6. The wave equation

Of the 3 basic equations derived in the previous section, we have already discussed the heat equation,

(1) $\theta_t = \kappa\theta_{xx} + Q(x,t).$

In this section we discuss the wave equation,

(2) $\theta_{tt} = c^2\theta_{xx} + Q(x,t)$

and its generalization to more space dimensions. First we discuss the initial value problem with Q=0 in the infinite domain:

$$\theta_{tt} = c^2\theta_{xx}, \quad -\infty < x < +\infty, \quad t > 0$$

(3)
$$\theta(x,0) = f(x)$$
$$\theta_t(x,0) = g(x)$$

In contrast to the heat equation we have 2 initial conditions. Eqn (3a) is almost unique among PDE in that it has a simple general solution:

(4) $\theta(x,t) = F(x+ct) + G(x-ct)$ [D'Alembert's solution]

where F and G are arbitrary functions of a single variable. This follows immediately by noting that (3a) may be written

(5) $$\frac{\partial^2\theta}{\partial\xi\,\partial\eta} = 0$$

where

(6) $\xi = x - ct$ and $\eta = x + ct$.

In fact, some books prefer (5), rather than (3a) as the standard form of the wave equation. For fixed ξ and η, the lines defined by (6) are called *characteristics*. By rewriting (5) yet again as

(7) $$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\theta = 0$$

we see that $R_1 = \theta_t + c\theta_x$ is conserved on the characteristics defined by $x + ct = const$, and $R_2 = \theta_t - c\theta_x$ is conserved on the characteristics defined by $x - ct = const$. The conserved quantities R_1 and R_2 are called *Riemann invariants*. Special solutions of (7) correspond to the cases in which R_1 or R_2 is uniform. For example, if R_1 is uniform then (7) is satisfied. However uniform R_1 means that

(8) $\theta_t + c\theta_x = const$

which brings us back to first-order PDE.

Now we return to the initial value problem (3). Subsituting (4) into the initial conditions we obtain

(9) $$F(x)+G(x)=f(x)$$

and

(10) $$cF'(x)-cG'(x)=g(x)$$

The integral of (10) is

(11) $$F(x)-G(x)=\frac{1}{c}\int_{x_1}^{x} g(x_0)\,dx_0$$

where x_1 is an undetermined constant. It follows from (9) and (11) that

(12) $$F(x)=\frac{f(x)}{2}+\frac{1}{2c}\int_{x_1}^{x} g(x_0)\,dx_0$$

and

(13) $$G(x)=\frac{f(x)}{2}-\frac{1}{2c}\int_{x_1}^{x} g(x_0)\,dx_0\,.$$

Substituting (12-13) into (4) we finally obtain the solution of (3) in the form

(14) $$\theta(x,t)=\frac{1}{2}\big(f(x+ct)+f(x-ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(x_0)\,dx_0\,.$$

Note that x_1 has disappeared.

From (14) we see that that the solution at (x,t) depends on the initial conditions in the range $(x-ct,x+ct)$. Conversely, the initial condition at x_0 affects the solution at t in the range (x_0-ct,x_0+ct).

We can play the same games with (14) as we did with the Green's function for the heat equation. First consider the problem

(15) $$\begin{aligned}&\theta_{tt}=c^2\theta_{xx}+Q(x,t), \qquad -\infty<x<+\infty, \qquad t>0\\ &\theta(x,0)=\theta_t(x,0)=0\end{aligned}$$

of forced waves. We start by letting $Q(x,t)=Q(x)\delta(t)$. Then the solution must correspond to (14) with initial conditions

(16) $$f(x)=0 \qquad \text{and} \qquad g(x)=Q(x).$$

That is, the solution of (15) with $Q(x,t) = Q(x)\,\delta(t)$ must be

$$\theta(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} Q(x')dx' . \tag{17}$$

Now suppose $Q(x') = \delta(x' - x_0)$. Then (17) becomes

$$\theta(x,t) = \frac{1}{2c}\Big(H(x+ct-x_0) - H(x-ct-x_0)\Big) \tag{18}$$

where $H(x)$ is the Heaviside function:

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{19}$$

According to (18) $\theta = 1/2c$ if $x - ct < x_0 < x + ct$ and zero otherwise. Finally, it follows from (18) that the solution to

$$\begin{aligned} &\theta_{tt} = c^2\theta_{xx} + \delta(x-x_0)\,\delta(t-t_0) \\ &\theta = \theta_t = 0 \qquad \text{at } \ t = 0 \end{aligned} \tag{20}$$

is

$$\begin{aligned} \theta(x,t) &= \frac{1}{2c}\Big(H\big(x - x_0 + c(t-t_0)\big) - H\big(x - x_0 - c(t-t_0)\big)\Big) \\ &\equiv G(x-x_0, t-t_0) \end{aligned} \tag{21}$$

