3 Many waves

In the previous lecture, we considered the case of two basic waves propagating in one horizontal dimension. However, Proposition #2 lets us have as many basic waves as we want. Suppose we want to have \( N \) waves. If \( N \) waves are present, the surface elevation takes the form

\[
\eta(x,t) = \sum_{i=1}^{N} \hat{A}_i \cos(k_i x - \omega(k_i) t + \alpha_i) \tag{3.1}
\]

where the \( \hat{A}_i, k_i, \) and \( \alpha_i \) represent \( 3N \) arbitrary parameters. We put the hats on \( \hat{A}_i \) so we can use un-hatted \( A_i \) for something else. As always, the \( \omega(k_i) \) are determined by the dispersion relation,

\[
\omega(k_i) = \sqrt{gk_i \tanh(k_i H)} \tag{3.2}
\]

and are always positive. If we define

\[
\theta_i = k_i x - \omega(k_i) t \tag{3.3}
\]

then (3.1) can be written in the form

\[
\eta(x,t) = \sum_{i=1}^{N} \hat{A}_i \cos(\theta_i + \alpha_i) = \sum_{i=1}^{N} \hat{A}_i \cos \alpha_i \cos(\theta_i) - \hat{A}_i \sin \alpha_i \sin \theta_i \tag{3.4}
\]

Then, defining

\[
A_i = \hat{A}_i \cos \alpha_i \quad \text{and} \quad B_i = -\hat{A}_i \sin \alpha_i \tag{3.5}
\]

we have

\[
\eta(x,t) = \sum_{i=1}^{N} A_i \cos(k_i x - \omega(k_i) t) + B_i \sin(k_i x - \omega(k_i) t) \tag{3.6}
\]

The form (3.6) is equivalent to (3.1), but it is sometimes more useful. The arbitrary parameters in (3.6) are \( A_i, B_i, \) and \( k_i \). In both (3.1) and (3.6), the sign of each \( k_i \) determines the propagation direction of the corresponding wave.

If you have studied mechanics, then you know that the evolution of any mechanical system is determined by: (a) Newton’s laws of motion, and (b) the location and velocity of every component of the system at some initial time, say \( t=0 \). The corresponding statement for us is that \( \eta(x,t) \) is determined by: (a) Propositions #1 and #2, and (b) the initial conditions \( \eta(x,0) \) and \( \partial \eta / \partial t(x,0) \). Suppose the latter are given to be \( F(x) \) and
$G(x)$ respectively. Then the solution of the initial value problem corresponds to choosing the arbitrary parameters in (3.6) to satisfy

$$\eta(x,0) = F(x) \quad \text{and} \quad \frac{\partial \eta}{\partial t}(x,0) = G(x)$$  \hspace{1cm} (3.7)

We can do this with $N$ waves if we can choose the wavenumbers and wave amplitudes to satisfy the equations

$$F(x) = \sum_{i=1}^{N} A_i \cos(k_i x) + B_i \sin(k_i x)$$  \hspace{1cm} (3.8)

and

$$G(x) = \sum_{i=1}^{N} A_i \omega(k_i) \sin(k_i x) - B_i \omega(k_i) \cos(k_i x)$$  \hspace{1cm} (3.9)

Is it ever possible to do this? If so, how many waves are needed?

Well, you might get lucky. For particularly simple $F(x)$ and $G(x)$, it might turn out that you need only a few waves. For example, if $F(x) = 5 \cos(3x)$ and $G(x) = 5 \omega(3) \sin(3x)$, then you need only a single wave. However, we are interested in the general case, in which $F(x)$ and $G(x)$ are completely arbitrary; they can be anything whatsoever. It turns out that, in the general case, you need every possible wavenumber. Thus, the general solution takes the form

$$\eta(x,t) = \int_{-\infty}^{+\infty} dk \left[ A(k) \cos(kx - \omega(k)t) + B(k) \sin(kx - \omega(k)t) \right]$$  \hspace{1cm} (3.10)

