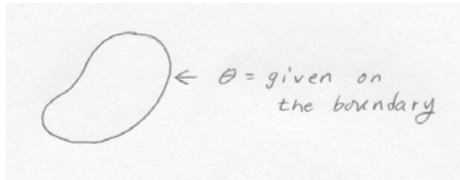
3. First-order linear equations as the limiting case of second-order equations

We consider the "advection-diffusion" equation

(1) $\mathbf{v} \cdot \nabla\theta = \varepsilon\nabla^2\theta$



on a bounded domain, with boundary conditions of prescribed θ. The coefficients

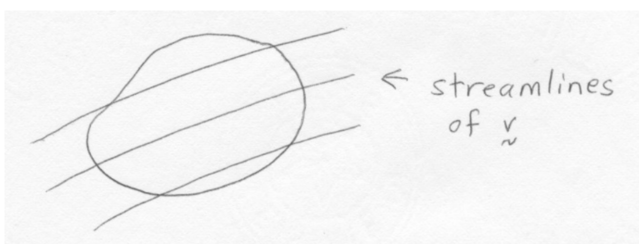
(2) $\mathbf{v} = \left(u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x})\right)$

are given functions which we regard as the components of a velocity $\mathbf{v}$. $\nabla \equiv \left(\partial / \partial x_1, \ldots, \partial / \partial x_n\right)$. We assume, on physical grounds, that this problem is well-posed; the solution exists and is unique.

We are interested in the case $\varepsilon \to 0$, in which the solutions of (1) must, in some sense, approach the solutions of the first-order equation

(3) $\mathbf{v} \cdot \nabla\theta = 0$

studied in the previous section. The characteristics of (3)—the streamlines of $\mathbf{v}$—are called *subcharacteristics* of (1). Consider the case in which these subcharacteristics intersect the boundary:



In that case, the limit $\varepsilon \to 0$ must be subtle, because the theory of (3) tells us that we cannot prescribe θ at 2 points on the same charcteristic. What really happens as $\varepsilon \to 0$?

Consider the one-dimensional example,

(4) $$u\theta_x = \varepsilon\theta_{xx}, \quad -1 < x < +1$$
$$\theta(-1) = a, \quad \theta(+1) = b$$

In the case u=constant, the solution is

(5) $\theta = A + B\, e^{ux/\varepsilon}$

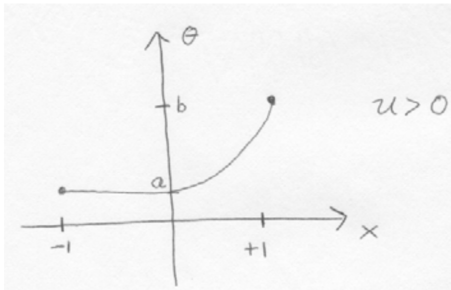
where the constants A and B are determined by the boundary conditions. We find that

(6) $$A = \frac{ae^{u/\varepsilon} - be^{-u/\varepsilon}}{e^{u/\varepsilon} - e^{-u/\varepsilon}}, \qquad B = \frac{b-a}{e^{u/\varepsilon} - e^{-u/\varepsilon}}$$

The nature of the solution depends very much on the sign of *u*. If *u*>0, $e^{u/\varepsilon} \to \infty$ and $e^{-u/\varepsilon} \to 0$, so $A \to a$, $B \to (b-a)e^{-u/\varepsilon}$ and

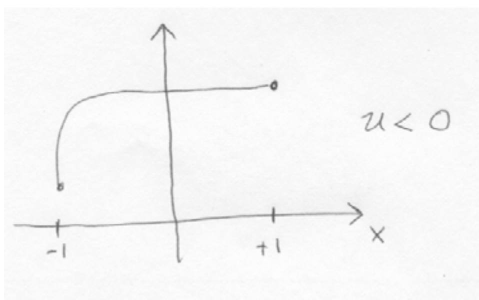
(7) $$\theta \to a + (b-a)e^{\frac{u}{\varepsilon}(x-1)}$$

