \approx \frac{1}{2}y^2 \int k^2 E(k)\mathrm{d}k = c_1 \Omega y^2 \longrightarrow K = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\overline{y^2} = c_2 T^{-1}\overline{y^2}$$

Non-local dispersion (dominated by largest eddies). Exponential growth of pair separation.

# 2-D TURBULENT FLOW CONTINUED



Inertial Subrange for 2-D turbulence:

**Inverse energy cascade ($\kappa^{-5/3}$)**

$$\overline{y^2} \propto \epsilon t^3, \quad K \propto \epsilon^{1/3} y^{4/3}, \quad ku(y) = \text{Const.}$$

**Enstrophy cascade ($\kappa^{-3}$)**

$$\overline{y^2} \propto \exp(c_3 \eta^{1/3} t), \quad K \propto \overline{y^2}, \quad ku(y) \propto \exp(c_4 \eta^{1/3} t)$$

Separation has exponential growth in time if initial separation is smaller than the deformation radius and cubic once the separation is larger.

# SHEAR DISPERSION

$$dy = \mathscr{K}^{1/2} dw$$
$$dx = \gamma y dt$$

$$\Big\downarrow$$

$$\langle y^2 \rangle = \cancel{2}\mathscr{K} t, \quad \langle xy \rangle = \frac{\mathscr{K}}{2} \gamma t^2$$
$$\langle x^2 \rangle = x_0^2 + \frac{\overset{1}{\cancel{2}}}{3} \mathscr{K} \gamma^2 t^3 .$$

Simpler to solve from the Fokker-Plank equation:

$$\partial_t P + \partial_x (\gamma y P) - \partial_y^2 \left( \frac{\kappa}{2} P \right) = 0$$
$$\int \partial_t y^2 P + \int \cancel{\partial_x (\gamma y^3 P)} - \int \partial_y^2 \left( y^2 \frac{\kappa}{2} P \right) = 0$$
$$\partial_t \langle y^2 \rangle - \kappa = 0$$
$$\langle y^2 \rangle = \kappa t$$

Shear dispersion produces Richardson-type growth, exactly as in a turbulent cascade. But pair velocities are uncorrelated, displacement PDF is Gaussian.

# FSLES

**Basic Idea:** Compute the time taken for two trajectories with some finite initial separation distance to reach a larger finite separation distance.

$$\lambda = \frac{1}{\tau} \log \frac{\delta_f}{\delta_0}$$

$\delta_0$ is the separation of the initial positions, $\delta_f$ is the prescribed final separation, $\tau$ is the first time at which the separation $\delta_f$ is reached.

If dispersion has a power law dependence on time, the FSLE has a power law dependence on separation.

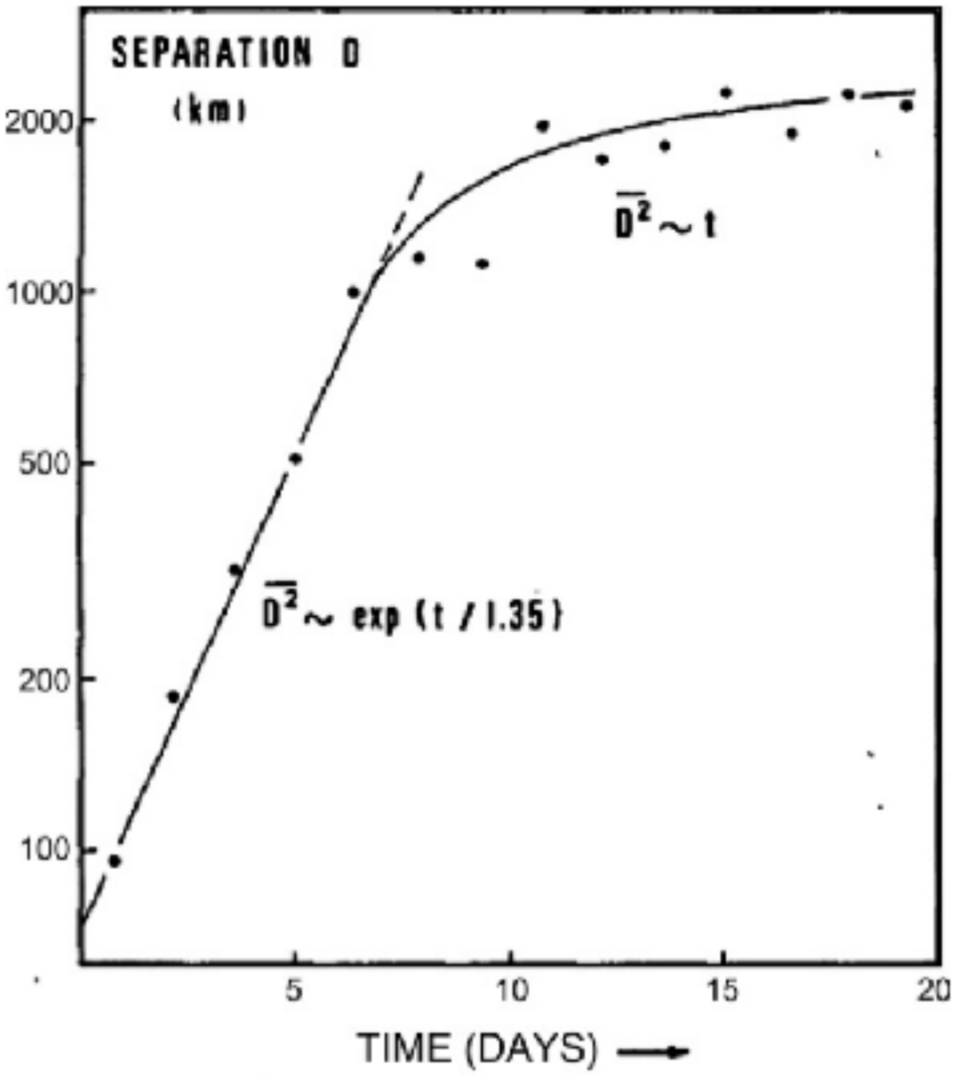
$$y^2 \propto t^\alpha \longrightarrow \lambda \propto y^{-2/\alpha}$$

