4 Waves generated by a distant storm

Imagine the following situation. Far across the Pacific, a powerful storm occurs within a specific region and a specific time interval. The wind churns up the waves, and they begin to propagate toward San Diego. Idealizing to the one-dimensional case, we let the location of the storm be (near) $x=0$, and we let the time of the storm be (around) $t=0$. The precise way in which the storm generates waves is quite complicated. We shall say more about it in the next section. For present purposes, we imagine that the storm kicks up the waves near $(x,t)=(0,0)$, and then stops. We pick up the problem at that point. Thus the problem to be solved is this: given the sea state just after the storm (as initial condition), find the sea state at San Diego at a much later time, when the waves from the storm finally reach our shore.

The ‘initial condition’ in the aftermath of the storm will be quite complicated, but we shall idealize it as a ‘motionless hump’ with the same spatial symmetry as considered in the previous lecture. Then the solution to the problem is given by (3.28-30) with the amplitudes $A(k)$ determined by the exact form of the initial hump. To get the answer, we need only evaluate the integrals in (3.29) and (3.30).

The problem we have set is a severely idealized one, but it can teach us a great deal about the general situation. The key assumption—the assumption that allows us to work out the answer—is that the storm takes place at a specific place and time, and one that is well separated from the location and time at which we want to know the answer.

Suppose that, in our one-dimensional world, San Diego lies to the right of the storm, at large positive $x$. Then we need only keep track of the right-moving waves. We need only evaluate

$$\eta_R(x,t) = \int_0^\infty dk \left[ A(k) \cos(kx - \omega(k)t) \right]$$

at large positive $x$ and $t$—large $t$, because the waves generated at $t=0$ take a long time to cross the ocean. With this in mind, we re-write (4.1) in the form

$$\eta_R(x,t) = \int_0^\infty dk \left[ A(k) \cos(t\phi(k)) \right]$$

where

$$\phi(k) \equiv k \frac{x}{t} - \omega(k)$$

We shall evaluate (4.2) for $t \to \infty$, but with the quotient $x/t$ in (4.3) held fixed. In that case, $\phi(k)$ depends only on $k$. This is the mathematically cleanest way to proceed. Of
course $x/t$ can have any value we want, and, by allowing $x/t$ to take all possible values after we have performed the integral, we will have the solution for all large $x$ and $t$.

What do we use for $A(k)$? The amplitudes $A(k)$ are determined by the initial conditions—the sea surface elevation just after the storm—via Fourier’s theorem. We shall assume only that $A(k)$ depends smoothly on $k$. This turns out to be a critical assumption, and it deserves further comment. But the comment will make better sense after we have finished the calculation. For now we simply emphasize: $A(k)$ depends smoothly on $k$.

For large enough $t$, even small changes in $\phi(k)$ will cause rapid oscillations in the factor $\cos(t\phi(k))$ as $k$ increases inside the integral (4.2). However, if $A(k)$ is smooth, these oscillations produce canceling contributions to (4.2). This is true everywhere except where $\phi'(k) = 0$; there changes in $k$ produce no change in $\phi(k)$. We therefore assume that, as $t \to \infty$, the dominant contribution to (4.2) comes from wavenumbers near the wavenumber $k_0$ at which

$$\phi'(k_0) = \frac{x}{t} - \omega'(k_0) = 0 \quad (4.4)$$

According to (4.4), $k_0$ is the wavenumber of the wave whose group velocity $c_g$ satisfies $x = c_g t$.

With $k_0$ defined by (4.4) for our chosen $x/t$, we approximate

$$\eta_R(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) \cos(t\phi(k)) \, dk = A(k_0) \int_{k_0-\Delta k}^{k_0+\Delta k} \cos(t\phi(k)) \, dk \quad (4.5)$$
where $\Delta k$ is a small, arbitrary constant. In (4.5) we have replaced $A(k)$ by $A(k_0)$ because, assuming that $A(k)$ is a smooth function, it changes very little within the narrow range of wavenumbers between $k_0 - \Delta k$ and $k_0 + \Delta k$.