in which the sum in (3.6) has been replaced by an integral. The quantities $A(k)dk$ and $B(k)dk$ in (3.10) are analogous to the quantities $A(k_i)$ and $B(k_i)$ in (3.6). The initial value problem (3.7) is solved if we can find functions $A(k)$ and $B(k)$ that satisfy

$$F(x) = \int_{-\infty}^{+\infty} dk \left[ A(k) \cos(kx) + B(k) \sin(kx) \right]$$  \hspace{1cm} (3.11)

and

$$G(x) = \int_{-\infty}^{+\infty} dk \left[ A(k) \omega(k) \sin(kx) - B(k) \omega(k) \cos(kx) \right]$$  \hspace{1cm} (3.12)

To find these $A(k)$ and $B(k)$, we make use of a very powerful theorem in mathematics.
Fourier’s theorem. For almost any function $f(x)$,

$$f(x) = \int_{0}^{\infty} dk \left[ a(k) \cos(kx) + b(k) \sin(kx) \right]$$

(3.13)

where

$$a(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, f(x) \cos(kx) \quad \text{and} \quad b(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, f(x) \sin(kx)$$

(3.14)

for all positive $k$.

In essence, Fourier’s theorem says that you can express any function $f(x)$ as the sum of sines and cosines, provided that you assign the right weight to each sine and each cosine; (3.14) tells you how to assign the weights. If you are given $f(x)$, then $a(k)$ and $b(k)$ are determined by (3.14). Conversely, if you are given $a(k)$ and $b(k)$, then $f(x)$ is determined by (3.13). The functions $a(k)$ and $b(k)$ are said to be the Fourier transform of $f(x)$. The function $f(x)$ is the inverse transform of $a(k)$ and $b(k)$. The transform and its inverse transform constitute a Fourier transform pair. Either member of the pair determines the other member.

You might be wondering why the transform contains two functions, $a(k)$ and $b(k)$, while $f(x)$ is only one function. That seems unfair! The reason is that $a(k)$ and $b(k)$ are defined only for positive $k$, while $f(x)$ is defined for both positive and negative $x$.

This is a good place to say that Fourier’s theorem can be stated in a great many equivalent but somewhat dissimilar forms. For example, instead of (3.13) many books write

$$f(x) = \int_{-\infty}^{\infty} dk \, \hat{a}(k) e^{ikx}$$

(3.15)

where $\hat{a}(k)$ is complex. In fact, if you look up Fourier’s theorem in a math book, you are probably more likely to find (3.15) than (3.13). In this course, we will use (3.13-14) and we will not confuse things by discussing other forms. Your math courses will teach you all the delicate points about Fourier analysis. The goal here is to develop your physical intuition.

We shall “prove” Fourier’s theorem by showing that it holds in one particular case, and then invite you to test any other cases that you like. Our test case will be the function

$$f(x) = e^{-\beta x^2}$$

(3.16)
where $\beta$ is a positive constant. First we use (3.14) to calculate the weights. We find that

$$a(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, e^{-\beta x^2} \cos(kx) = \frac{2}{\pi} \int_{0}^{\infty} dx \, e^{-\beta x^2} \cos(kx) = \frac{1}{\sqrt{\pi \beta}} e^{-k^2/4\beta}$$  \hspace{1cm} (3.17)

and

$$b(k) = 0.$$  \hspace{1cm} (3.18)

In working out (3.17) we have used the general formula

$$\int_{0}^{\infty} e^{-q x^2} \cos(px) \, dx = \frac{1}{2} \sqrt{\frac{\pi}{q}} e^{-p^2/4q}$$  \hspace{1cm} (3.19)

which can be found in any table of definite integrals. The result (3.18) follows quickly from the fact that the product of an even function and an odd function is an odd function.