by scale analysis

Growth in the Richardson regime has an FSLE which decreases like $y^{-2/3}$

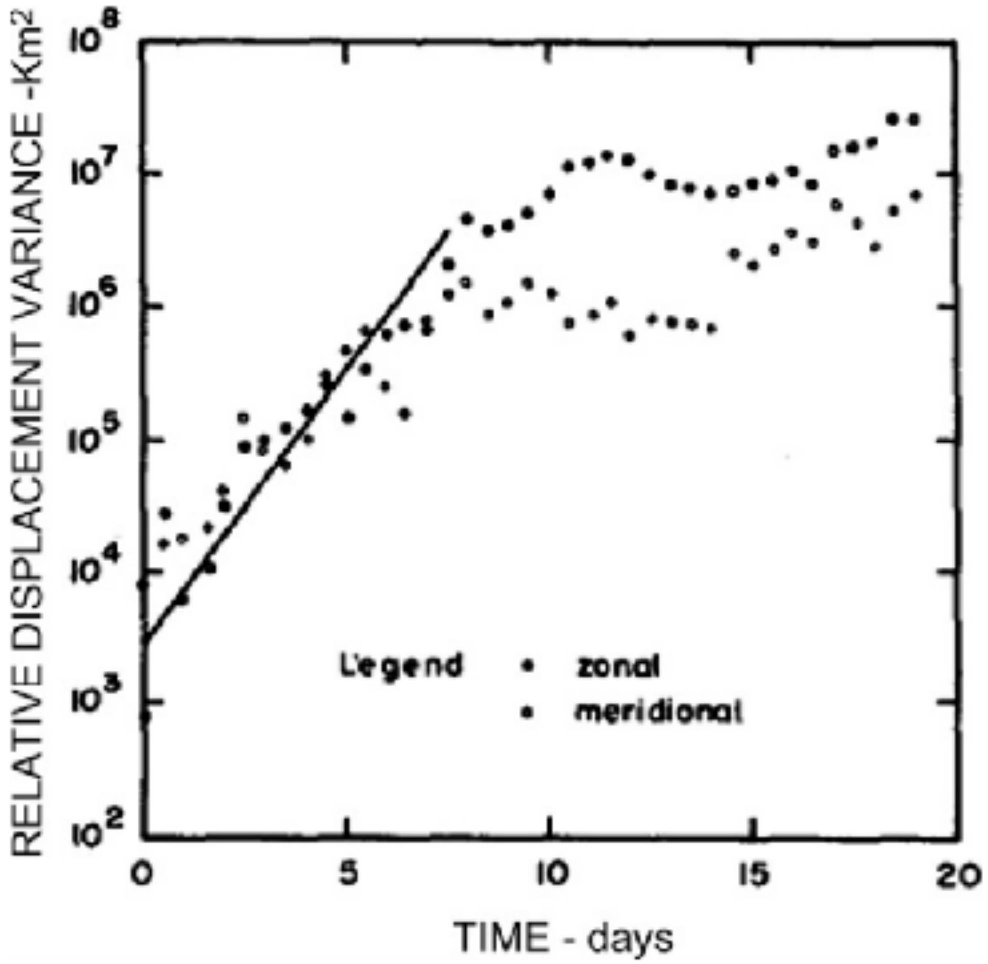
# RELATIVE DISPERSION IN THE ATMOSPHERE

**EOLE**



Morel and Larcheveque (1974)

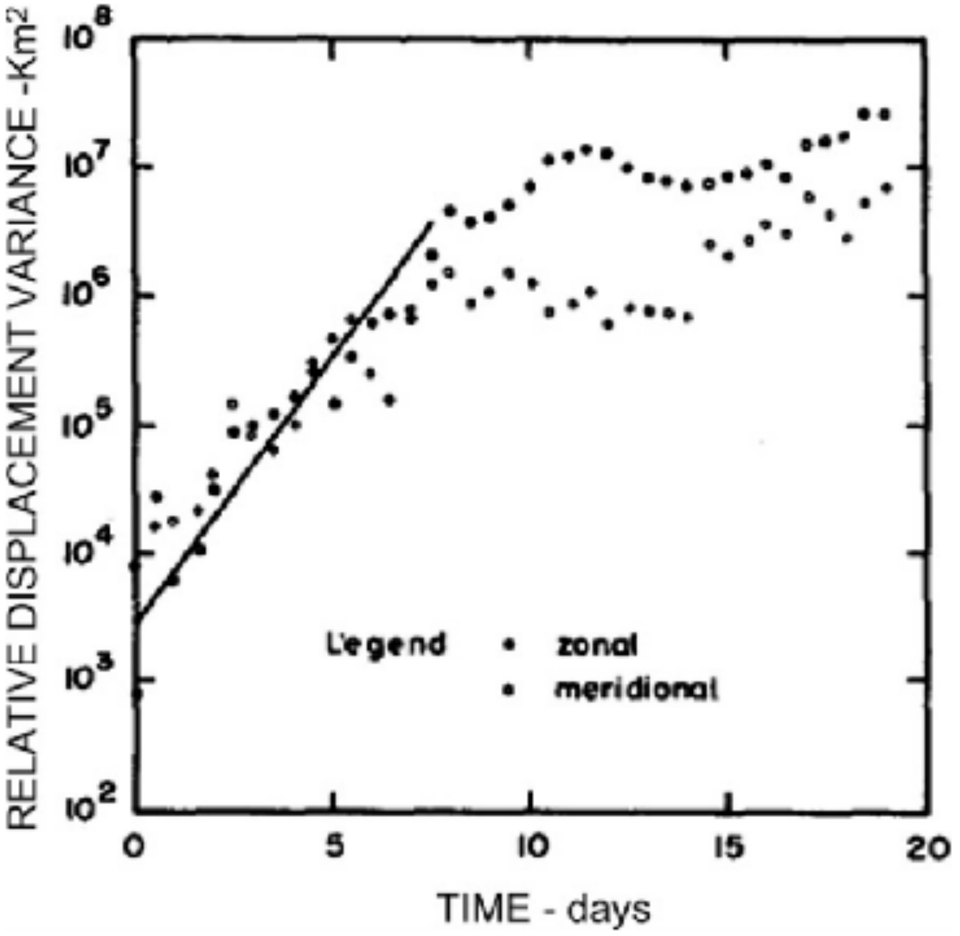
**TWERLE**



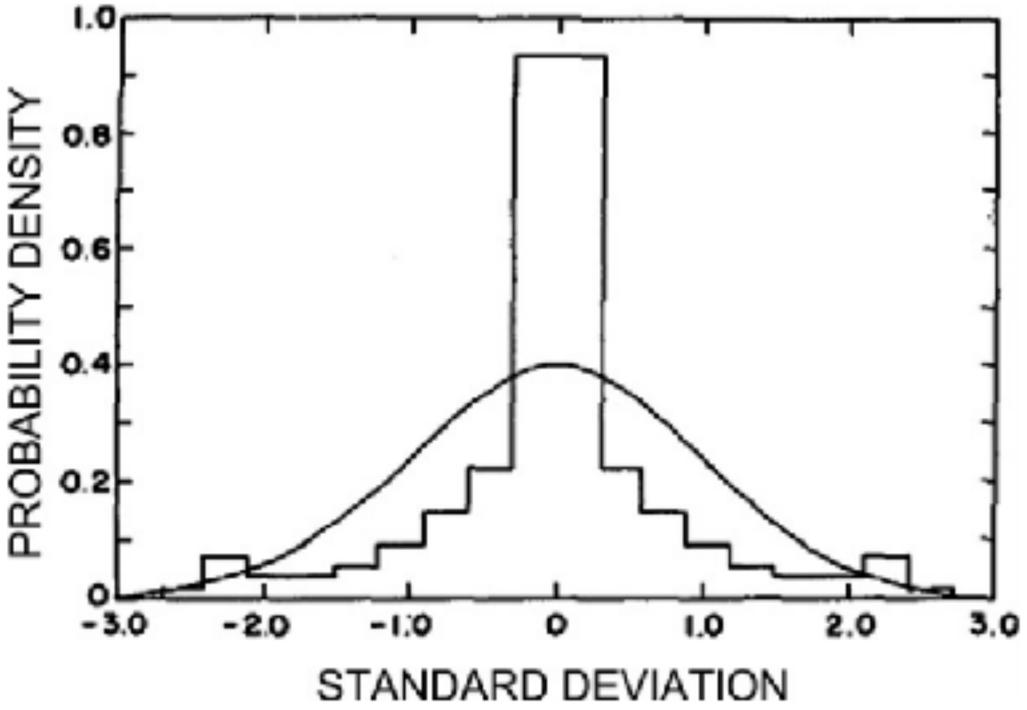
Er-el and Peskin (1981)

Exponential growth in separation distance up to 1000 km. EOLE suggests linear growth at larger scales, TWERLE suggest $t^3$ dependence.

