Statistics from Lagrangian observations (single particle analysis)

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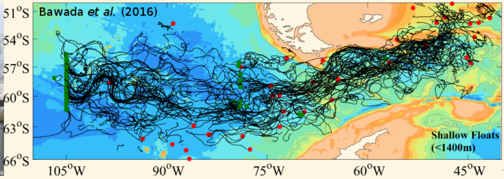

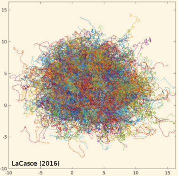

André Palóczy

PO Theory Seminar (SIOC219), Spring/2019

Today's discussion

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Review

Statistics from Lagrangian observations

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ABSTRACT

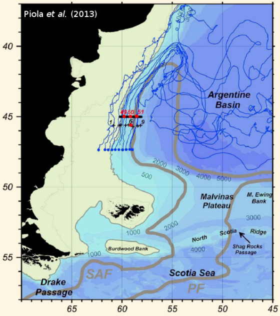
We review statistical analyses of Lagrangian data from the ocean. These can be grouped into studies involving single particles and those with pairs or groups of particles. Single particle studies are the most common. The prevalent analysis involves binning velocities geographically to estimate the Eulerian means and lateral diffusivities. However single particle statistics have also been used to study Rossby wave propagation, the influence of bottom topography and eddy heat fluxes. Other studies have used stochastic models to simulate dispersion, calculated Lagrangian frequency spectra and examined the relation between Lagrangian and Eulerian integral scales. Studies involving pairs of particles are fewer, and the results are not well-established yet. There are indications that pair separations grow exponentially in time below the deformation radius, as is also the case in the stratosphere. The behavior at larger scales is less clear, indicating either a turbulent cascade or dispersion by the sheared large-scale circulation. In addition, three or more particles can be used to measure relative vorticity and divergence.

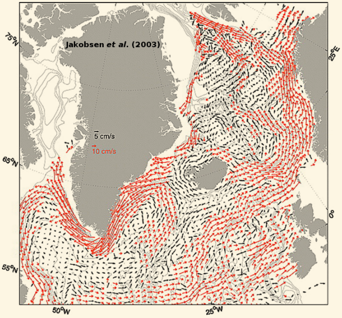
Contents

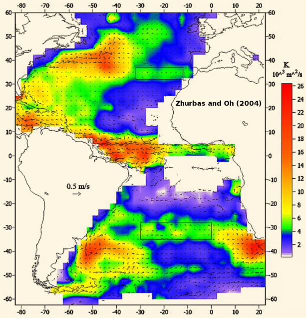
Outline

Some applications of single-particle Lagrangian analysis

- Visualization of parcel origin and fate;
- Mapping of the Eulerian mean flow and eddy kinetic energy;
- Estimation of Lagrangian diffusivities;
- Direct measurement of eddy fluxes, *e.g.*, $\overline{u'v'}$, $\overline{u'T'}$;