The solution looks like this:



If, on the other hand, *u*<0, then $A \to b$, $B \to (a-b)e^{u/\varepsilon}$, and

(8) $$\theta \to b + (a-b)e^{\frac{u}{\varepsilon}(x+1)}$$



Thus, in both cases, as $\varepsilon \to 0$ the solution approaches the solution of $u\theta_x = 0$ (namely θ=*constant*) almost everywhere, with the constant chosen to satisfy the boundary condition at the "inflow" boundary. A *boundary layer* of thickness ε occurs at the "outflow" boundary. In the boundary layer, θ changes rapidly to satisfy the "outflow" boundary condition.

Next we consider the slightly more interesting cases of $u = \pm x$. In the case *u*=*x* of diverging flow,

$$x\theta_x = \varepsilon\theta_{xx} \quad \Rightarrow \quad \frac{x}{\varepsilon} = \partial_x \ln\theta_x \quad \Rightarrow \quad \frac{x^2}{2\varepsilon} = \ln x + const$$

(9) $$\Rightarrow \quad \theta = A\int_0^x e^{y^2/2\varepsilon}dy + B.$$

Once again we choose *A* and *B* to satisfy the boundary conditions, finding that

(10) $$A = \frac{b-a}{2\int_0^1 e^{y^2/2\varepsilon}\,dy}, \qquad B = \frac{a+b}{2}.$$

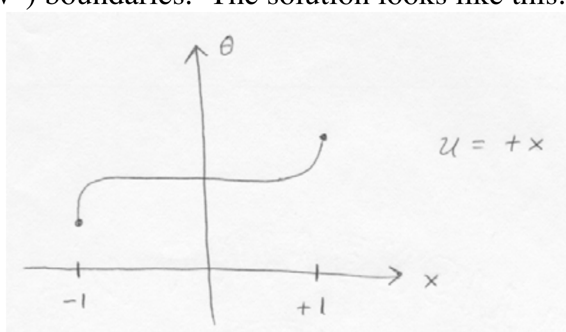
Thus

(11) $$\theta = \frac{b-a}{2} R(x,\varepsilon) + \frac{a+b}{2}$$

where

(12) $$R(x,\varepsilon) = \frac{\int_0^x e^{y^2/2\varepsilon}\,dy}{\int_0^1 e^{y^2/2\varepsilon}\,dy} \qquad (u = +x)$$

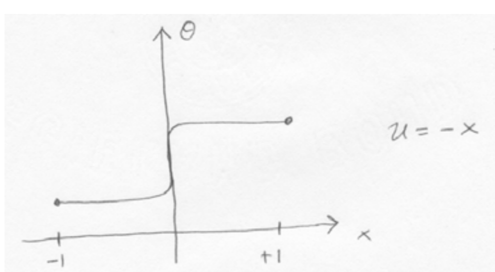
As $\varepsilon \to 0$, the ratio *R* is tiny except near $x = \pm 1$. Thus, throughout the interior of the domain, θ is uniform at the average of the 2 boundary values. Boundary layers occur at both ("outflow") boundaries. The solution looks like this:



In the case $u = -x$ of converging flow, we obtain the solution simply by replacing ε by $-\varepsilon$ in (12). Thus the solution of (4) with $u = -x$ is (11) with

(13) $$R(x,\varepsilon) = \frac{\int_0^x e^{-y^2/2\varepsilon}\,dy}{\int_0^1 e^{-y^2/2\varepsilon}\,dy} \qquad (u = -x)$$

As $\varepsilon \to 0$ R changes rapidly from -1 to $+1$ as x passes through zero. The solution looks like this:

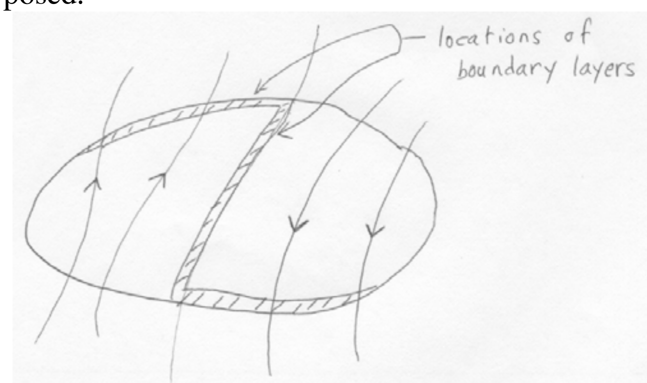


There are no boundary layers at $x = \pm 1$, but there is an *internal* boundary layer at x=0. As $\varepsilon \to 0$ (11,13) may, by a change of variables, be written as

$$\theta = \frac{(b-a)}{2} \frac{\int_0^{x/\varepsilon^{1/2}} e^{-y^2/2} dy}{\int_0^{\infty} e^{-y^2/2} dy} + \frac{(a+b)}{2} \qquad (u = -x) \tag{14}$$

Thus the internal boundary layer has a thickness $\varepsilon^{1/2}$. In the diverging (u=x) case, the boundary layer thickness is ε, the same as for the case of constant u. This follows from (12). These thicknesses could also be obtained by a crude balance of terms.

The emerging picture is one in which the limit $\varepsilon \to 0$ corresponds to a solution of $\mathbf{v} \cdot \nabla\theta = 0$ *almost everywhere*, with the value of θ on each streamline (i.e. subcharacteristic) equal to the value at the boundary point of inflow. The diffusion $\varepsilon\theta_{xx}$ is negligible outside of the boundary layers, but it determines *which* boundary influences the interior. That is, the diffusion determines the *direction* in which the boundary information travels along subcharacteristics. When ε=0 neither direction is preferred, and the boundary value problem is simply ill-posed.



The internal boundary layer of example (11,13) corresponds to the near-discontinuity that develops when information flows to the same point from widely different locations on the same boundary. This also happens in 2 or more dimensions. However, in 2 or more dimensions new phenomena occur:

1.) The boundary can *coincide* with a subcharacteristic.

2.) Closed subcharacteristics can occur within the domain.

We illustrate the first of these with an example dear to the hearts of oceanographers: Stommel's theory of ocean circulation. In this example, it is especially important to distinguish between the "velocity" **v** and the real, physical velocity to which the problem refers.

Example. The equations of motion for a rotating, two-dimensional fluid are

$$\frac{\partial u}{\partial t} - fv = -\frac{\partial p}{\partial x} - \varepsilon u + F^x$$

$$\text{(15)} \qquad \frac{\partial v}{\partial t} + fu = -\frac{\partial p}{\partial y} - \varepsilon v + F^y$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where (u,v) is the fluid velocity in the (east, north) direction, p is the pressure, ε is a drag coefficient, and (F^x, F^y) is the prescribed wind force. The "Coriolis parameter"

$$\text{(16)} \qquad f = 2\frac{2\pi}{day}\sin(latitude)$$

equals twice the local vertical component of the Earth's rotation vector. As a rough approximation, $f = f_0 + \beta y$. To satisfy the last of (15) we take $(u,v) = (-\psi_y, \psi_x)$. Then, eliminating p between the first and second of (15) we obtain the vorticity equation

$$\text{(17)} \qquad \frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = -\varepsilon\nabla^2\psi + W(x,y)$$

where $W = \partial F^y / \partial x - \partial F^x / \partial y$ is a given function. We seek steady solutions of

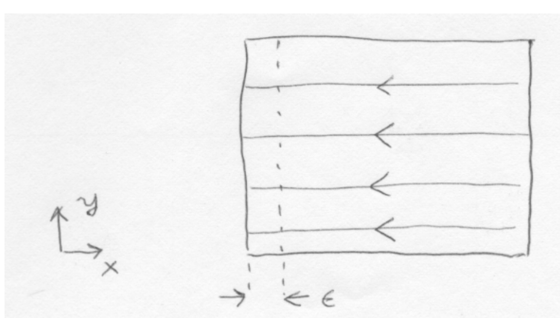
$$\text{(18)} \qquad -\beta\frac{\partial \psi}{\partial x} = \varepsilon\nabla^2\psi - W$$

in a square ocean $0<x,y<L$. The boundary condition is $\psi=0$.