The solution to (15) is

$$\theta(x,t) = \int_0^t dt_0 \int_{-\infty}^{+\infty} dx_0 \;\, G(x-x_0, t-t_0)\,Q(x_0,t_0) \tag{22}$$

What is the solution of

$$\begin{aligned} &\theta_{tt} = c^2\theta_{xx} + Q(x,t) \\ &\theta(x,0) = f(x) \qquad\qquad ? \\ &\theta_t(x,0) = g(x) \end{aligned} \tag{23}$$

We could continue in the manner of Section 1 covering Green's (Riemann's) functions for the semi-infinite and finite space domain, using images or eigenfunctions. We shall not do this!

Consider the initial conditions

(24) $$f(x) = e^{-|x|}, \qquad g(x) = 0$$

which have a cusp (discontinuous first derivative) at x=0. By (14) this corresponds to the solution

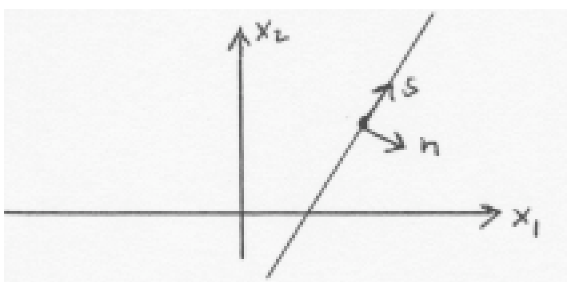
(25) $$\theta(x,t) = \frac{1}{2}e^{-|x-ct|} + \frac{1}{2}e^{-|x+ct|}$$

in which cusps propagate along the characteristics $x \pm c\,t = 0$.

We shall show that such discontinuities in first derivatives only occur along characteristics. For this it is worthwhile to consider the general 2nd-order equation in 2 independent variables,

(26) $$\sum_{i,j=1}^{2} A_{ij} \frac{\partial^2 \theta}{\partial x_i \partial x_j} + l.o.d. = \cdots$$

in the notation of Section 5. Suppose that first derivatives are discontinuous at a line in the x_1 - x_2 plane:



Let n be the normal distance from the line, and let s be the distance in the tangential direction. Then

(27) $$\frac{\partial \theta}{\partial x_i} = \frac{\partial \theta}{\partial n}\frac{\partial n}{\partial x_i} + \frac{\partial \theta}{\partial s}\frac{\partial s}{\partial x_i} \approx \frac{\partial \theta}{\partial n}\frac{\partial n}{\partial x_i}$$

because $\partial\theta / \partial n$ is large compared to $\partial\theta / \partial s$. By the same reasoning,

(28) $$\frac{\partial^2 \theta}{\partial x_i \partial x_j} \approx \frac{\partial^2 \theta}{\partial n^2}\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_j}.$$

Thus the largest terms in (26) are

(29) $$\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_j}\right)\frac{\partial^2 \theta}{\partial n^2}$$

and, since no other terms are so large, we must have

$$(30)\qquad \sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial n}{\partial x_i}\frac{\partial n}{\partial x_j}\right)=0.$$

In matrix notation this is

$$(31)\qquad \mathbf{v}^T\mathbf{A}\,\mathbf{v}=0 \qquad \text{where} \qquad \mathbf{v}\equiv\begin{pmatrix}\partial n/\partial x_1\\ \partial n/\partial x_2\end{pmatrix}.$$

In summary, if discontinuities are present they must occur at lines perpendicular to **v**, where **v** satisfies (31).

In Section 5 we showed how a rotation of the coordinates transforms (31) into the form

$$(32)\qquad {v_1}^2\lambda_1+{v_2}^2\lambda_2=0$$

where λ_1 and λ_2 are the eigenvalues of **A**. This is the same diagonalizing transformation we used to produce the normal form of the equation. From (32) we see that **v**=0 (no discontinuity possible) if λ_1 and λ_2 have the same sign (the elliptic case). Thus solutions of $\theta_{xx}+\theta_{yy}+\cdots$ cannot have discontinuous derivatives.

If λ_1 and λ_2 have opposite signs (the hyperbolic case) then nontrivial **v** exist. For example, the wave equation $\theta_{tt}-c^2\theta_{xx}=0$ corresponds to

$$(33)\qquad \left(\frac{\partial n}{\partial t}\right)^2-c^2\left(\frac{\partial n}{\partial x}\right)^2=0$$

which implies

$$(34)\qquad \begin{pmatrix}\partial n/\partial t\\ \partial n/\partial x\end{pmatrix}\propto\begin{pmatrix}\pm c\\ 1\end{pmatrix}$$

which are the perpendiculars of $x\pm c\,t=const$. This leads to a general definition of a characteristic: Characteristics are the lines along which discontinuities in first derivative may occur.

