Fundamentals of Lagrangian Analysis and its applications

as told by Paola Cessi

Definition of a "particle"

The fluid is considered as a continuous field of particles, which are averages of many molecules of water (or air for the atmosphere).

The velocity at a point *x,y,z* and time *t* is the mass-weighted average over many molecules, with mass m_i and velocities $\mathbf{v}_i$, centered around the point (*x,y,z)*

$$\mathbf{v}(x,y,z,t) = \frac{\sum_i m_i \mathbf{v}_i}{\sum_i m_i}$$

The volume must be much smaller than the size of the fluid, i.e. several mm or more, but much larger than the separation between molecules, about 10^{-10} m.

The velocity the particles (or parcels) is then considered to obey the same laws of motion as a point mass (a molecule), with both position and velocity taking continuous (rather than discrete) values.

Eulerian vs. Lagrangian descriptions

In the Eulerian description the independent variables are *x,y,z* and time *t*. Dependent variables are **v***(x,y,x,t)*, *T(x,y,z,t)*, *S(x,y,z,t)*, *etc…*

In the Lagrangian description the independent variables are particle labels *(a,b,c)* and time τ=*t*. Dependent variables are *x(a,b,c, τ)*, *y(a,b,c, τ)*, *z(a,b,c, τ)*.

Particle labels vary continuously through the fluid, but they stay fixed as a particle moves through the fluid. They can be thought of as the initial position of the particle, which varies as you pick a different particle in the fluid, but is kept fixed for each particle. The time is denoted by τ because we keep *(a,b,c)* fixed when varying τ.

The connection between the time rate of change in the Lagrangian and Eulerian description is

$$\frac{\partial F}{\partial \tau} \equiv \frac{DF}{Dt} \quad \text{or} \quad \frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial t}\frac{\partial t}{\partial \tau} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \tau} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \tau},$$

By definition

$$\mathbf{v} \equiv \left(\frac{\partial x}{\partial \tau}, \frac{\partial y}{\partial \tau}, \frac{\partial z}{\partial \tau}\right) \equiv (u, v, w),$$

and thus

$$\frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = \frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F.$$

Mass conservation

Let's define the particle labels such that $dV_{abc} = da\,db\,dc = d(mass)$,

dV_{abc} is an infinitesimal volume in label-space enclosing the infinitesimal mass $d(mass)$

In Eulerian space we have

$d(mass) = \rho dV_{xyz}$, with $dV_{xyz} = dx\,dy\,dz$

The two volumes are connected by $$\frac{dV_{abc}}{dV_{xyz}} = \frac{\partial(a,b,c)}{\partial(x,y,z)},$$

and thus $$\rho = \frac{\partial(a,b,c)}{\partial(x,y,z)} \equiv \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}.$$

In the Boussinesq approximation ρ is constant,

and thus volume is conserved following a particle.

For an incompressible fluid, the infinitesimal volume assigned to a particle at t=0 is conserved
It can be assigned at t=0, and then conserved following the path