Now forget everything you know about the physics of (18). The important mathematical fact is that (18) fits the general form of an advection-diffusion equation,

$$\text{(19)} \qquad \mathbf{v}\cdot\nabla\theta = \varepsilon\nabla^2\theta + Q$$

with velocity $\mathbf{v} = (-\beta, 0)$ and source $Q = -W$. Thus the subcharacteristics are lines of constant y, and boundary information flows from east to west. A boundary layer of thickness ε must occur at the western ("outflow") boundary.



Outside the boundary layer, ε is negligible, and

(20) $$-\beta \frac{\partial \psi}{\partial x} = -W(x,y),$$

corresponding to the characteristic equations,

(21) $$\frac{dx}{ds} = -\beta, \quad \frac{dy}{ds} = 0, \quad \frac{d\psi}{ds} = -W(x,y)$$

which imply

(22) $$\psi = \frac{1}{\beta}\int_L^x W(x',y)dx' \equiv \psi_I(x,y)$$

which is called the "interior solution". Note that the constant of integration in (22) has been chosen to satisfy the boundary condition at x=L.

In the western boundary layer we let $\psi = \psi_I + \psi_W$, where ψ_W is the "western boundary layer correction." ψ_W approximately obeys

(23) $$-\beta \frac{\partial \psi_w}{\partial x} = \varepsilon \frac{\partial^2 \psi_w}{\partial x^2},$$

the boundary condition $\psi_W = -\psi_I$ at x=0, and the matching condition $\psi_W \to 0$ as $x/\varepsilon \to \infty$. The solution is

(24) $$\psi_W = -\psi_I(0,y)e^{-x\beta/\varepsilon}.$$

If $W(x,0) \neq 0$, there must also be a southern boundary layer, because in that case the interior solution (22) does not satisfy the boundary condition at y=0. From a more mathematical viewpoint, the southern boundary coincides with a subcharacteristic, along which it is forbidden to prescribe ψ when ε=0. Let $\psi = \psi_I + \psi_S$ in the southern boundary layer. Then

(25) $$-\beta \frac{\partial \psi_S}{\partial x} = \varepsilon \frac{\partial^2 \psi_S}{\partial y^2}$$

with boundary condition

(26) $$\psi_S = -\psi_I(x,0) \quad \text{on} \quad y = 0$$

and "initial" condition

(27) $$\psi_S(L,y) = 0$$

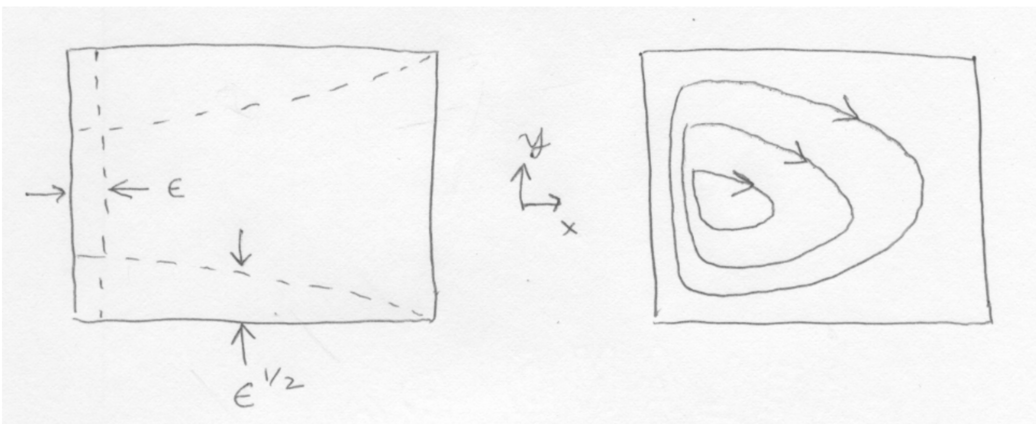
Except for the boundary at y=L, which is too distant to affect the solution for ψ_S, the problem (25-27) is equivalent to the problem of heat diffusion in a semi-infinite bar with a prescribed temperature at the end of the bar. The solution was given by eqns (52-53) in Section 1. The analogy is between:

$$\text{(28)} \qquad \psi_S \leftrightarrow \theta, \quad L - x \leftrightarrow t, \quad \frac{\varepsilon}{\beta} \leftrightarrow \kappa, \quad y \leftrightarrow x, \quad -\psi_I \leftrightarrow g$$

Thus

$$\text{(29)} \qquad \psi_S(x,y) = \int_x^L dx_0 \left[-\psi_I(x_0,0)\right] \frac{y}{\sqrt{\frac{4\pi\varepsilon}{\beta}}(x_0 - x)^{3/2}} \exp\left(-\frac{y^2}{4\frac{\varepsilon}{\beta}(x_0 - x)}\right)$$

From this we see that the boundary layer thickness is proportional to $\sqrt{\varepsilon(L-x)}$. The solution is summarized by the diagram on the left.



In the subtropical ocean, $W<0$ corresponding to $Q>0$. The ψ-lines look like as on the right, with the arrows pointing in the direction of the actual, physical velocity. The western boundary layer corresponds to the Gulf Stream.

It is an amazing fact that the fully 3-dimensional set of equations corresponding to *linear* ocean circulation theory can also be studied as an "advection-diffusion" problem. In the general 3d case, the "advected" quantity is the fluid pressure.

Example. Consider the "advection-diffusion" equation

$$\text{(30)} \qquad u(r,\phi)\frac{\partial \theta}{\partial r} + \frac{v(r,\phi)}{r}\frac{\partial \theta}{\partial \phi} = \varepsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \phi^2}\right)$$

on the annulus

$$\text{(31)} \qquad a < r < b, \qquad 0 < \phi < 2\pi .$$

with boundary conditions

(32) $$\theta(a,\phi) = \theta_a(\phi) \quad \text{and} \quad \theta(b,\phi) = \theta_b(\phi)$$

where $\theta_a(\phi)$ and $\theta_b(\phi)$ are given functions.

Solve this problem in the four distinct cases:

(i) $u \equiv 1, \quad v \equiv 0$

(ii) $u \equiv -1, \quad v \equiv 0$

(iii) $u \equiv 0, \quad v \equiv 1$

(iv) u,v such that the equation takes the form $-\theta_x = \varepsilon(\theta_{xx} + \theta_{yy})$ in Cartesian coordinates, but in the same domain, and with the same boundary conditions, as above.

Solutions.

(i) In the case

(33) $$\frac{\partial \theta}{\partial r} = \varepsilon \nabla^2 \theta$$

the interior equation is

(34) $$\frac{\partial \theta_I}{\partial r} = 0.$$

Thus $\theta_I = \theta_I(\phi)$. Since there is no boundary layer at $r=a$, the interior solution must satisfy the boundary condition there. Thus $\theta_I = \theta_a(\phi)$. This of course does not satisfy the boundary condition at $r=b$.

Near $r=b$, the balance must be

(35) $$\frac{\partial \theta}{\partial r} = \varepsilon \frac{\partial^2 \theta}{\partial r^2}$$

which implies

(36) $$\theta = A(\phi) + B(\phi) e^{(r-b)/\varepsilon}.$$

To match the interior we choose $A(\phi) = \theta_a(\phi)$, and to satisfy the boundary condition at $r=b$, we choose $B(\phi) = \theta_b(\phi) - \theta_a(\phi)$. The uniformly valid approximation is

(37) $$\theta = \theta_a(\phi) + \left(\theta_b(\phi) - \theta_a(\phi)\right) e^{(r-b)/\varepsilon}.$$

The boundary layer thickness is ε.