# RELATIVE DISPERSION IN THE ATMOSPHERE



Er-el and Peskin (1981)

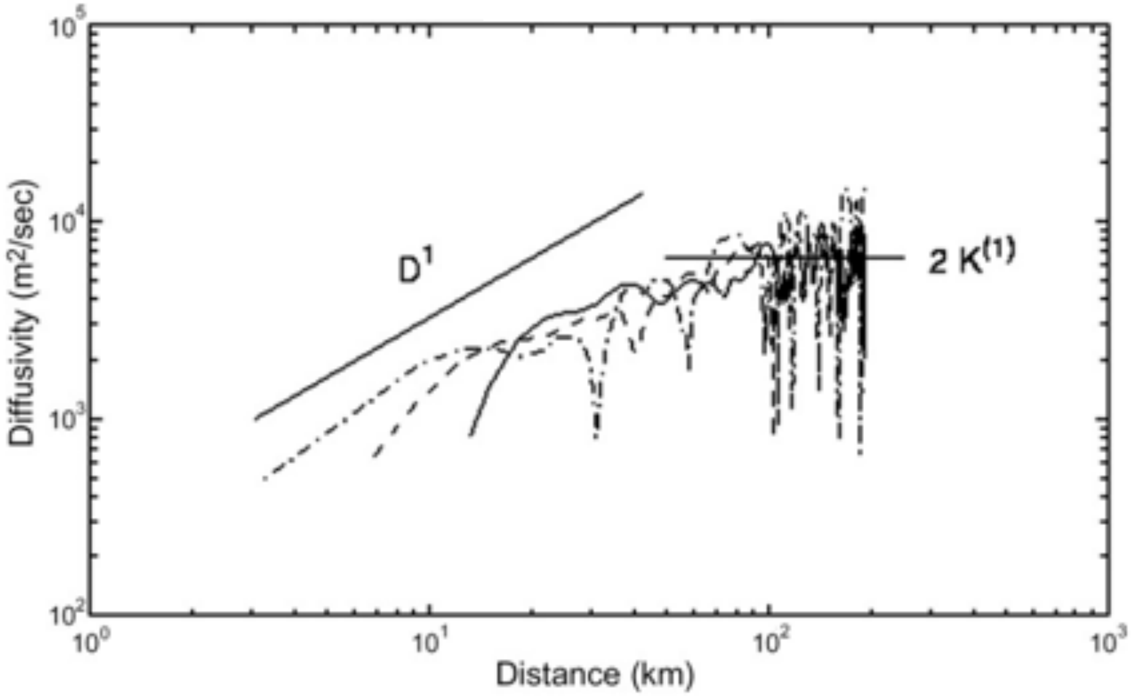


PDF of pair separations during the exponential growth phase (TWERLE) is non-Gaussian → non-local dispersion

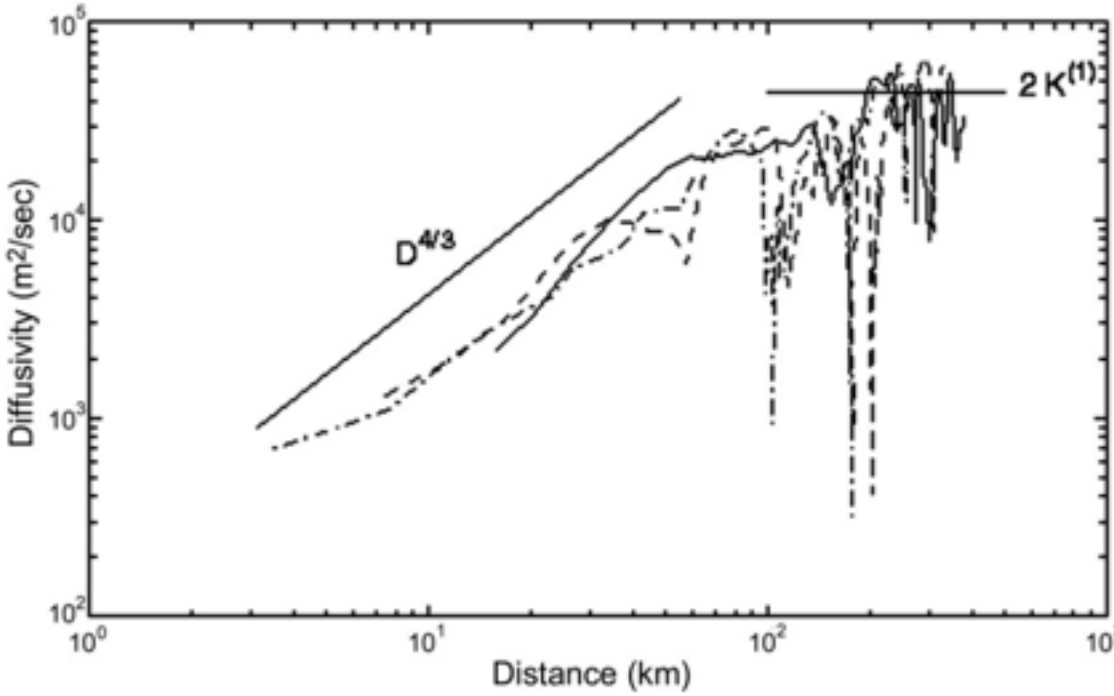
Dispersion was approximately isotropic in the exponential growth range and preferentially zonal at scales >1000 km