Determine the origin of waters in the upper limb of the AMOC

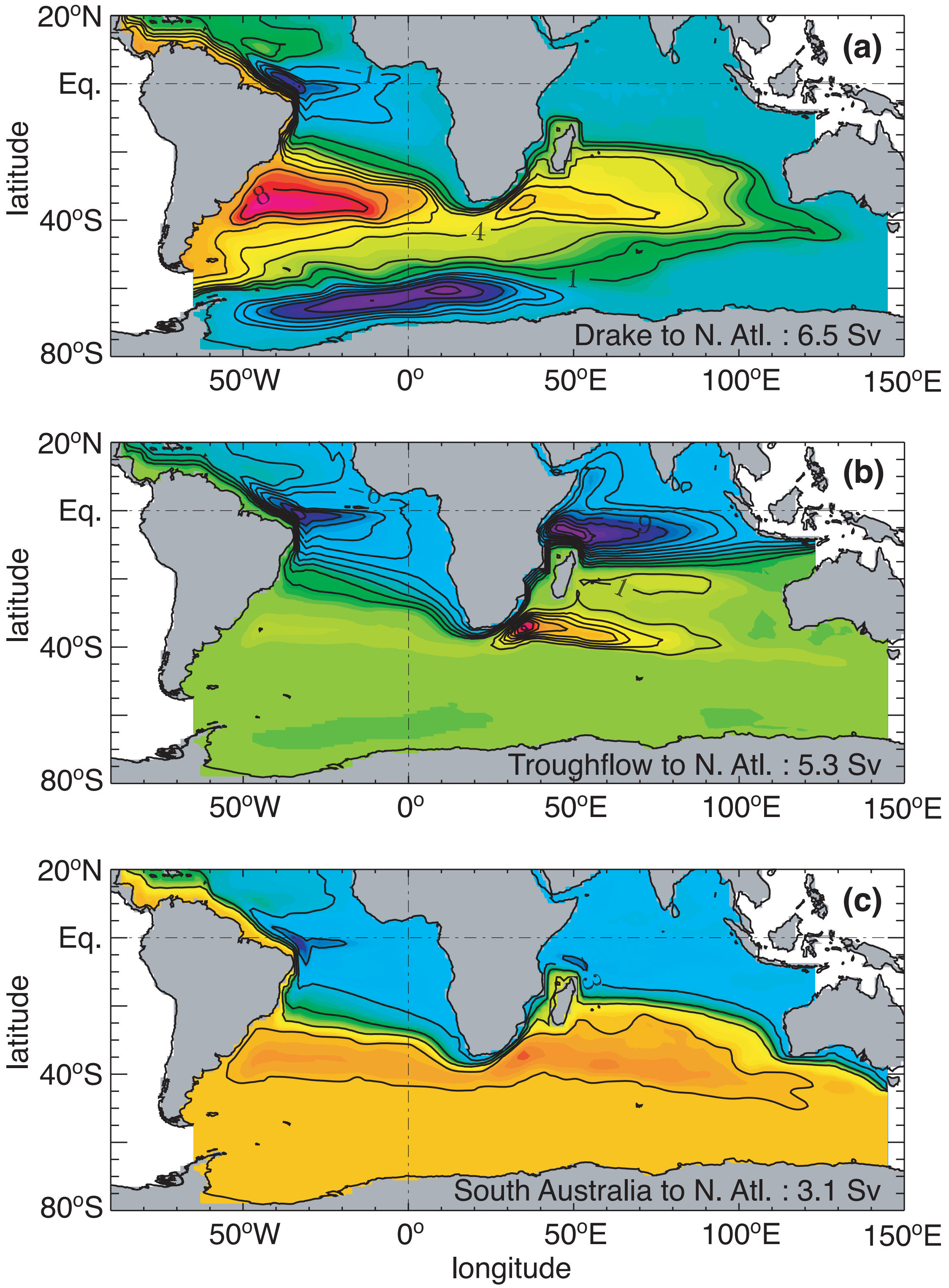


Plate 1. Horizontal streamfunction related to the vertically-integrated transport of the northward-transmitted warm waters to the North Atlantic (0-1200m) with origins a) in the Drake Passage b) in the Indonesian Throughflow c) South of Australia.

Trace particles trajectories backwards in time that cross 20°N, coming from outside the Atlantic (14.9 Sv)

(a) Particles originating from Drake Passage: 6.5Sv

(b) Particles originating in the Indonesian throughflow: 5.3Sv

(c) Particles originating between Australia and Antarctica at 145°E (not going through DP): 3.1Sv

A transport is associated with the initial particle position going through a grid cell of this 2°x2° model (ORCA), and then preserved during the particle trajectory. The model's T and S are relaxed to Levitus, as a crude form of data assimilation.