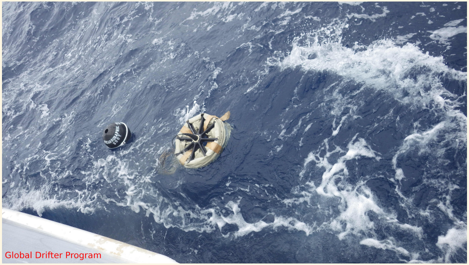


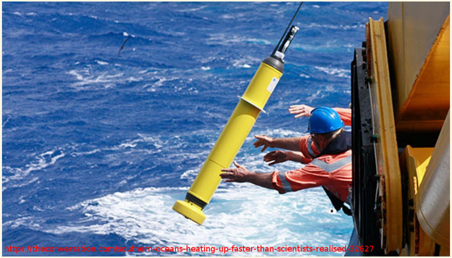




Instruments

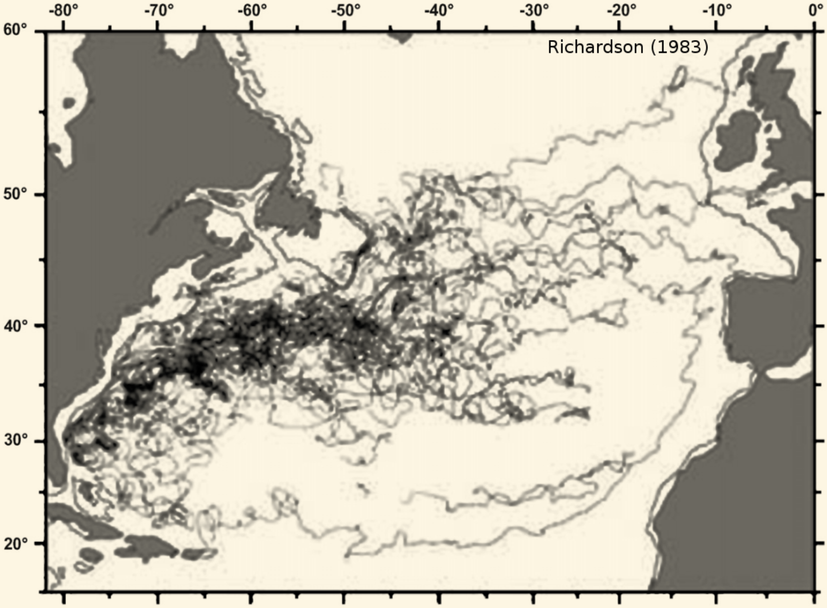

Photo: André Palóczy


Global Drifter Program


https://theconversation.com/southern-oceans-heating-up-faster-than-scientists-realised-32627

“Drifters float, floats sink”

Example: Drifter trajectories in the North Atlantic



Statistics of point clouds – metrics

- Center of mass' displacement

$$M_x(t) = \frac{1}{N}\sum_{i=1}^{N}[x_i(t) - x_i(0)]$$

- Cloud variance (a measure of relative dispersion/"cloud size")

$$D_x(t) = \frac{1}{N-1}\sum_{i=1}^{N}[x_i(t) - x_i(0) - M_x(t)]^2$$

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- Cloud variance in terms of the relative particle positions $x_i(t) - x_j(t)$ only:

$$D_x(t) = \frac{1}{2N(N-1)}\sum_{i\neq j}[x_i(t) - x_j(t)]^2$$

Statistics of point clouds – metrics

- Skewness (asymmetry, a normal distribution has sk = 0)

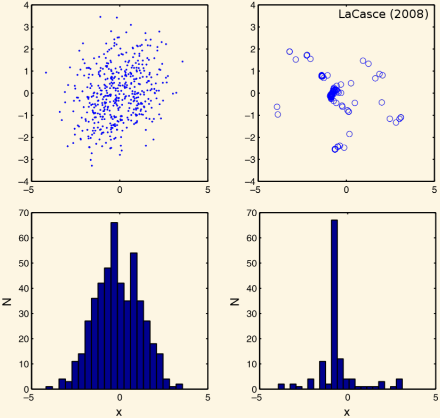
$$\mathrm{sk}(t) = \frac{\sum_i (x_i - x_i)^3}{\left(\sum_i (x_i - x_i)^2\right)^{\frac{3}{2}}}$$

- Kurtosis (tail length, a normal distribution has ku = 3)

$$\mathrm{ku}(t) = \frac{\sum_i (x_i - x_i)^4}{\sum_i (x_i - x_i)^2}$$

Statistics of point clouds – examples

•Advection by a random walk vs. by a realistic 2D turbulent flow:



•With assumptions (stationarity/homogeneity) all moments can be derived from the displacement PDF $Q(X, t)$.

Outline

Single-particle statistics – Single-particle PDFs

•Consider a particle located at $\boldsymbol{x_0}$ at t_0. Later, at $t = t_1$, the probability P of finding it at $\boldsymbol{x} = \boldsymbol{x_1}$ is

$$P(\boldsymbol{x_1}, t_1) = \int P(\boldsymbol{x_0}, t_0) Q(\boldsymbol{x_1}, t_1 | \boldsymbol{x_0}, t_0) \, d\boldsymbol{x_0},$$

where Q is the single-particle displacement PDF.