If Fourier’s theorem is correct, then it must by (3.13) be true that

$$e^{-\beta x^2} = \frac{1}{\sqrt{\pi \beta}} \int_{0}^{\infty} dk \, e^{-k^2/4\beta} \cos(kx)$$  \hspace{1cm} (3.20)

We leave it to you to verify (3.20)—by using the formula (3.19) again.

So what is all this good for? We can use Fourier’s theorem to obtain the general solution of our initial value problem. Recall that the problem was to find the wave amplitudes $A(k)$ and $B(k)$ that satisfy the initial conditions (3.11) and (3.12). To make (3.11) resemble (3.13), we rewrite (3.11) in the form

$$F(x) = \int_{0}^{\infty} dk \left[ (A(k) + A(-k)) \cos(kx) + (B(k) - B(-k)) \sin(kx) \right]$$  \hspace{1cm} (3.21)

We have changed the integration limits in (3.11) to match those in (3.13). Then Fourier’s theorem tells us that

$$A(k) + A(-k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, F(x) \cos(kx)$$  \hspace{1cm} (3.22a)

and

$$B(k) - B(-k) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \, F(x) \sin(kx)$$  \hspace{1cm} (3.22b)

Equations (3.22) are two equations in four unknowns. The four unknowns are the amplitudes $A(k)$ and $A(-k)$ of the right- and left-propagating cosine waves, and the
amplitudes $B(k)$ and $B(-k)$ of the right- and left-propagating sine waves. We get two more equations for the same four unknowns by applying Fourier’s theorem to our other initial condition (3.12). The result is

\[ A(k) - A(-k) = \frac{1}{\pi \omega(k)} \int_{-\infty}^{+\infty} dx \, G(x) \sin(kx) \]  
\[ B(k) + B(-k) = -\frac{1}{\pi \omega(k)} \int_{-\infty}^{+\infty} dx \, G(x) \cos(kx) \]  

Solving (3.22) and (3.23) for the four unknowns, we obtain

\[ A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, F(x) \cos(kx) + \frac{1}{2\pi \omega(k)} \int_{-\infty}^{+\infty} dx \, G(x) \sin(kx) \]  
\[ B(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \, F(x) \sin(kx) - \frac{1}{2\pi \omega(k)} \int_{-\infty}^{+\infty} dx \, G(x) \cos(kx) \]  

which hold for both positive and negative $k$. In summary, the wave field corresponding to the initial conditions (3.7) is (3.10) with the wave amplitudes given by (3.24).

This is a remarkable achievement. Using only our two propositions, we have shown how to find the wave field that results from any set of initial conditions. The result is a tribute to the power of Proposition #2, which allows us to add together as many waves satisfying Proposition #1 as we please. By adding together an infinite number of waves, we acquire the ability to handle the general case. Of course, (3.10) only gives us the surface elevation. But Proposition #1 tells us that the accompanying velocity field must be

\[ u(x,t) = \int_{-\infty}^{+\infty} dk \left[ A(k) \omega(k) \frac{\cosh(k(H+z))}{\sinh(kH)} \cos(kx - \omega(k) t) + B(k) \omega(k) \frac{\cosh(k(H+z))}{\sinh(kH)} \sin(kx - \omega(k) t) \right] \]  

The amplitudes in (3.25) are the same ones as in (3.10)—the amplitudes given by (3.24).

There is a catch to this, as you may already be suspecting. The integrals in (3.24) and (3.25) might be very hard to do. One could of course get lucky. For a particular $F(x)$ and $G(x)$ it might turn out that the integrals are easy to do, or to look up. More typically (3.24) are easy, but the integrals (3.10) and (3.25) are impossible. Even more typically, all the integrals are impossible to do exactly. But that is not really the point. Just by writing down (3.10), (3.24) and (3.25), we have solved the problem in principle. If we absolutely need a quantitatively accurate answer, we can always evaluate these integrals with the help of a computer, using numerical techniques. Sometimes, however, we don’t need a perfectly accurate answer, but we want to see what’s going on. What are these
formulas really telling us? In that case, a good method is to consider special cases for which the calculations are easier.