Travel times are fastest from Drake P. and Indonesian TF

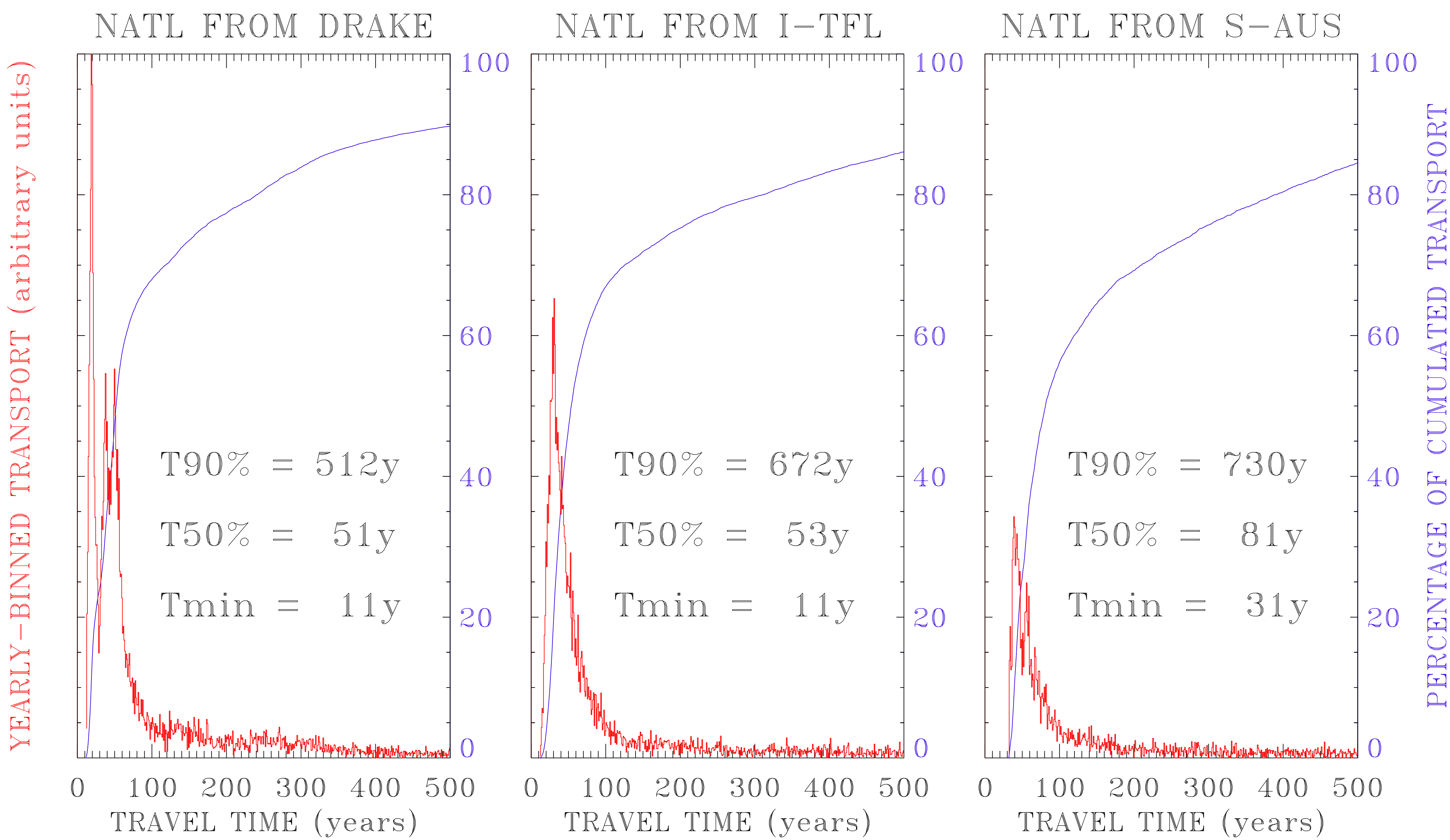
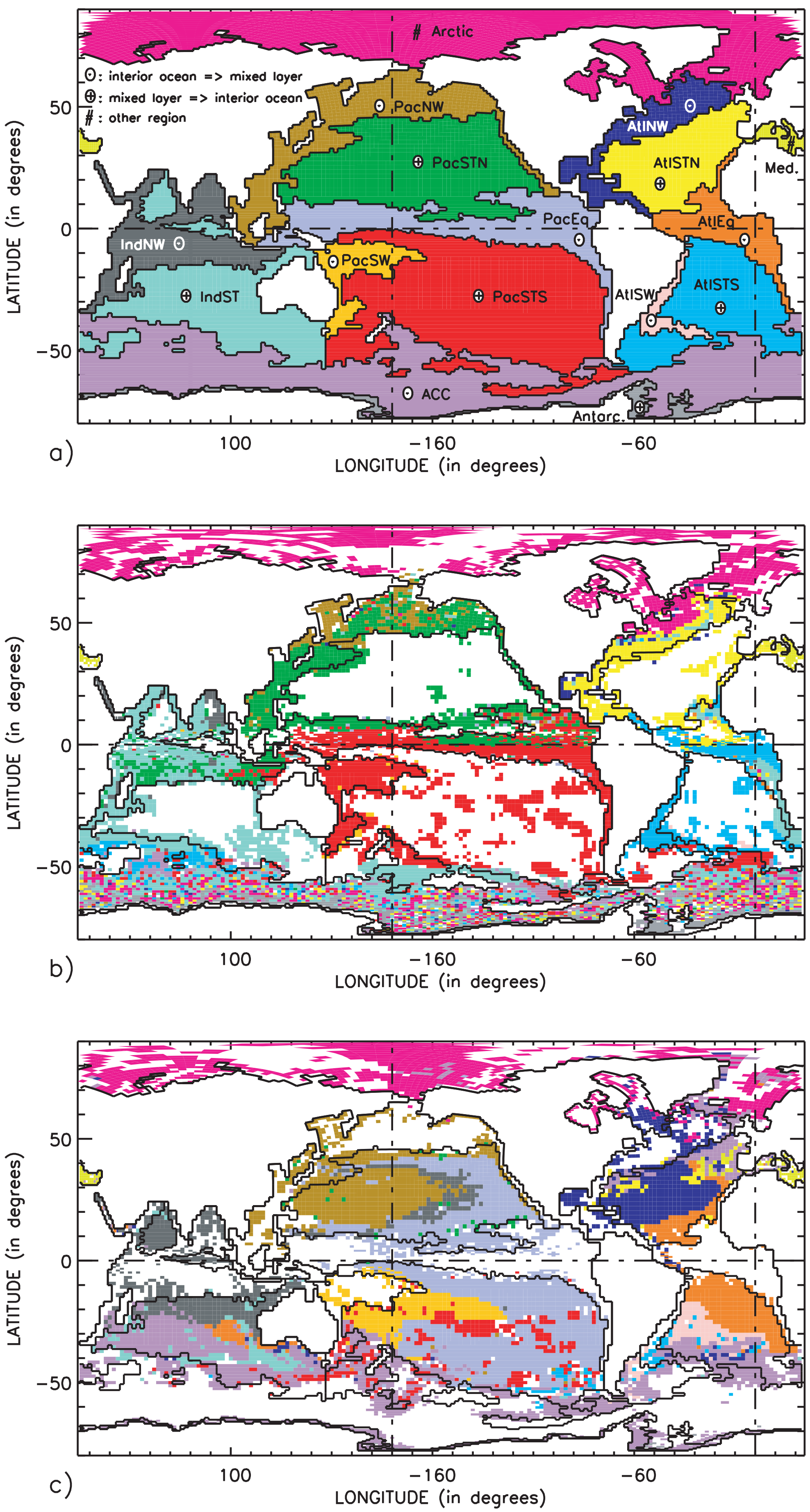