Since the integral (4.5) is over a narrow range of wavenumbers centered on $k_0$, we can approximate $\phi(k)$ by a truncation of its Taylor-series expansion about $k_0$. The Taylor series is

$$
\phi(k) = \phi(k_0) + \phi'(k_0)(k - k_0) + \frac{1}{2} \phi''(k_0)(k - k_0)^2 + \cdots
$$

(4.6)

where the second term vanishes on account of (4.4). Keeping only the first two non-vanishing terms in (4.6), substituting the result into (4.5), and using the trigonometric identity (2.2), we obtain

$$
\eta_R(x, t) = A(k_0)\cos(\theta_0) \cdot I_1(t) + A(k_0)\sin(\theta_0) \cdot I_2(t)
$$

(4.7)

where $\theta_0 = k_0x - \omega(k_0)t$.

$$
I_1(t) = \int_{k_0-\Delta k}^{k_0+\Delta k} dk \cos\left(\frac{1}{2} t\omega''(k_0)(k - k_0)^2\right)
$$

(4.8a)

and

$$
I_2(t) = \int_{k_0-\Delta k}^{k_0+\Delta k} dk \sin\left(\frac{1}{2} t\omega''(k_0)(k - k_0)^2\right)
$$

(4.8b)

The rest of the problem is just mathematics. To perform the integrals in (4.8), we change the integration variable from $k$ to

$$
\alpha = \sqrt{\frac{t}{2|\omega''(k_0)|}} (k - k_0)
$$

(4.9)

Then, keeping in mind that $\omega''(k_0)$ is negative in our case,

$$
I_1(t) = \sqrt{\frac{2}{t|\omega''(k_0)|}} \int_{-\Delta \alpha}^{+\Delta \alpha} d\alpha \cos(\alpha^2)
$$

(4.10a)

and

$$
I_2(t) = \sqrt{\frac{2}{t|\omega''(k_0)|}} \int_{-\Delta \alpha}^{+\Delta \alpha} d\alpha \sin(-\alpha^2)
$$

(4.10b)
where
\[ \Delta \alpha = \sqrt{\frac{t}{2}} |\omega'(k_0)| \Delta k \]  (4.11)

The limit \( t \to \infty \) corresponds to \( \Delta \alpha \to \infty \) for fixed \( \Delta k \). Since
\[ \lim_{\Delta \alpha \to \infty} \int_{-\infty}^{+\Delta \alpha} d\alpha \cos(\alpha^2) = \int_{-\infty}^{+\infty} d\alpha \cos(\alpha^2) = \sqrt{\frac{\pi}{2}} \]  (4.12a)
and
\[ \lim_{\Delta \alpha \to \infty} \int_{-\infty}^{+\Delta \alpha} d\alpha \sin(\alpha^2) = \int_{-\infty}^{+\infty} d\alpha \sin(\alpha^2) = \sqrt{\frac{\pi}{2}} \]  (4.12b)

we obtain
\[ I_1(t) = -I_2(t) = \sqrt{\frac{\pi}{t |\omega'(k_0)|}} \]  (4.13)

so, finally,
\[ \eta_R(x,t) = A(k_0) \sqrt{\frac{\pi}{t |\omega'(k_0)|}} (\cos \theta_0 - \sin \theta_0) \]
\[ = A(k_0) \sqrt{\frac{2\pi}{t |\omega'(k_0)|}} \cos (k_0 x - \omega(k_0) t + \pi/4) \]  (4.14)

The integrals in (4.12) can be looked up.

According to (4.14), the surface elevation far from the storm is a single, slowly varying wave. According to (4.4), its wavenumber \( k_0 \) is the wavenumber of the wave that travels, at its group velocity, the distance \( x \) between us and the storm, in the time \( t \) since the storm. Here is further evidence that the wave energy travels at the group velocity. We see a single wave because we are so far from the storm. Only one particular wave has the right wavenumber to reach our location in the time since the storm.

How has the vast separation between us and the storm been built into our calculation? In two ways. First, by our assumption that \( t \to \infty \) (for fixed \( x/t \)). And second, by our assumption that \( A(k) \) is a smooth function. The latter assumption is equivalent to the assumption that the initial surface distribution \( \eta(x,0) = F(x) \) is concentrated near \( x=0 \).