What happens in the case of the heat equation, $\theta_t=\kappa\theta_{xx}$? In that case,

$$(35)\qquad \kappa\left(\frac{\partial n}{\partial x}\right)^2=0.$$

Thus there is a "double" characteristic along each line *t*=*const*. Discontinuities may occur along such lines, as, for example, when a boundary condition changes suddenly. The sudden change is communicated instantaneously to all parts of the domain (even though it is exponentially small at large distances). This property of the heat equation—that signals propagate at infinite speed—is of course physically unrealistic.

Wave equation in higher space dimensions.

In 2d (using polar coordinates)

$$(36)\qquad \theta_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2}\right]$$

and in 3d (using spherical coordinates)

$$(37)\qquad \theta_{tt} = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) + \frac{1}{r^2\sin\alpha}\frac{\partial}{\partial\alpha}\left(\sin\alpha\frac{\partial\theta}{\partial\alpha}\right) + \frac{1}{r^2\sin^2\alpha}\frac{\partial^2\theta}{\partial\phi^2}\right]$$

where α is the colatitude. The radially symmetric cases

$$(38)\qquad \theta_{tt} = c^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)\right] \qquad \text{(2d)}$$

and

$$(39)\qquad \theta_{tt} = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right] \qquad \text{(3d)}$$

are the keys to finding the general solutions of (36) and (37). The 3d case is easiest because (39) can be rewritten in the form

$$(40)\qquad (r\theta)_{tt} = c^2(r\theta)_{rr}$$

which is just the one-dimensional wave equation in the variable $r\theta$. Thus (cf. (4)) the general solution of (39) or (40) is

$$(41)\qquad \theta(r,t) = \frac{1}{r}F(r-ct) + \frac{1}{r}G(r+ct)$$

where F and G are arbitrary functions. F represents the outgoing wave and G represents the incoming wave. Before proceeding to the general solution of (37), we pause for a radially symmetric example.

The balloon break.

The governing equations are the ideal fluid equations. In spherical coordinates:

$$(42)\qquad \begin{aligned} (\rho u)_t &= -p_r \\ (r^2\rho)_t &= -(r^2\rho u)_r \\ p &= p(\rho) \end{aligned}$$

with initial conditions

(43) $$p = \begin{cases} p_0, & r < r_0 \\ 0, & r > r_0 \end{cases} \quad \text{and } u = 0, \qquad 0 < r$$

Eqn (42) imply

(44) $$p_{tt} = c^2 \frac{1}{r^2}\left(r^2 p_r\right)_r$$

Thus p obeys (39) and therefore

(45) $$p = \frac{F(r - ct)}{r} + \frac{G(r + ct)}{r}.$$

It only remains to determine the functions F and G. To know the solution on r>0 and t>0 we must determine F for *all* values of its argument, and we must determine G for all *positive* values of its argument.

The functions F and G are determined by the initial conditions (43). Since u=0 implies $p_t = 0$ we have

(46) $$F(r) + G(r) = \begin{cases} rp_0, & 0 < r < r_0 \\ 0, & r_0 < r \end{cases}$$

and $-F'(r) + G'(r) = 0$, i.e.

(47) $$-F(r) + G(r) = c_1, \qquad 0 < r$$

where c_1 is a constant. Eqns (46-47) determine F and G for *positive* values of their arguments. However, the solution (45) requires F at negative values of its argument.

To avoid a mass source at r=0, we must insist that $r^2 \rho u \to 0$ as $r \to 0$. This in turn requires that $r^2 p_r \to 0$ as $r \to 0$. By (45) this implies that $F(-ct) + G(ct) = 0$, i.e.

(48) $$F(-\xi) + G(\xi) = 0, \qquad \xi > 0.$$

Eqn (48) determines the values of F for negative argument in terms of the values of G for positive argument.

Solving (46-48) for F and G, we find that

(The constant c_1 cancels out when F and G are substituted into (45), and therefore is irrelevant. We have simply set it to zero.)

The solution (45) consists of $1/r$ times F translated a distance ct to the right, plus $1/r$ times G translated a distance ct to the left. Thus

$$p = \frac{p_0}{2r}\left(r - ct\right) S_F + \frac{p_0}{2r}\left(r + ct\right) S_G \tag{49}$$

where the "switches" are defined by

$$S_F = \begin{cases} 1, & -r_0 < r - ct < +r_0 \\ 0, & otherwise \end{cases} \quad \text{and} \quad S_G = \begin{cases} 1, & 0 < r + ct < +r_0 \\ 0, & otherwise \end{cases} \tag{50}$$

Thus, at the point $r < r_0$ inside the balloon, $p = p_0$ until the time $t = (r_0 - r)/c$ at which G switches off. At that moment (which corresponds to the time required for a sound wave to travel from the balloon's surface) the pressure drops suddenly to

$$p = \frac{p_0}{2r}\left(r - \left(r_0 - r\right)\right) = \frac{2r - r_0}{r}\frac{p_0}{2} \tag{51}$$

and then decreases smoothly to the value $-p_0 r_0 / 2r$ at time $t = (r + r_0)/c$. At that moment (which corresponds to the time for a sound wave to travel from the balloon's surface to the center and then back to r), the pressure suddenly vanishes, and remains zero. (Zero pressure corresponds to the pressure far from the balloon.)