# RELATIVE DISPERSION IN THE ATLANTIC

**Eastern North Atlantic**



**Western North Atlantic**

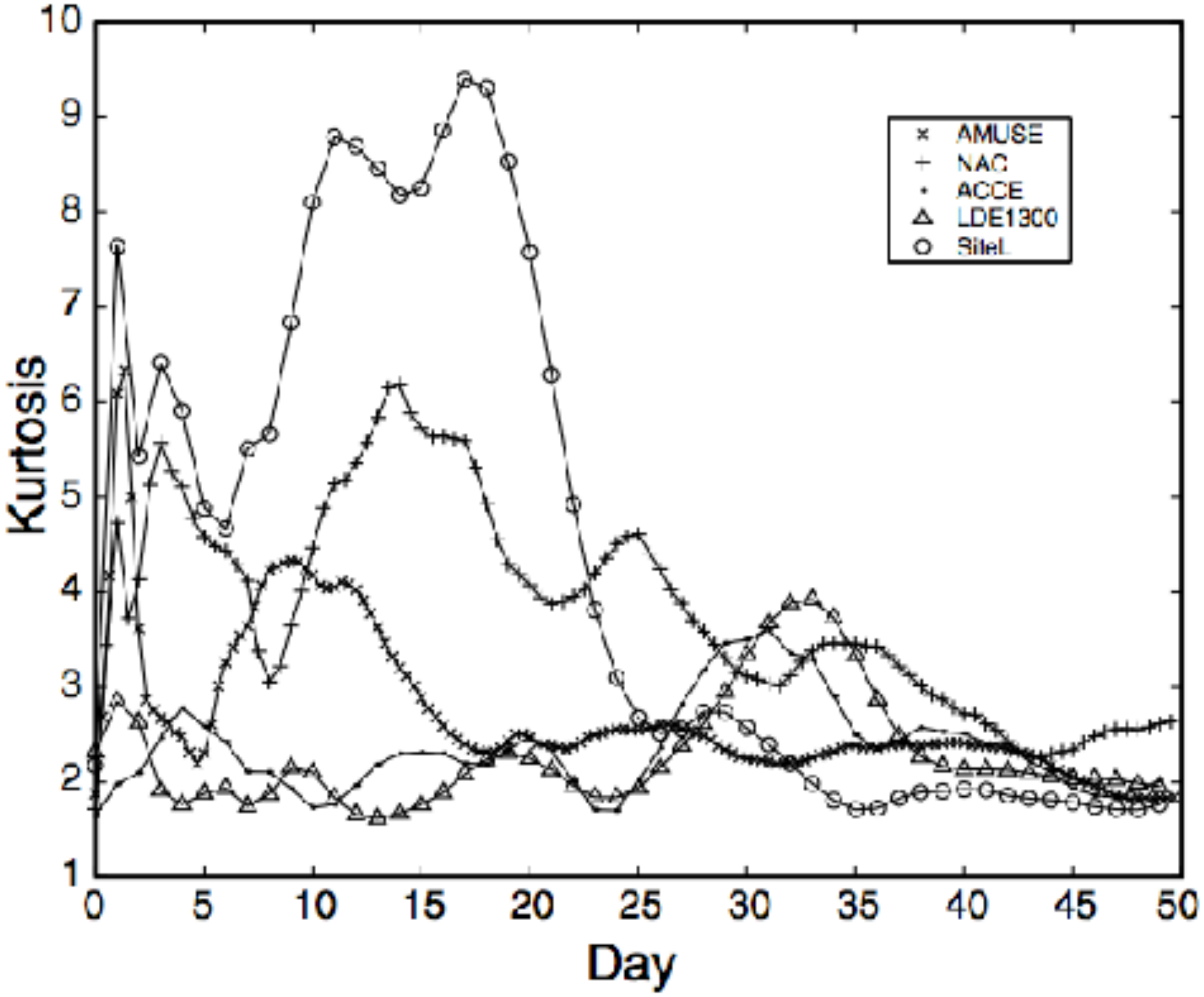


LaCasce and Bower (2000)

Relative dispersion from SOFAR and RAFOS floats was isotropic in all regions and at all scales. In eastern Atlantic, pair velocities were uncorrelated at all scales. In western Atlantic, pair velocities were correlated for separations up to 100-200 km (relative dispersion had $y^{4/3}$ dependence).

# RELATIVE DISPERSION IN THE ATLANTIC

## Kurtoses of Relative Displacements

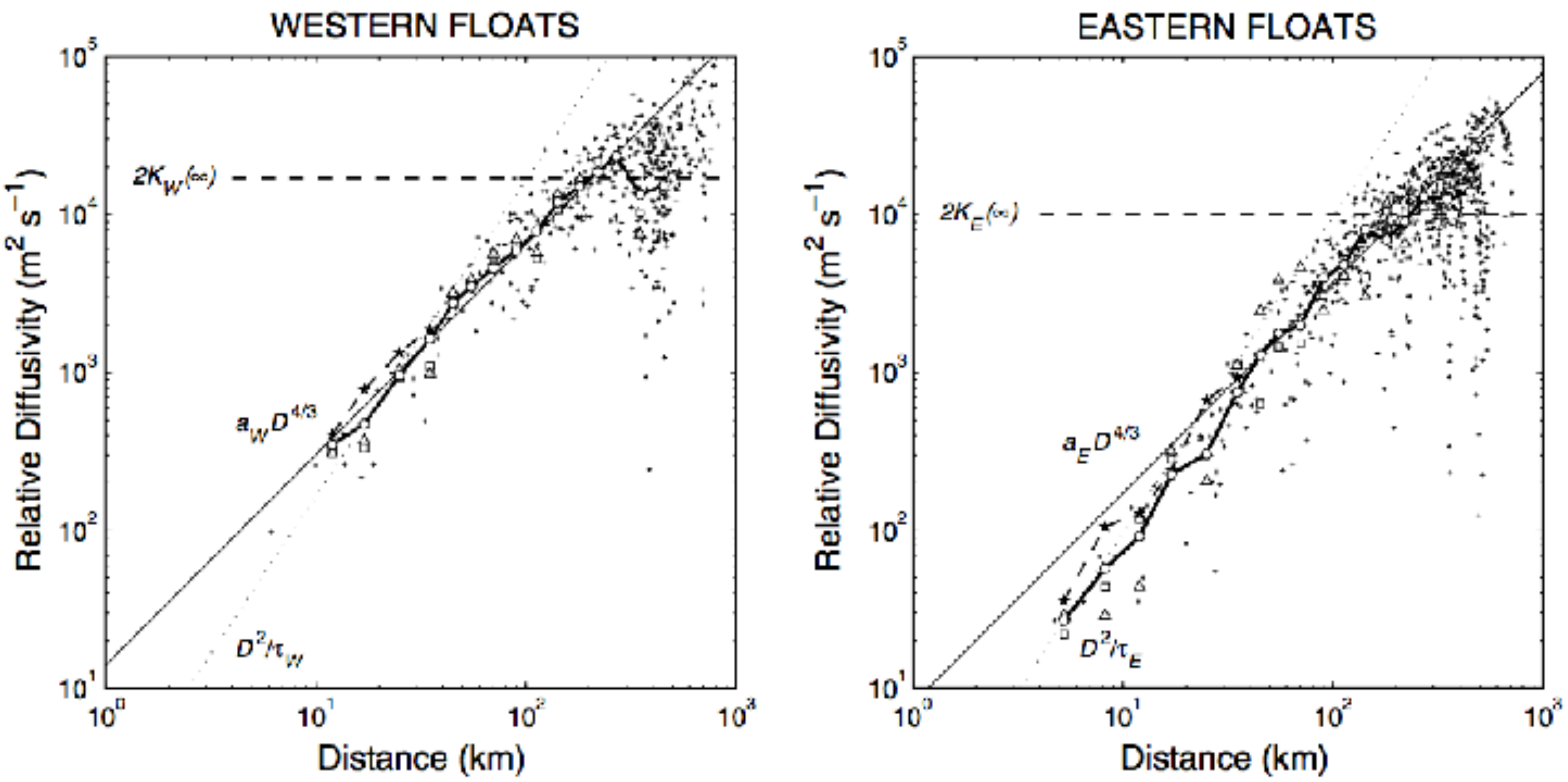


LaCasce and Bower (2000)

Eastern floats have kurtoses near 3 (Gaussian), western floats have elevated kurtoses that fall back towards 3 after 20-30 days.

Richardson-type dispersion in the west could be from (1) inverse energy cascade driven by baroclinic instability in the Gulf Stream (2) shear dispersion in the Gulf Stream.