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•If the flow is statistically **homogeneous**, only the spatial lag $X \equiv x_1 - x_0$ matters, not x_1 and x_0:

$$Q(\boldsymbol{x_1}, t_1 | \boldsymbol{x_0}, t_0) = Q(X, t_1, t_0)$$

•If the flow is statistically **stationary**, only the temporal lag $t \equiv t_1 - t_0$ matters, not t_1 and t_0:

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•If both homogeneous and stationary:

$$\boxed{Q(\boldsymbol{x_1}, t_1 | \boldsymbol{x_0}, t_0) = Q(X, t)}$$

Single-particle statistics – Definitions

$\longrightarrow$If the flow is homogeneous and stationary and $Q(X,t)$ is known:

•First moment $\longrightarrow$ **mean displacement**:

$$\overline{X}(t) = \int XQ(X,t)\,dX.$$

•Second moment $\longrightarrow$ **Single-particle absolute dispersion**:

$$\overline{X^2}(t) = \int X^2Q(X,t)\,dX.$$

•Discussion: What about the physical interpretation of **higher moments** (*e.g.*, skewness, kurtosis)?

Single-particle statistics – Definitions

•The absolute dispersion $\overline{X^2}(t)$ can be written in terms of ν and $R(\tau)$ instead of X and Q:

$$\overline{X^2} = 2\nu^2 \int_0^\infty (t-\tau)R(\tau)\, d\tau,$$

where ν is the RMS velocity and $R(\tau)$ is the normalized velocity autocorrelation.

•$\overline{X^2}$ first grows quadratically, then linearly.

Single-particle statistics – Definitions

• The diffusivity κ represents how fast particles disperse:

$$\kappa(t) \equiv \frac{1}{2}\frac{d}{dt}\overline{X^2} = \overline{X(t)u(t)} = \int_0^t \overline{u(X,t)u(X,\tau)}\, d\tau \underbrace{= \nu^2 \int_0^t R(\tau)\, d\tau,}_{\text{if flow is stationary}}$$

Single-particle statistics – Definitions

•The Lagrangian integral timescale T_L measures the characteristic decorrelation time for the particle velocities:

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•The Lagrangian integral timescale T_L measures the characteristic decorrelation time for the particle velocities:

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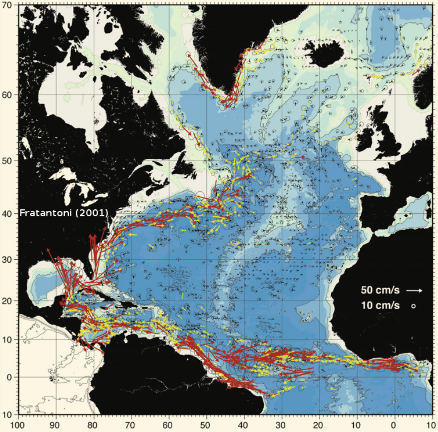
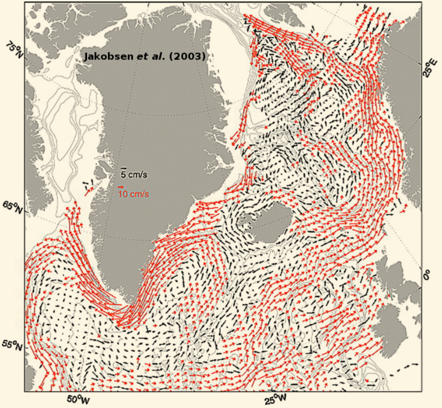
•The Lagrangian frequency spectrum $L(\omega)$ and $R(\tau)$ are Fourier transform pairs:

$$L(\omega) \equiv 2\nu^2 \int_0^{\infty} R(\tau) \cos(2\pi\omega\tau)\, d\tau$$

Outline

Eulerian mean flow from bin-averaged Lagrangian velocities





- Caveats:
  - Uneven drifter coverage (array bias)
  - Statistical significance

Diffusivity mapping

- The diffusivity *tensor* κ_{jk} is (Davis, 1991)

$$\kappa_{jk}(\boldsymbol{x}, t) = \int_{-t}^{0} \langle u'_j(t_0|\boldsymbol{x}, t_0) u'_k(t_0 + \tau|\boldsymbol{x}, t_0)\rangle \, d\tau, \quad u'(t_0) = u(t_0) - U(\boldsymbol{x}),$$

where $U(\boldsymbol{x})$ is the time-mean flow.

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where $U(\boldsymbol{x})$ is the time-mean flow.

•In terms of the residual displacements $d'(t) = d(t) - d_m(\boldsymbol{x}, t)$:

$$\kappa_{jk}(\boldsymbol{x}, t) = -\langle u'_j(t_0|\boldsymbol{x}, t_0) d'_k(t_0 - t|\boldsymbol{x}, t_0)\rangle$$

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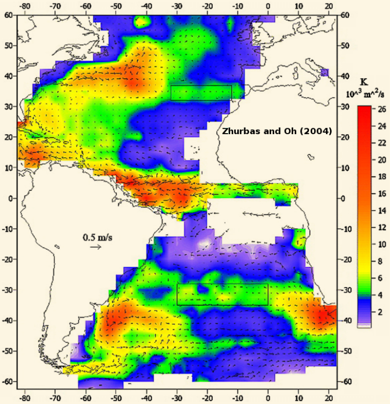

Diffusivity – sensitivity to choice of mean

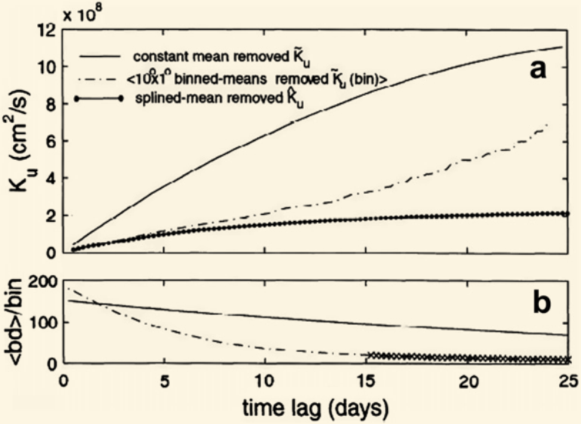

Fig. 6. The diffusivity plotted against lag from drifter data in a region of the tropical Pacific. Three different mean fields were used to calculate the residual velocities: a constant one, one obtained from averaging in $10^\circ \times 1^\circ$ rectangles and one derived from spline-fitting. The latter method produces the best convergence. From Bauer et al. (1998), with permission.

- Results are sensitive to the definition of mean velocity field.

Outline

Stochastic models

Order	Stochastic variable	Model equations	Fokker-Planck equation
0^{th}	position	$\frac{dx_i}{dt} = U_i + \sqrt{2}\nu_i \frac{dw_i}{dt}$	$\partial_t P + \boldsymbol{U} \cdot \boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot (\boldsymbol{\kappa} \boldsymbol{\nabla} P)$

- P is the position PDF. $P = P(\boldsymbol{x}, t)$
- x_i, u_i and a_i are position, velocity and acceleration.
- U_i is the time-mean background velocity.
- $T_i = T_{vi}$ and T_{ai} are velocity and acceleration integral timescales.
- $\boldsymbol{\nabla}_{\boldsymbol{u}} = \hat{\boldsymbol{u}}\partial_u + \hat{\boldsymbol{v}}\partial_v$ is the gradient operator in *velocity* space.
- In the 2^{nd} order model, the F-P eqn. would have diffusion in $\boldsymbol{x}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ spaces.