How do we see this? Suppose, for example, that \( F(x) = F_0 e^{-\beta x^2} \) with \( F_0 \) and \( \beta \) positive constants. We have used this example before! The larger the constant \( \beta \), the more \( F(x) \) is concentrated near \( x=0 \). For this choice of \( F(x) \), (3.26) gives...
\[ A(k) = \frac{1}{\pi} \int_{0}^{\infty} dx \, F(x) \cos(kx) = \frac{F_0}{2\sqrt{\pi\beta}} e^{-k^2/4\beta} \] (4.15)

where we have used the formula (3.19). We can describe this \( A(k) \) as a hump whose width (in \( k \)-space) is inversely proportional to the width of \( F(x) \) (in \( x \)-space). Thus, as \( \beta \to \infty \) (\( F(x) \) sharply concentrated at \( x=0 \)), \( A(k) \) becomes an infinitely wide hump. That is, \( A(k) \) becomes very smooth.

What is true in this particular example is also true in general. The more concentrated \( F(x) \), the more spread out is its Fourier transform \( A(k) \). When you study quantum mechanics, you will see that Heisenberg’s Uncertainty Principle corresponds to this same mathematical fact. In that context, and speaking very, very roughly, \( F(x) \) is the probability of finding a particle at location \( x \), and \( A(k) \) is the probability that the particle has velocity \( k \) (never mind the wrong units—Planck’s constant fixes that!). Just remember that you heard it first in a course on ocean waves.

To better admire our final answer, we re-write (4.14) in the form of a slowly varying wavetrain,

\[ \eta_r(x,t) = A_{SV}(x,t) \cos(k(x,t)x - \omega(x,t)t - \pi/4) \] (4.16)

Here,

\[ A_{SV}(x,t) = A(k(x,t)) \sqrt{\frac{2\pi}{t |\omega''(k(x,t))|}} \] (4.17)

is the slowly varying amplitude; \( k(x,t) \) is the slowly varying wavenumber; and \( \omega(x,t) \) is the slowly varying frequency. By slowly varying we mean that these three quantities change by only a small percentage over each wavelength or period.

Let \((x,t)\) be given. The slowly varying wavenumber at \((x,t)\) is determined as the solution to (4.4), namely

\[ \omega'(k) = x/t \] (4.18)

For deep water waves, this would be

\[ \text{DW} \quad \frac{1}{2} \sqrt{\frac{g}{k}} = x/t \quad \text{so} \quad k(x,t) = \frac{gt^2}{4x^2} \] (4.19)

If \( x \) and \( t \) are large, it is obvious that \( k(x,t) \) changes very little when \( x \) changes by a wavelength, or when \( t \) changes by a wave period. With \( k(x,t) \) thus determined, \( \omega(x,t) \) is determined by the dispersion relation.
Obviously, $\omega(x,t)$ is also a slowly varying function. Finally, the slowly varying amplitude is obtained by substituting $k(x,t)$ into (4.17).

If all these things very slowly, what is it that changes rapidly? The answer, of course, is $\eta(x,t)$ itself; it changes by 100% in each wavelength and period.

How do we understand the form (4.17) taken by the slowly varying amplitude? What is it telling us? Imagine that you and a friend each have speed boats, and you decide to play a little game with this slowly varying wavetrain. Each of you picks a fixed wavenumber value, and each of you decides to drive your boat at just the right speed to always observe your chosen wavenumber. If you choose the wavenumber $k_1$, then the location of your boat must satisfy $\omega'(k_1) = x/t$. In other words, you must drive your boat at the group velocity corresponding to $k_1$. If your friend chooses the value $k_2$, she must drive her boat at the group velocity $\omega'(k_2)$ corresponding to $k_2$. We suppose that $k_2 < k_1$; your friend has decided to follow a longer wavelength than yours.

Driving these boats will take some skill. You can’t be fooled into following your wavenumber by keeping up with the crests and troughs. If you do that, you will notice that the wavelength you observe will gradually get longer. You will have left your assigned wavenumber far behind. To keep pace with your assigned wavenumber, you must drive your boat at half the speed of the crests and troughs, because in deep water the group velocity is half the phase velocity. Up ahead, your friend must do the same for her assigned wavenumber $k_2$. But since, $k_2 < k_1$ her boat will move faster than yours. The two boats gradually draw apart.