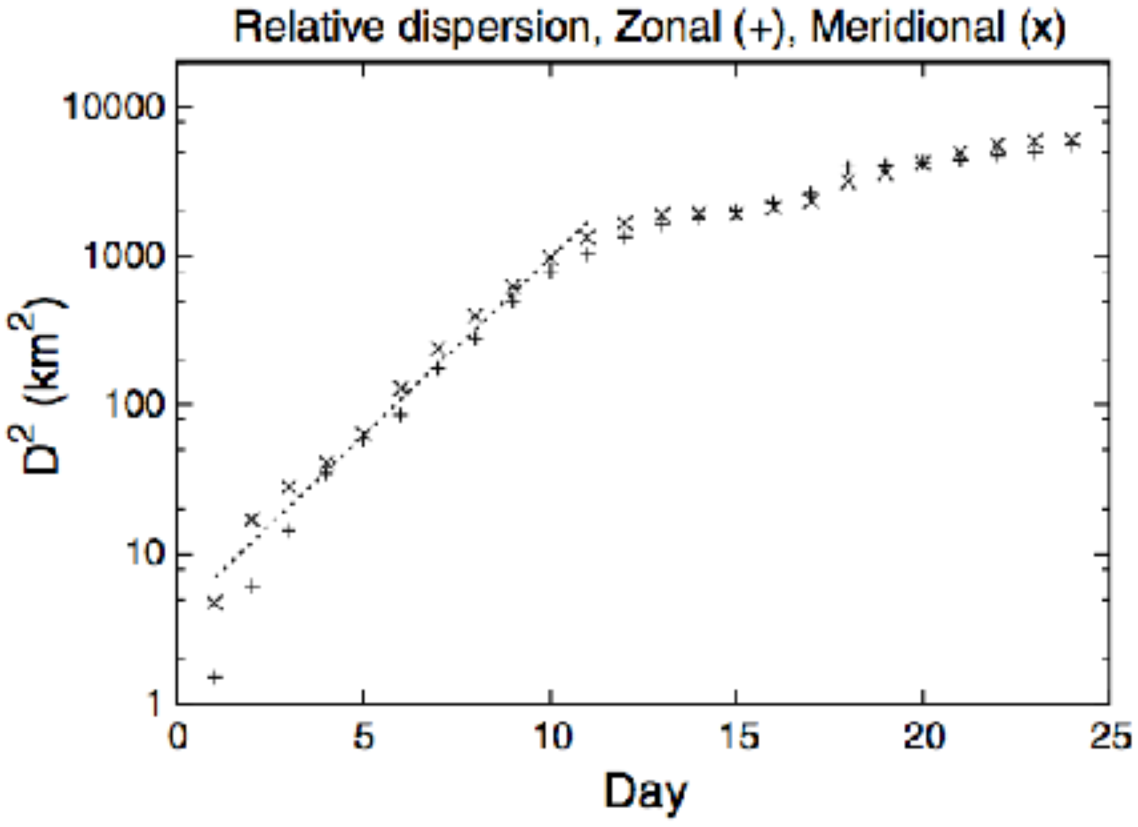
# RELATIVE DISPERSION

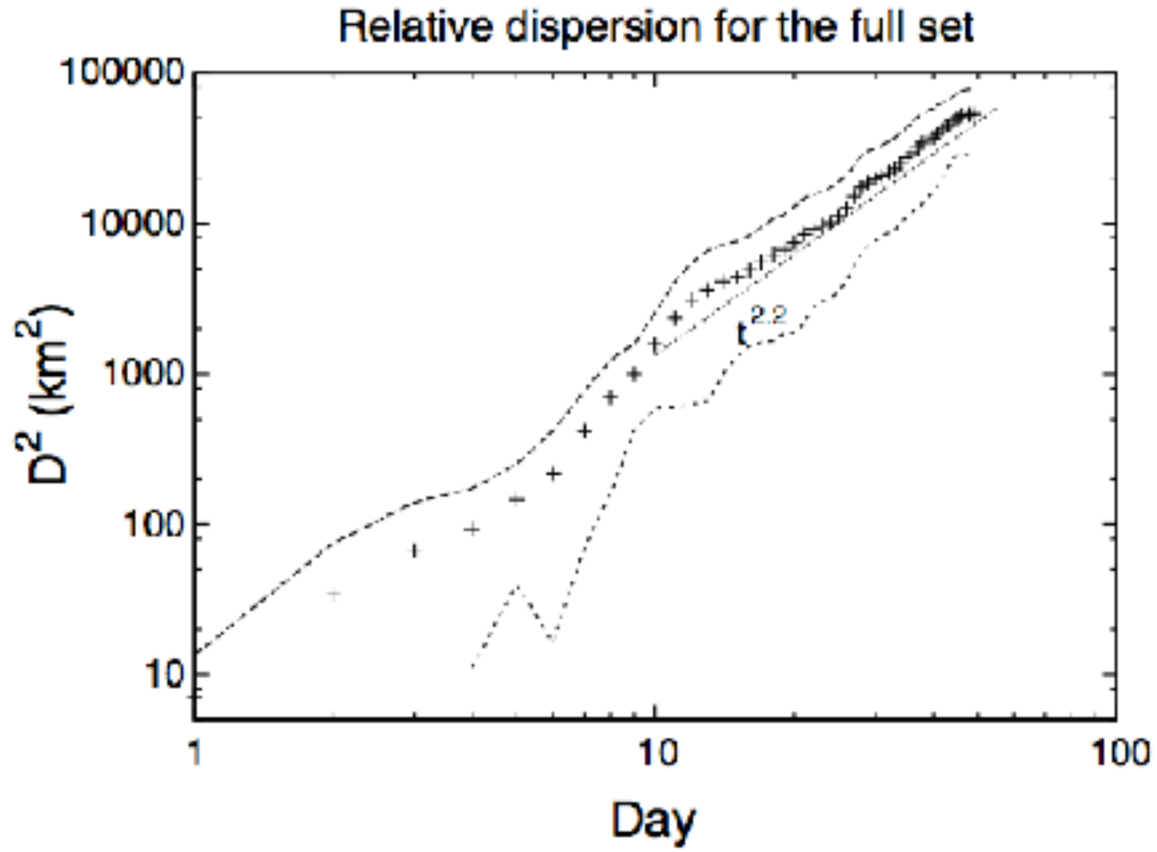


Ollitrault *et al.* (2005)

Richardson-type dispersion up to 200-300 km in both the western and eastern Atlantic with exponential growth below the deformation radius.