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1^{st}	velocity	$\frac{dx_i}{dt} = U_i + u_i$ $\frac{du_i}{dt} = -u_i/T_i + \sqrt{\frac{2}{T_i}}\nu_i \frac{dw_i}{dt}$	$\partial_t P + (\boldsymbol{U} + \boldsymbol{u}) \cdot \boldsymbol{\nabla} P = \boldsymbol{\nabla_u} \cdot (\boldsymbol{u} P/T) + \boldsymbol{\nabla_u} \cdot (\boldsymbol{\kappa} \boldsymbol{\nabla_u} P)$

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2^{nd}	acceleration	$\frac{dx_i}{dt} = U_i + u_i$ $\frac{du_i}{dt} = a_i - u_i/T_{vi}$ $\frac{da_i}{dt} = -a_i/T_{ai} + \left(\frac{2(T_{ai}+T_{vi})}{T_{ai}T_{vi}}\right)\nu_i \frac{dw_i}{dt}$	?

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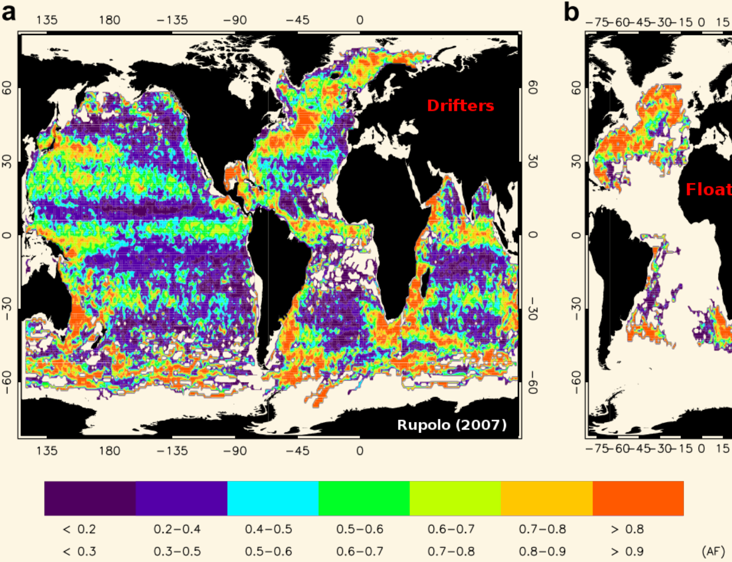
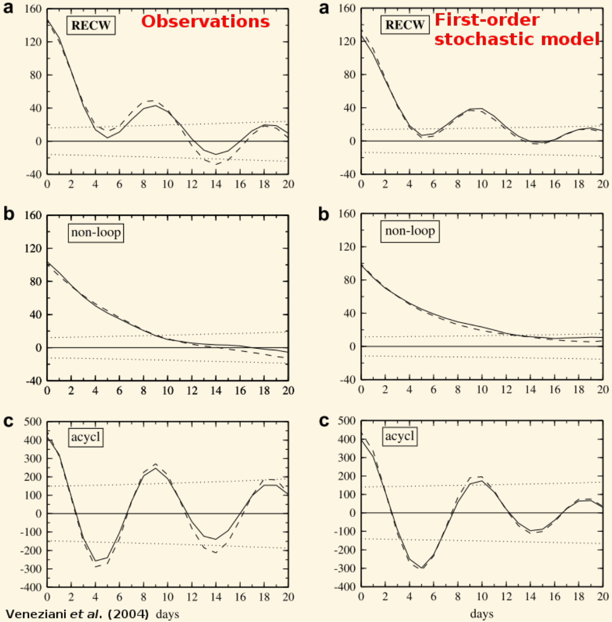
- In the 2^{nd} order model, the F–P eqn. would have diffusion in $\boldsymbol{x}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ spaces.

Acceleration (T_{ai}) and velocity (T_{vi}) timescales

- The ratio T_{ai}/T_{vi} is closer to unity in energetic regions.