Plate 2. Yearly binned transport (in red) and time integrated transport (in blue) as function of travel times for the northward-transmitted warm waters to the North Atlantic (0-1200m) with origins a) in the Drake Passage b) in the Indonesian Throughflow c) South of Australia.

Determine the origin of ventilated waters



Trace particles trajectories escaping from and returning to the mixed layer in a 2°x2° model (OPA). The model's T and S are relaxed to Levitus, as a crude form of data assimilation.

(a) Color coding of the region of origin or of final destination.

(b) Water mass origin from interior to mixed layer: notice the rim of the supergyre taking S.Atl. water to Indian, and the interior N. H. subtropical gyres taking water to the WBC.

(c) Water mass destination: N.H. WBC waters ending in the subtropical gyres. and NH Indian ocean water ending at rim of N.Pac. subtropical gyre (gray coloring).

Time-scale for ventilation

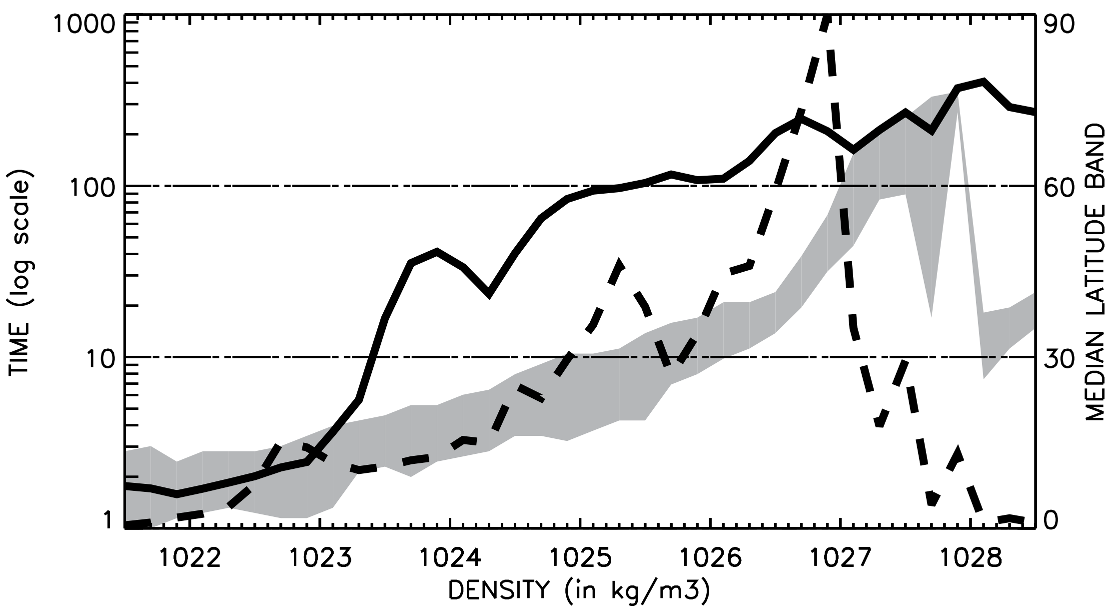


Figure 1. Mean renewal time scales in years (solid line, left-hand axis) as a function of surface density (in kg/m^3), with corresponding median bands of longitude (shaded area, right-hand axis) and ventilation fluxes (dashed line, arbitrary unit).

Bin the travel time from the mixed layer at a certain density until the water comes back to the mixed layer (somewhere else) (solid line).

Lightest waters (equatorial) have short transit times, and abyssal waters (high latitudes) have the longest times.

This analysis caps the longest travel times to 400 years (this would be the time of the abyssal overturning cell)