# RELATIVE DISPERSION



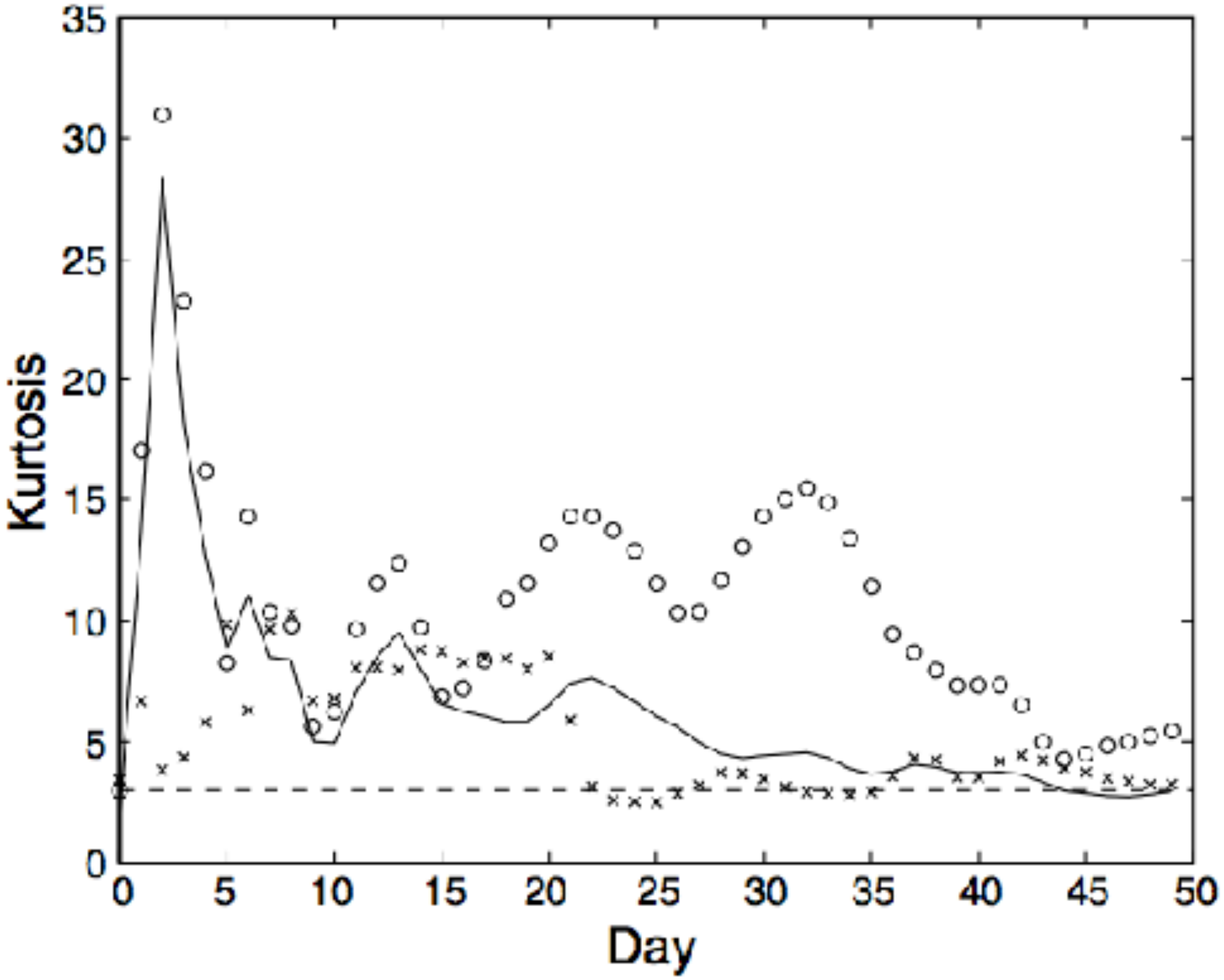


LaCasce and Ohlmann (2003)

Exponential growth up to 50 km (~first 10 days). Dispersion at late times is proportional to $t^{2.2}$ (less than expected for Richardson regime).

# RELATIVE DISPERSION

**Kurtoses of Relative Displacements**



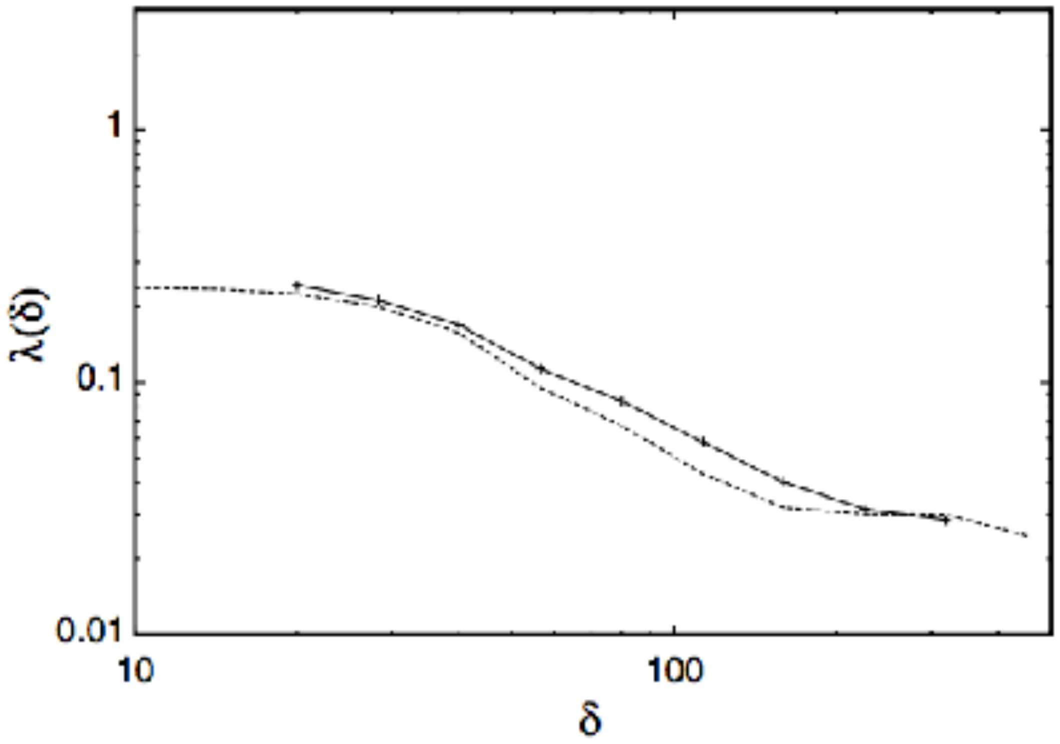
LaCasce and Ohlmann (2003)

Rapid growth of kurtoses (highly non-Gaussian) in first 3 days consistent with non-local dispersion.

Relative dispersion is isotropic but kurtoses are anisotropic (larger values in the meridional direction)?

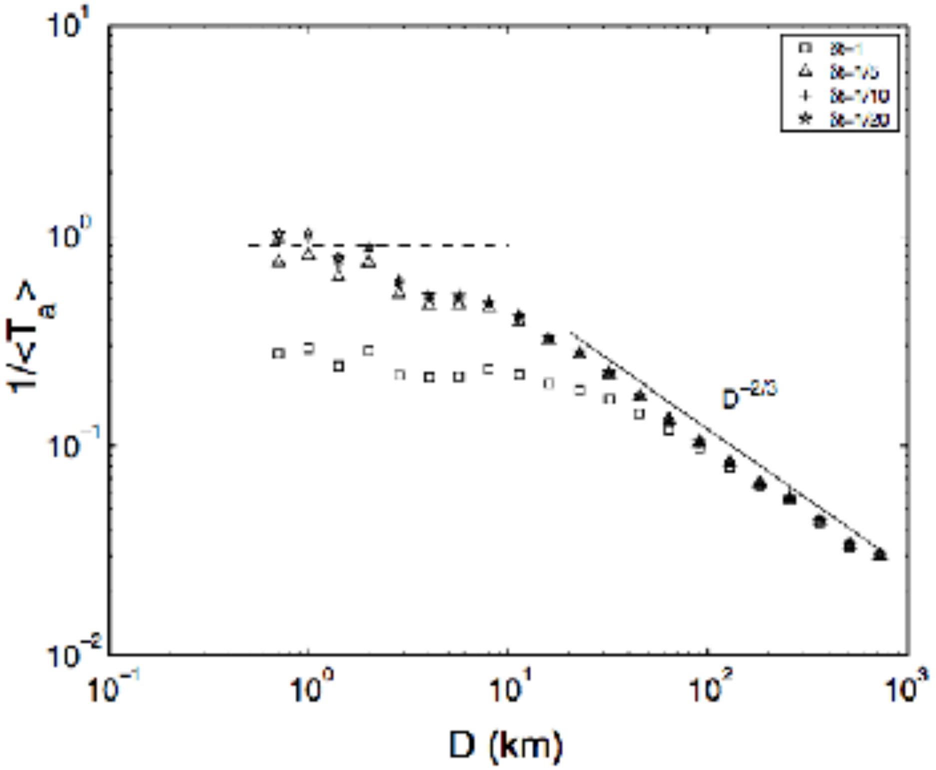
# FSLES

**Adriatic Sea FSLE**



Lacorata *et al.* (2001)

**SCULP FSLE**



LaCasce and Ohlmann (2003)

2 regimes with a constant FSLE at scales less than 50km and greater than 100 km.