(*ii*) The case

(38) $$-\frac{\partial\theta}{\partial r} = \varepsilon\nabla^2\theta$$

is similar except that the boundary layer is at $r=a$, and the boundary layer equation is

(39) $$-\frac{\partial\theta}{\partial r} = \varepsilon\frac{\partial^2\theta}{\partial r^2}.$$

We find that

(40) $$\theta = \theta_b(\phi) + \left(\theta_a(\phi) - \theta_b(\phi)\right)e^{-(r-a)/\varepsilon}.$$

(*iii*) In the case

(41) $$\frac{1}{r}\frac{\partial\theta}{\partial\phi} = \varepsilon\nabla^2\theta,$$

the interior solution $\theta_I = \theta_I(r)$ satisfies neither boundary condition. Thus we anticipate boundary layers at both boundaries.

The most straight-forward approach is to expand the ϕ-dependence in a Fourier series,

(42) $$\theta(r,\phi) = \sum_{n=-\infty}^{\infty}\theta_n(r)e^{in\phi}$$

where

(43) $$\theta_n(r) = \frac{1}{2\pi}\int_0^{2\pi}\theta(r,\phi)e^{-in\phi}d\phi$$

and similarly for the boundary conditions, e.g.

(44) $$\theta_a(\phi) = \sum_{n=-\infty}^{\infty}(\theta_a)_n e^{in\phi}.$$

Note that the $n=0$ component corresponds to the azimuthal average,

(45) $$\theta_0(r) = \frac{1}{2\pi}\int_0^{2\pi}\theta(r,\phi)d\phi \equiv \langle\theta\rangle.$$

As we have already seen, the $n=0$ component is the only component present outside the boundary layers.

For each n, the exact equation takes the form

(46) $$\frac{1}{r} in\theta_n(r) = \varepsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_n}{\partial r}\right) - \frac{n^2}{r^2}\theta_n(r)\right)$$

with boundary conditions

(47) $$\theta_n(a) = \left(\theta_a\right)_n \equiv \frac{1}{2\pi}\int_0^{2\pi}\theta_a(\phi)e^{-in\phi}\,d\phi$$

and

(48) $$\theta_n(b) = \left(\theta_b\right)_n \equiv \frac{1}{2\pi}\int_0^{2\pi}\theta_b(\phi)e^{-in\phi}\,d\phi.$$

The case n=0 is clearly special. If n=0,

(49) $$0 = \varepsilon\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_0}{\partial r}\right) - 0\right)$$

and the diffusion term is important throughout the domain. Two integrations and use of the boundary conditions give us the solution

(50) $$\theta_0(r) = \frac{\langle\theta_a\rangle\ln(r/b) - \langle\theta_b\rangle\ln(r/a)}{\ln(a/b)}.$$

For $n \neq 0$, $\theta_n(r) = 0$ in the interior (as we have already seen), but this satisfies neither boundary condition. In the boundary layer near r=a,

(51) $$\frac{1}{a} in\theta_n(r) = \varepsilon\frac{\partial^2\theta_n}{\partial r^2}$$

which has solutions of the form $e^{\lambda r}$ where

(52) $$\lambda^2 = i\frac{n}{a\varepsilon}.$$

Thus if n>0,

(53) $$\lambda = \pm\sqrt{\frac{n}{a\varepsilon}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

and if n<0,

(54) $$\lambda = \pm\sqrt{\frac{|n|}{a\varepsilon}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right).$$

We cover both cases by writing

(55) $$\lambda = \pm\sqrt{\frac{|n|}{2a\varepsilon}}\left(1 + i\,\mathrm{sgn}(n)\right).$$

Only the minus-alternative produces an acceptable match, and once again we find that the amplitude is determined by the boundary condition. Thus

(56) $$\theta_n(r) = (\theta_a)_n \exp\left[-\sqrt{\frac{|n|}{2a\varepsilon}}(1 + i\,\mathrm{sgn}(n))(r-a)\right].$$

We note that reality of $\theta_a(\phi)$ implies that $(\theta_a)_n$ is conjugate symmetric; it then follows from the equation above that θ_n is conjugate symmetric; hence θ is real. The other boundary layer proceeds similarly and we obtain the uniformly valid approximation