In that spirit, we suppose that the initial surface elevation is a ‘motionless hump.’ That is, we suppose that \( G(x) = 0 \). We further suppose that the hump is symmetric about \( x = 0 \), i.e. that \( F(x) = F(-x) \). An example of such a function is \( F(x) = e^{-\beta x^2} \), the same function we used to illustrate Fourier’s theorem. With these restrictions on the initial conditions, the wave amplitudes (3.24) take the simple form

\[
A(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \ F(x) \cos(kx) \quad \text{and} \quad B(k) \equiv 0
\]

(3.26)

so our solution is

\[
\eta(x,t) = \int_{-\infty}^{+\infty} dk \left[ A(k) \cos(kx - \omega(k)t) \right]
\]

(3.27)

with \( A(k) \) given by (3.26). The solution (3.27) contains no sine waves, and, because (3.26) tells us that \( A(k) = A(-k) \), the amplitude of the left-moving wave equals the amplitude of the right-moving wave at the same wavelength. Let us rewrite (3.27) to emphasize that fact. We write

\[
\eta(x,t) = \eta_L(x,t) + \eta_R(x,t)
\]

(3.28)

where

\[
\eta_R(x,t) = \int_{0}^{+\infty} dk \left[ A(k) \cos(kx - \omega(k)t) \right]
\]

(3.29)

represents the right-moving wave, and

\[
\eta_L(x,t) = \int_{-\infty}^{0} dk \left[ A(k) \cos(kx - \omega(k)t) \right] = \int_{0}^{+\infty} dk \left[ A(-k) \cos(-kx + \omega(-k)t) \right]
\]

(3.30)

represents the left-moving wave. In simplifying (3.30), we have used the facts that the cosine is an even function, \( \cos(s) = \cos(-s) \), and that the frequency is always positive, \( \omega(k) = \omega(-k) > 0 \). By the even-ness of the cosine we see that
Thus the left-moving waves are a mirror image of the right-moving waves. Of course, this is a consequence of the symmetry of our initial conditions.

Our results hold for any kind of wave—deep water waves, shallow water waves, or in between. But suppose we are dealing with shallow water waves. This is subtle, because the integrations in (3.29) and (3.30) run over all k, and high enough k will certainly violate \( kH << 1 \). Their wavelengths will be less than the fluid depth. What we are really assuming is that \( A(k) \) is small for those large k. And that, in turn, is an assumption about our initial conditions. Our ‘motionless hump’ must be very broad.

In the case of shallow water waves (3.27) becomes

\[
\eta(x,t) = \eta_L(x,t) + \eta_R(x,t)
\]

\[
\text{SW} = \int_0^\infty dk \left[ A(k) \cos(k(x + ct)) \right] + \int_0^\infty dk \left[ A(k) \cos(k(x - ct)) \right] \quad (3.32)
\]

where \( c = \sqrt{gH} \) is the shallow water phase speed, because by (3.21)

\[
F(\xi) = 2 \int_0^\infty dk \left[ A(k) \cos(k\xi) \right]. \quad (3.33)
\]

For the particular initial condition

\[
\eta(x,0) = F(x) = e^{-\beta x^2} \quad (3.34)
\]

the shallow-water solution is

\[
\text{SW} \quad \eta(x,t) = \frac{1}{2} e^{-\beta(x+ct)^2} + \frac{1}{2} e^{-\beta(x-ct)^2} \quad (3.35)
\]

Thus the initial hump splits symmetrically, into two, mirror-image parts, which move apart without changing their shape. This property of ‘not changing shape’ is a peculiar property of nondispersive waves. It depends critically on the fact that all the basic, cosine waves move at the same speed, regardless of wavelength. Thus it applies only to shallow water waves. For the more interesting case of deep water waves, no simplification like (3.32) is possible; we must attack the general solution (3.29-30) by other means. That is our next assignment.