If energy really moves at the group velocity, then the total amount of energy between the two boats must always be the same. From the previous lecture we know that the energy per unit horizontal distance is proportional to the square of the slowly varying wave amplitude. Therefore, if energy moves at the group velocity, it must be true that

$$\int_{x_i(t)}^{x_2(t)} dx \ A_{SV}^2(x,t) = \text{constant} \tag{4.21}$$

where $x_i(t)$ is the location of your boat, and $x_2(t)$ is the location of your friend’s boat. Suppose that the difference between $k_1$ and $k_2$ is very small. Then the difference between $x_i(t)$ and $x_2(t)$ is also very small, and (4.21) becomes

$$\left(x_2(t) - x_i(t)\right) A_{SV}^2 = \left(\omega'(k_2) - \omega'(k_1)\right) t A_{SV}^2 = \text{constant} \tag{4.22}$$

where $\omega'(k_2) - \omega'(k_1)$ is the difference in the boat speeds. If the difference between $k_i$ and $k_2$ is very small, then
\[ \omega'(k_2) - \omega'(k_1) = \omega''(k_1) (k_2 - k_1) \]  

(4.23)

Thus (4.22) implies that

\[ A_{sv}^2 \propto \frac{1}{|\omega''(k_1)|} \]  

(4.24)

which agrees with (4.17) and provides an explanation for it. The square of the slowly varying amplitude—the energy density—is inversely proportional to the separation between the two boats, and it decreases as the boats diverge. The energy per unit distance decreases because the same amount of energy is spread over a wider area.

Our calculation shows why, if you are surfer, it is better to be far away from an intense, wave-producing storm than close to it. If the storm were just over the horizon, the waves reaching you could not yet have dispersed. You would be seeing a superposition of all the wavelengths produced by the storm. The surf would be a jumble. In the case of a distant storm, you see only the wavelength whose group velocity matches your location and time. Even if the storm covers a wide area or lasts for a significant time, you will still see a single wavetrain if the distance between you and the storm is sufficiently great.

If the situation is more complicated, our solution still has value, because Proposition #2 says that waves superpose. For example, suppose there are two storms, well separated from one another, but both very far away. Then the sea state at your location will be a superposition of two slowly varying wavetrains, each determined by its distance in space and time from its source. These two wavetrains will interfere, just like the two waves traveling in one direction that we considered in Chapter 2. The result could be a series of wave groups—sets—like those described in Chapter 2. The sets move at about the average group velocity of the two waves.

If you are beginning to get the idea that group velocity is all-important, then you are absorbing one of the central ideas of this course. In dispersive waves like DW, the group velocity is the key concept, and it is much more important than the phase velocity. In nondispersive waves like SW, the group velocity and the phase velocity are the same, so there is no need to introduce the concept of group velocity. In nondispersive systems the phase velocity assumes great importance. For example, the solution (3.32) involves the phase velocity.

The two types of waves most frequently encountered in engineering and physics courses are electromagnetic waves and acoustic waves. Both of these are, to good approximation, nondispersive waves. Electromagnetic waves are exactly nondispersive in vacuum, but become slightly dispersive in material media. The phenomenon of chromatic aberration in optics is one manifestation of the slight dispersion of electromagnetic waves in glass. However, dispersive effects in electromagnetism and acoustics are usually sufficiently small that the concept of group velocity isn’t invoked. This explains why you may never
have heard of it, even if you have studied these waves. In water waves, group velocity becomes the key concept. There is no way to avoid it.

The mathematical techniques for dealing with these two classes of waves—dispersive and nondispersive—differ greatly. For dispersive waves like DW, the primary techniques are Fourier analysis and the superposition of waves—the very techniques you have begun to learn. Of course these methods also work—as a special case—for nondispersive waves like SW. But for nondispersive waves there are, in addition, very powerful and specialized mathematical methods that apply only to nondispersive waves. These specialized methods are useful for studying shocks, which occur commonly in nondispersive systems. (The SW analog of the shock is the bore, which we will discuss in Chapter 10; breaking ocean waves often turn into turbulent bores.) However, the specialized methods, besides being limited to nondispersive waves, are mathematically rather advanced, and are somewhat beyond our scope.