FSLE constant at scales less than 10 km and $y^{-2/3}$ dependence at larger scales (Richardson regime).

# 3+ PARTICLES

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

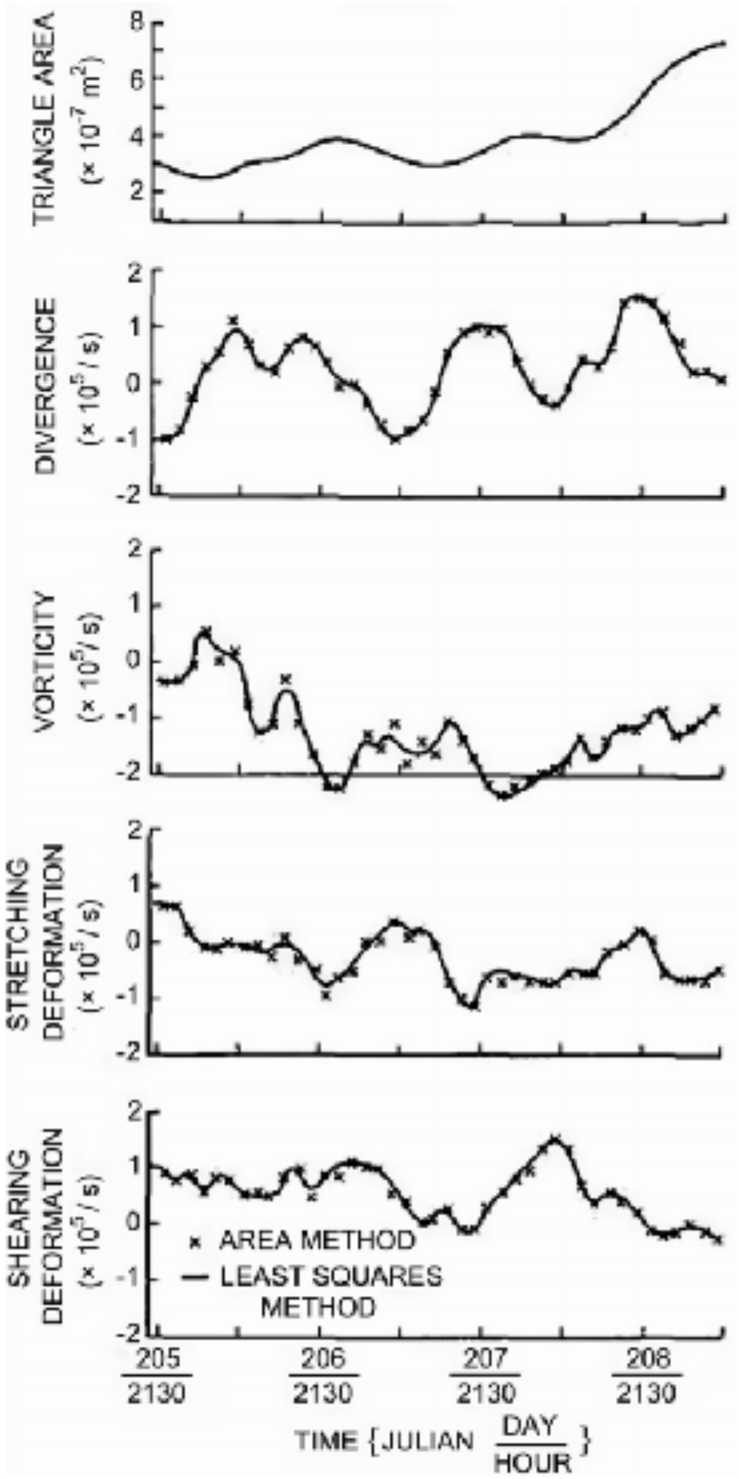
Surface divergence can be related to cluster area because if drifters are close together, the material inside the polygon formed by them will be conserved.

This can be related to vorticity by rotating the instantaneous velocity vectors of the drifters using the transformation:

$$u \rightarrow v', \quad v \rightarrow u' \quad \longrightarrow \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = \frac{1}{A'}\frac{\mathrm{d}A'}{\mathrm{d}t}$$

If you rotate both *u* and *v* then how does vorticity become a divergence?

# 3+ PARTICLES



Molinari and Kirwan (1975)

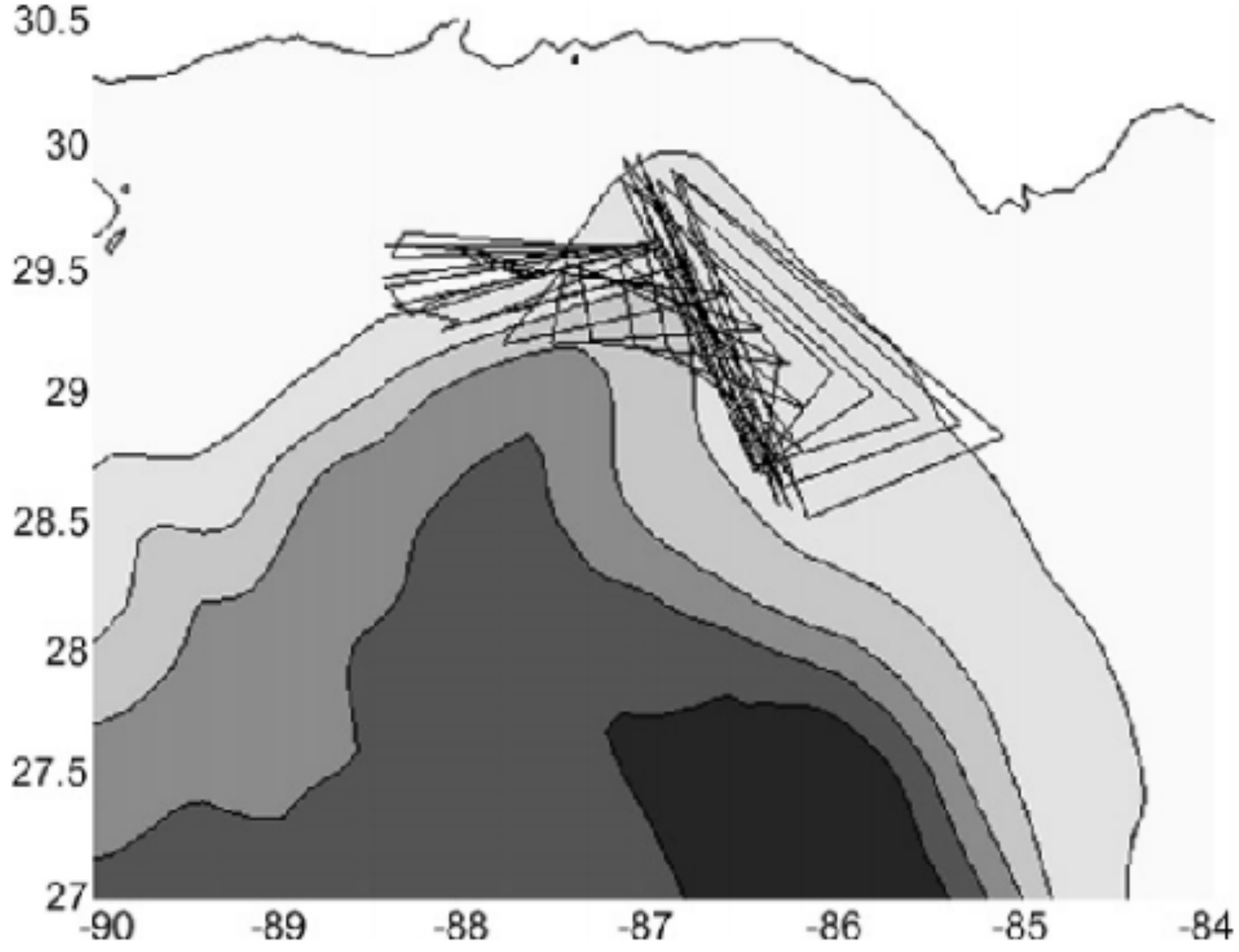
Example of calculated triangle area, divergence, vorticity, stretching and shearing deformations from a triplet of drifters.