Velocity autocorrelations

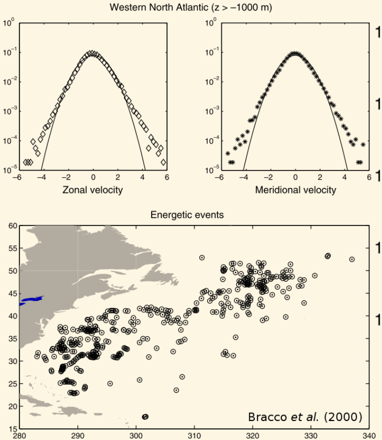


Veneziani *et al.* (2004)

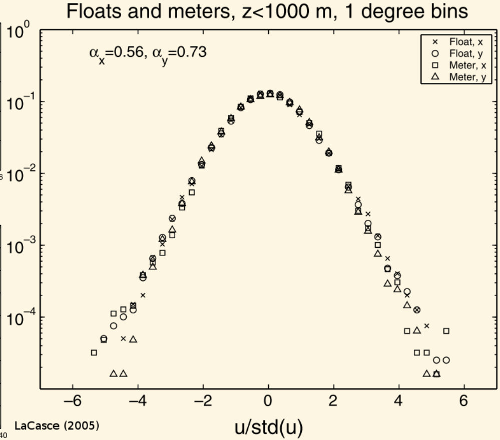
Outline

Velocity PDFs in the ocean

- What do Lagrangian velocity PDFs look like in the ocean?
  - California Current: Gaussian away from coast (Swenson & Niiler, 1996)
  - North Atlantic: Non-Gaussian, longer tails during energetic events [Bracco *et al.* (2000)]



Bracco *et al.* (2000)



LaCasce (2005)

Outline

Lagrangian particles and the Taylor–Proudman constraint

- Taking curl of the the depth-averaged linear momentum equation and defining a transport streamfunction $(U, V) \equiv (-\psi_y, \psi_x)$:

$$\partial_t \zeta + J\left(\psi, \frac{f}{H}\right) = \nabla \times \left(\frac{\boldsymbol{\tau_s} - \boldsymbol{\tau_b}}{\rho H}\right)$$

Lagrangian particles and the Taylor–Proudman constraint

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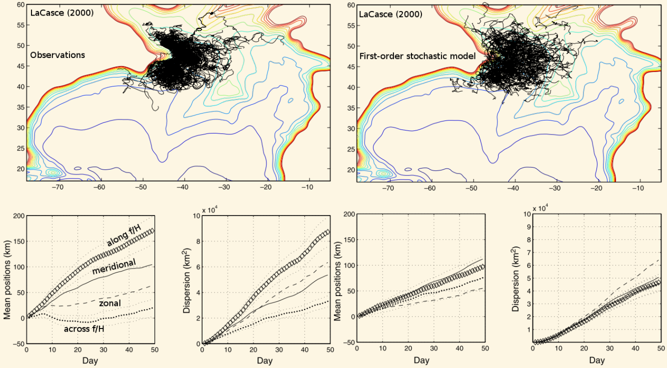


- Floats preferentially displace and spread *along* f/H rather than across.

Outline

Correlations with scalar fields and eddy fluxes (*e.g.*, temperature)

$$\partial_t T + \nabla \cdot (\overline{\boldsymbol{U}}\,\overline{T}) + \nabla \cdot (\overline{\boldsymbol{U}'T'}) = \text{sources} + \text{mixing}$$

•$\overline{\boldsymbol{U}'T'}$ can be estimated with drifter data.
•Example of an application: Experimental testing of diffusion parameterizations based on mean gradients:

$$\nabla \cdot (\overline{\boldsymbol{U}'T'}) = -\nabla \cdot (\kappa \nabla \overline{T})$$

- Accurate in the California Current System (Swenson and Niiler, 1996);
- Not accurate in the Southern Ocean (Gille, 2003).