Let’s rewrite (4.21) in a slightly different form, as

$$\int_{x_1(t)}^{x_2(t)} E(x,t) \, dx = \text{constant}$$  \hspace{1cm} (4.25)

where $E(x,t)$ is the average wave energy per unit area. The time derivative of (4.25) is

$$0 = \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} E(x,t) \, dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t} E(x,t) \, dx + E(x_2,t) \frac{dx_2}{dt} - E(x_1,t) \frac{dx_1}{dt}$$  \hspace{1cm} (4.26)

$$= \int_{x_1}^{x_2} \frac{\partial}{\partial t} E(x,t) \, dx + E(x_2,t) c_g(x_2,t) - E(x_1,t) c_g(x_1,t)$$

because the speed boats are moving at the group velocity. The last line can be written in the equivalent form

$$0 = \int_{x_1}^{x_2} \frac{\partial}{\partial t} E(x,t) \, dx + F(x_2,t) - F(x_1,t)$$  \hspace{1cm} (4.27)

where $F(x,t) = c_g(x,t)E(x,t)$ is the energy flux toward positive $x$. This can be written yet again as

$$0 = \int_{x_1}^{x_2} \left[ \frac{\partial}{\partial t} E(x,t) + \frac{\partial}{\partial x} F(x,t) \right] \, dx$$  \hspace{1cm} (4.28)
Finally, since (4.28) must hold for every value of $x_1$ and $x_2$—that is, for every pair of constant wavenumbers selected by the two speed boat drivers—it must be true that

$$\frac{\partial}{\partial t} E(x,t) + \frac{\partial}{\partial x} F(x,t) = 0$$

(4.29)

everywhere. Equivalently,

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} (c_2 E) = 0.$$  

(4.30)

We have seen this equation before!

Our calculation has been for the case of one space dimension, but the ocean’s surface is two-dimensional. In two dimensions, our calculation corresponds to an infinitely long storm located along the $y$-axis. It is more realistic to regard the storm as a disturbance located near the point $(x,y)=(0,0)$ at time $t=0$. In this more realistic case, the waves spread away from the storm in a pattern of concentric circles. As in the one-dimensional case, energy moves at the group velocity, but in two dimensions the slowly varying amplitude is given by

$$A_{SV}^2 \propto \frac{1}{t^2\omega'(k_1)\omega''(k_1)}$$

(4.31)

instead of (4.24).

To understand (4.31), we must use our speed boats again. The first speed boat driver, traveling at just the right speed to always observe wavenumber $k_1$, must stay at radius $r_1 = \omega'(k_1)t$ measured from the center of the storm. The second speed boat driver, always observing $k_2$, must stay at $r_2 = \omega'(k_2)t$. The energy in the annulus between $r_1$ and $r_2$ is conserved. Hence, assuming that the difference between $r_1$ and $r_2$ is small,
\[
\text{constant} = 2\pi r_1 (r_2 - r_1) A_{sv}^2 \\
= 2\pi \left( \omega'(k_1) t \left| \omega^{*}(k_1) \right| t \Delta k \right) A_{sv}^2 \tag{4.32}
\]

where \( \Delta k = |k_2 - k_1| \). In two dimensions, the wave amplitude decreases faster than in one dimension because the same amount of energy is spread over an even larger area than in one dimension. Part of the enlargement is caused by the dispersion of waves (increasing separation of the speed boats) in the radial direction of wave propagation, just as in the one-dimensional case. However, now there is an additional enlargement caused by the fact that the circles themselves get larger.

We should admit that in talking about wave energy we have gone somewhat beyond the authority of Propositions #1 and #2. We have introduced the additional assumption that the average energy density—the energy per unit area of ocean surface—is proportional to the square of the wave amplitude. In a Section 2 we tried to justify this assumption with an incomplete calculation of the kinetic and potential energies, but we were mainly relying upon what you already know about energy. Propositions #1 and #2 say nothing at all about energy! When we eventually get around to justifying Propositions #1 and #2 by considering the general equations for a fluid, we will need to verify energy conservation as well.