Ventilation rate: geographical distribution

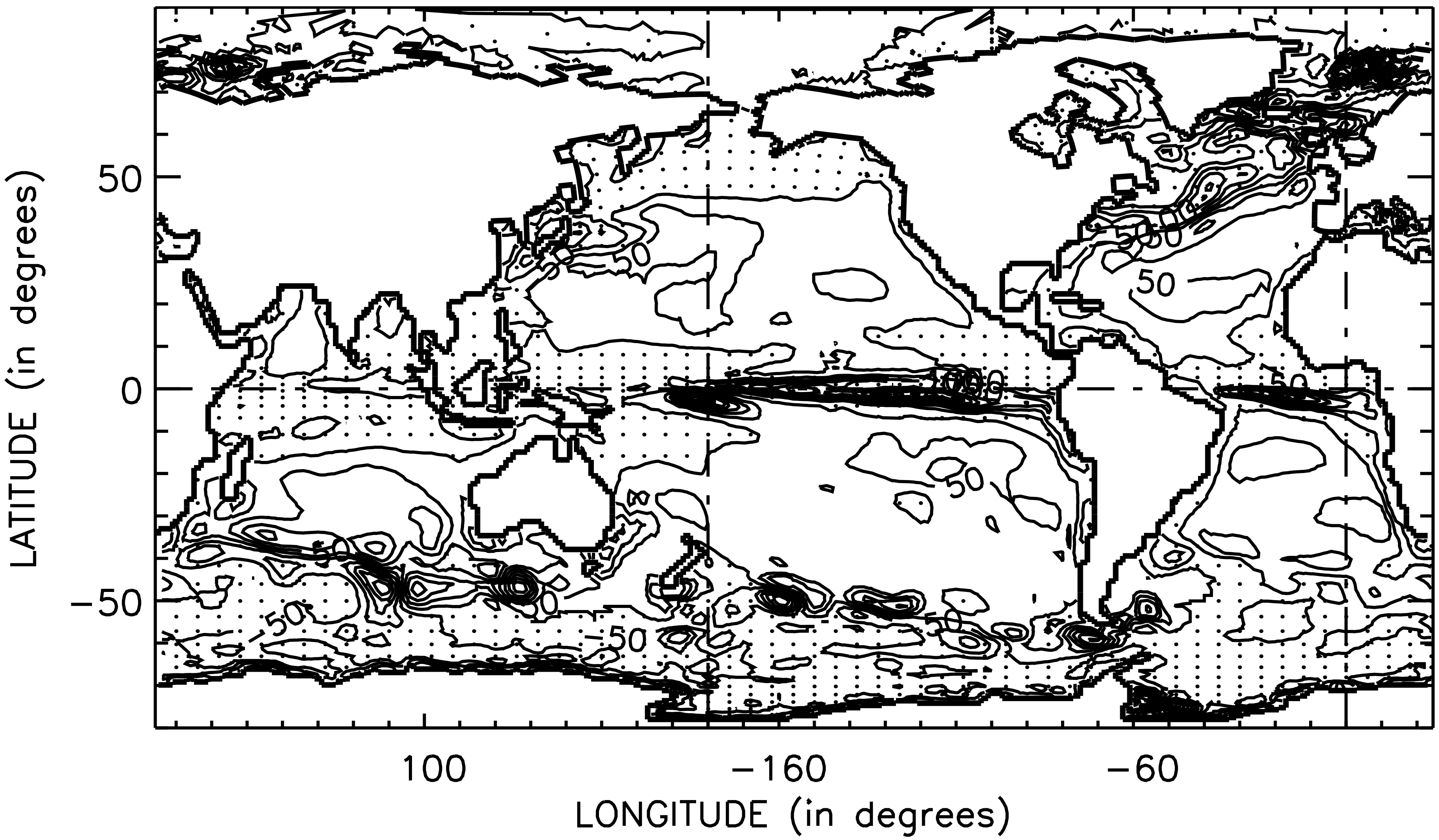


Figure 2. Net ventilation rate for the global ocean contoured with a 50 m/yr contour interval, as diagnosed from the initial and final positions of the trajectories documenting the ventilation process. Dotted areas refer to movements from the interior ocean to the surface mixed layer.

Divide the downward transport by the cell size to get the positive ventilation rate. Divide the upward transport by the cell size to get the upwelling rate (dotted lines). Contour size is 50m/yr