(57) $$\begin{aligned}\theta(r,\phi) &= \frac{\langle\theta_a\rangle \ln(r/b) - \langle\theta_b\rangle \ln(r/a)}{\ln(a/b)} \\ &+ \sum_{n\neq 0}(\theta_a)_n \exp\left[-\sqrt{\frac{|n|}{2a\varepsilon}}(1 + i\;\mathrm{sgn}(n))(r-a) + in\phi\right] \\ &+ \sum_{n\neq 0}(\theta_b)_n \exp\left[+\sqrt{\frac{|n|}{2b\varepsilon}}(1 + i\;\mathrm{sgn}(n))(r-b) + in\phi\right]\end{aligned}$$

where the first line represents the interior solution and the second and third lines are the boundary layer corrections. We see that the boundary layers have thickness $\sqrt{\varepsilon}$. This solution can be written in various ways. If we take $(\theta_a)_n = A_n + iB_n$, then the second term can be written as

(58) $$\sum_{n\neq 0}(A_n + iB_n)\exp\left[-\sqrt{\frac{|n|}{2a\varepsilon}}(r-a)\right](\cos\alpha_n + i\sin\alpha_n)$$

where

(59) $$\alpha_n(r,\phi) = -\sqrt{\frac{|n|}{2a\varepsilon}}\,\mathrm{sgn}(n)(r-a) + n\phi$$

Since the sum is clearly real, it can be written as

(60) $$\sum_{n>0} 2\exp\left[-\sqrt{\frac{n}{2a\varepsilon}}(r-a)\right](A_n\cos\alpha_n - B_n\sin\alpha_n)$$

where

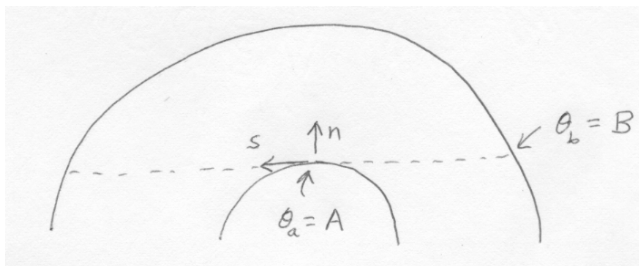
(61) $$\alpha_n = -\sqrt{\frac{n}{2a\varepsilon}}(r-a) + n\phi.$$

Physically, diffusion in the boundary layers wipes out the ϕ-dependence of θ there, allowing only the radially averaged value of θ to penetrate to the interior. The behavior in the boundary layers is an *oscillatory* decay—oscillatory because the ϕ-dependence of the boundary condition is swirled many times around the boundary layer before the small diffusion wipes it out.

(*iv*) This case is very similar to the ocean circulation problem done in lecture. There are ε-thickness boundary layers on the "west coast" of the basin and on the "east coast" of the central island. These "western" boundary layers thicken as one goes north or south from the center of the domain.

The new features are the boundary layers that emanate from the top and bottom of the island and extend to the west coast as shown:

These layers arise to smooth the discontinuity between the boundary values *A* and *B* shown on the sketch:



In the (s,n) coordinates depicted, the boundary layer equation is the heat equation

(62) $$\frac{\partial \theta}{\partial s} = \varepsilon \frac{\partial^2 \theta}{\partial n^2}$$

with "initial condition"

(63) $$\theta(0,n) = \begin{cases} A & n < 0 \\ B & n > 0 \end{cases}$$

Using the basic Green's function for the heat equation, the solution is

(64) $$\theta(s,n) = \frac{1}{\sqrt{4\pi\varepsilon s}} \left\{ A\int_{-\infty}^{0} e^{-(n-n_0)^2/4\varepsilon s} dn_0 + B\int_0^\infty e^{-(n-n_0)^2/4\varepsilon s} dn_0 \right\}$$

(This is strictly valid only for $s \gg \varepsilon^{1/2}$; the small region near s=0 demands special treatment.)

References. Bender and Orszag. Cole.