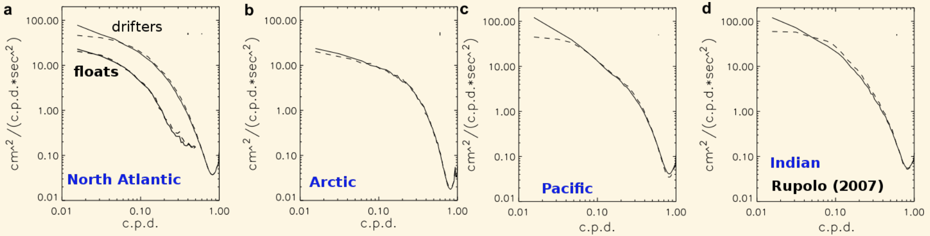
Outline

Frequency spectra

•Fourier-transforming an exponentially-decaying autocorrelation with timescale T_L predicts a flat spectrum at low ω and a ω^{-2} power-law dependence at high ω:

$$u(\omega), v(\omega) = \int_0^\infty e^{-t/T_L} cos(2\pi\omega t)\, dt = \frac{2T_L^{-1}}{T_L^{-2} + 4\pi^2\omega^2} =$$

$$\frac{\Omega_L}{2\pi\omega^2}\left(\frac{1}{1+\Omega_L^2/\omega^2}\right) = \frac{\Omega_L}{2\pi\omega^2}\left(1 - \Omega_L^2/\omega^2 + \Omega_L^4/\omega^4 + \cdots\right)$$



Outline

Eulerian vs. Lagrangian scales

- How are Lagrangian and Eulerian time/length scales related?

$$\frac{T_L}{T_E} \approx \frac{q}{(q^2 + T_E^2/T_{\mathrm{adv}}^2)^{\frac{1}{2}}}. \qquad \text{Middleton (1985)}$$

$T_{\mathrm{adv}} \equiv L_E/\nu, \quad q \equiv \sqrt{\pi/8}, \quad \nu \equiv$ rms velocity

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•Also, with $L_L = \nu T_L$,

$$\frac{L_L}{L_E} \approx \frac{T_L}{T_E} \times \frac{T_E}{T_{\text{adv}}}$$

Eulerian vs. Lagrangian scales

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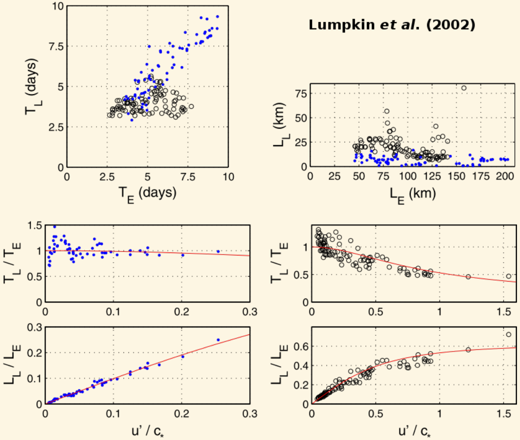

- Middleton's (1985) simple prediction tests well against observations.

Eulerian vs. Lagrangian scales

- Consider the nondimensional advection-diffusion equation for a scalar S:

$$\underbrace{\frac{\hat{S}}{T_E}\partial_{t'}S'}_{❶} + \underbrace{\frac{\nu\hat{S}}{L_E}\boldsymbol{u}'\cdot\boldsymbol{\nabla}'S'}_{❷} = 0$$

- The ratio ❷/❶ scales as $\nu T_E/L_E = T_E/T_{\text{adv}} = \alpha$.

- If $T_E = T_{\text{adv}}$, $\alpha = 1$.
- If $\alpha \ll 1$, mean flow is sluggish. Eddies determine scales $\rightarrow$ "fixed float" limit.
- If $\alpha \gg 1$, mean flow is fast and carries eddies $\rightarrow$ "frozen turbulence" limit.

Does diffusivity scale with other variables?

- Two hypotheses:
  - Eddy kinetic energy, $\boxed{\kappa \propto \nu^2 T}$: If T is constant ("Fixed–float" regime).
  - RMS velocity, $\boxed{\kappa \propto \nu L}$: If L is constant ("frozen turbulence" regime).