**Lagrangian vorticity balance:**

$$\frac{\mathrm{d}}{\mathrm{d}t}(\zeta+f)+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\text{resid.}$$

*Molinari and Kirwan* (1975) evaluated the Lagrangian vorticity balance from drifter clusters. Residuals were large but did indicate that the terms on LHS were balancing.

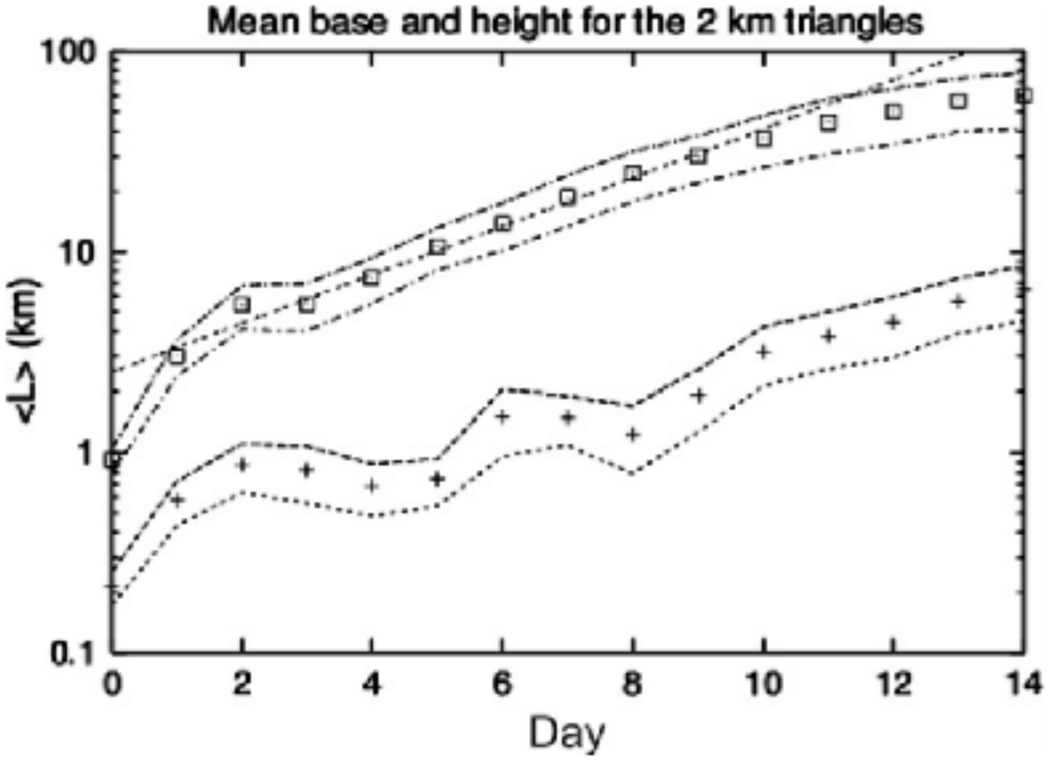
# NONDIVERGENT FLOW TRIANGLES



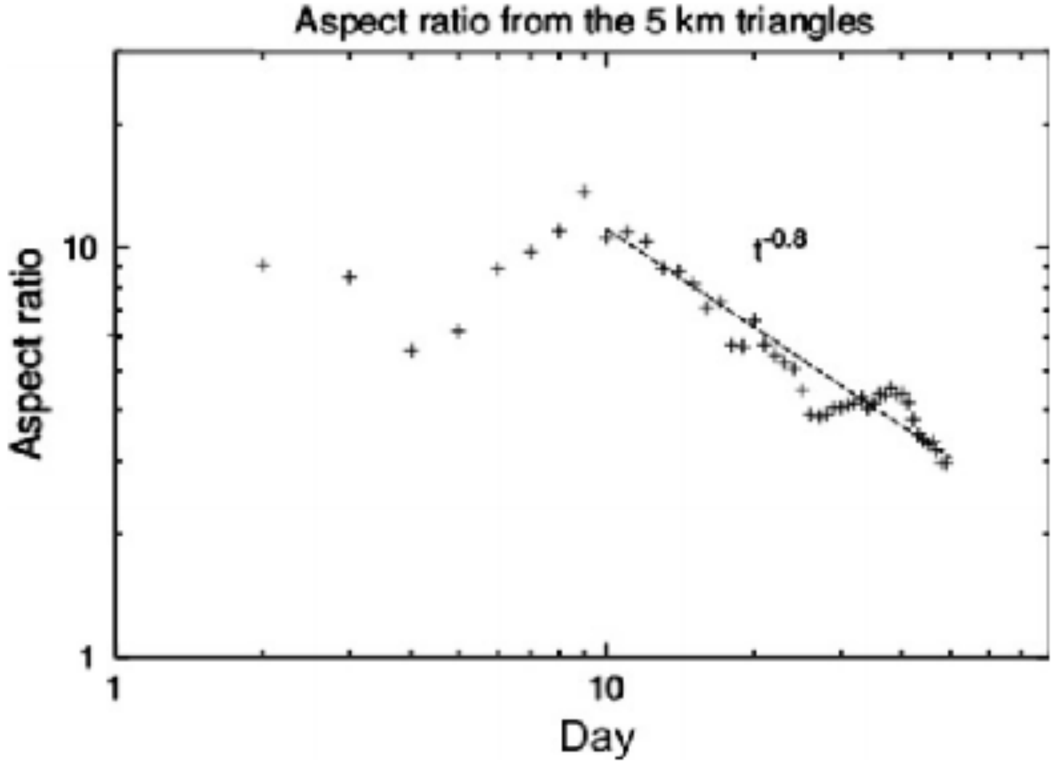
LaCasce and Ohlmann (2003)

If there's no divergence, the area of the triangle formed by 3 particles is conserved. Exponential growth of separation between 2 of the particles must be compensated by the third particle (triangle collapses onto a line). ~30 of these triangles in the SCULP drifter dataset.

# NONDIVERGENT FLOW TRIANGLES



Mean triangle base (longest leg) grew exponentially in time for first 10 days (consistent with mean pair dispersion). Triangle height also increases but not significantly different from 1 km for first 8 days.



Triangle aspect ratio (base divided by height) decreases, shifting toward a more equilateral shape. Why is this expected?

# CONCLUSIONS

- Relative dispersion between particle pairs can quantify how a tracer spreads. At small and large scales, relative dispersion behaves like absolute dispersion.
- If the slope of the energy spectrum E(k) is steep (e.g. enstrophy cascade for 2D turbulence: scales smaller than the deformation radius), pair separation increases exponentially in time.
- If the slope of the energy spectrum E(k) is between 1 and 3 (e.g. energy cascade for 2D turbulence: scales larger than the deformation radius), pair separation increases like $t^3$.
- Clusters of 3+ particles can be used to estimate divergence, vorticity, stretching and shearing deformations.