Updated origin of waters in the upper limb of the AMOC

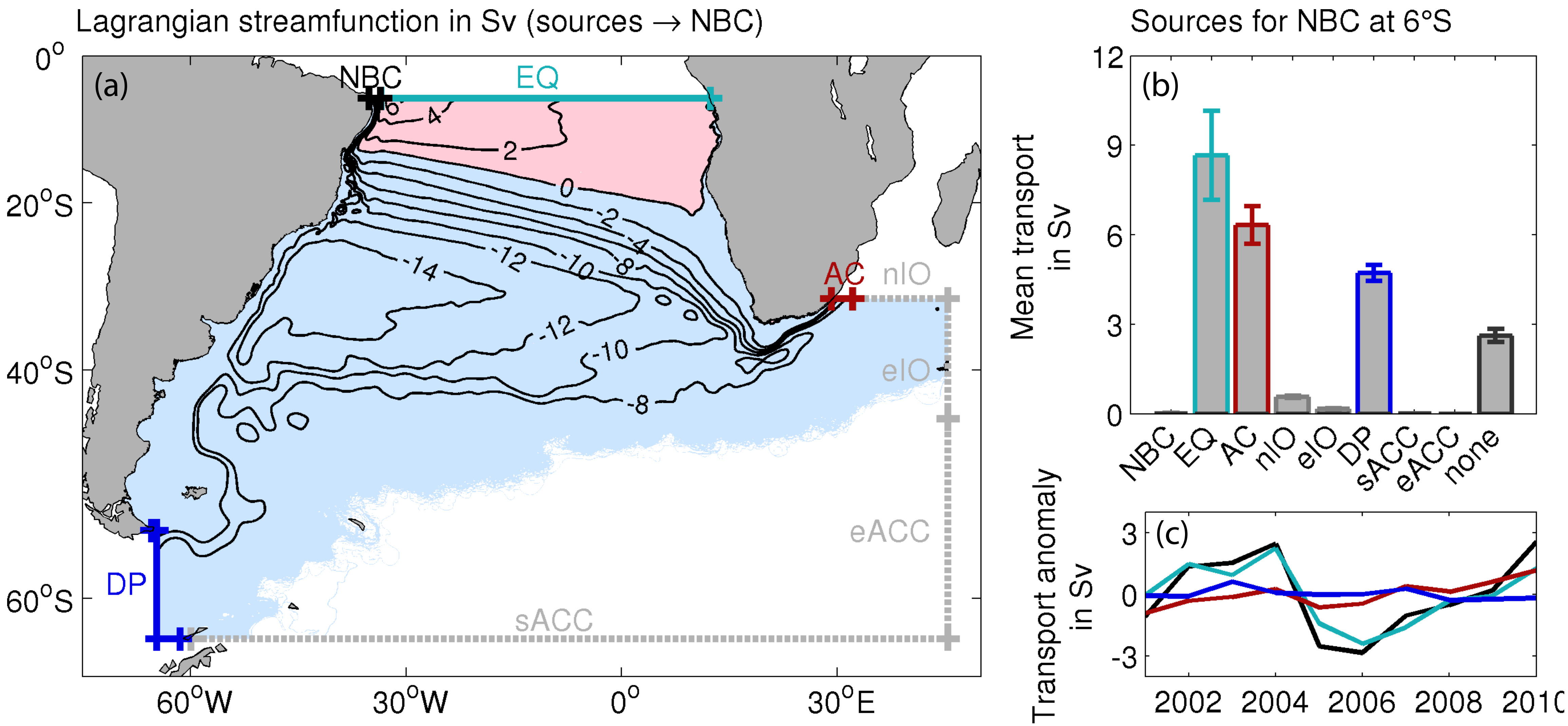


Figure 3. Sources for NBC transport inferred from the REF set of 10 Lagrangian experiments for which particles were released in the NBC at 6 °S every 5 days for years 2000 to 2009 and then traced backwards in time towards the indicated source sections for maximum 40 years. (a) Mean Lagrangian streamfunction representing volume transport pathways from all source sections towards the NBC. (b) Mean volumetric contributions of the individual sources to the NBC; whiskers indicate the range of transport estimates. (c) Timeseries of interannual variability of the total NBC transport (black line) and its individual contributions, that are, volumetric contribution of each Lagrangian experiment plotted against the respective release year (colored lines).

Use eddy-resolving model to trace back origin of 14.4Sv waters in the North Brazil Current (NBC): 6.3Sv from Aghulas Current (AC), 4.7Sv from Drake Passage (DP) and 0.8Sv from Tasman Leakage (eIO).