At points $r > r_0$ outside the balloon, only the F-wave is felt. Thus the pressure rises suddenly, then falls gradually to a negative value, and then vanishes.

Next we consider the general initial-value problem in 3 dimensions:

$$\begin{aligned} &\theta_{tt} = c^2\left(\theta_{xx} + \theta_{yy} + \theta_{zz}\right), \qquad -\infty < x, y, z < \infty \\ &\theta(\mathbf{x}, 0) = f(\mathbf{x}), \qquad \theta_t(\mathbf{x}, 0) = g(\mathbf{x}) \end{aligned} \tag{52}$$

From (45) we expect the initial conditions at a particular point $\mathbf{x}_0$ to create an outward traveling wave that decays as the inverse of the distance from $\mathbf{x}_0$. Such a solution is given by

$$\theta_1(\mathbf{x}, t) = \frac{1}{ct} \iint_{|\mathbf{x} - \mathbf{x}_0| = ct} \psi\left(\mathbf{x}_0\right) dS_0 \tag{53}$$

where $\psi(\mathbf{x}_0)$ is an arbitrary function, and the integration is over the surface of the sphere located a distance ct from $\mathbf{x}$. dS_0 is the element of surface area. (Show that (53) satisfies the wave equation for any $\psi(\mathbf{x}_0)$.)

As $t \to 0$ the radius of this sphere vanishes and

(54) $$\iint\limits_{|\mathbf{x}-\mathbf{x}_0|=ct} \psi(\mathbf{x}_0)\,dS_0 = \psi(\mathbf{x})\cdot 4\pi c^2 t^2 + O(t^4)$$

provided that $\psi(\mathbf{x}_0)$ is Taylor expandable about **x**:

(55) $$\psi(\mathbf{x}_0) = \psi(\mathbf{x}) + \frac{\partial \psi}{\partial x}(\mathbf{x})\,(x_0 - x) + \frac{\partial \psi}{\partial y}(\mathbf{x})\,(y_0 - y) + \cdots$$

No O(t^3) terms appear in (54) because the integral of the first-order derivatives in (55) vanishes by symmetry. Thus as $t \to 0$

(56) $$\theta_1(\mathbf{x},t) \sim 4\pi c t\ \psi(\mathbf{x}) + O(t^3)$$

and

(57) $$\frac{\partial \theta_1}{\partial t}(\mathbf{x},t) \sim 4\pi c\ \psi(\mathbf{x}) + O(t^2)$$

If we choose $\psi(\mathbf{x}_0) = g(\mathbf{x}_0)/4\pi c$ then

(58) $$\theta_1(\mathbf{x},t) = \frac{1}{4\pi c^2 t} \iint\limits_{|\mathbf{x}-\mathbf{x}_0|=ct} g(\mathbf{x}_0)\,dS_0 \equiv t\langle g\rangle_0$$

solves the initial value problem (52) with $f \equiv 0$. We have half of the desired result! Note that $\langle g\rangle_0$ denotes the *average* of *g* over the sphere $|\mathbf{x}-\mathbf{x}_0| = ct$. In this notation (53) takes the form

(59) $$\theta_1(\mathbf{x},t) = 4\pi c t\ \langle \psi \rangle_0$$

To solve the other half of the initial value problem, we note that if θ_1 is a solution of the wave equation, then so is

(60) $$\theta_2(\mathbf{x},t) \equiv \frac{\partial}{\partial t}\theta_1 = \frac{\partial}{\partial t}\left(4\pi c t\langle \psi\rangle_0\right)$$

However, by (57) we see that

(61) $$\theta_2 \to 4\pi c\,\psi(\mathbf{x}) \qquad \text{and} \qquad \frac{\partial \theta_2}{\partial t} \to 0$$

as $t \to 0$. Thus if we choose $\psi(\mathbf{x}_0) = f(\mathbf{x}_0)/4\pi c$, we have the solution

(62) $$\theta_2(\mathbf{x},t) = \frac{\partial}{\partial t}\left(t\langle f\rangle_0\right)$$

of the initial value problem (52) with $g \equiv 0$. By superposition, the solution of the general initial value problem (52) must be

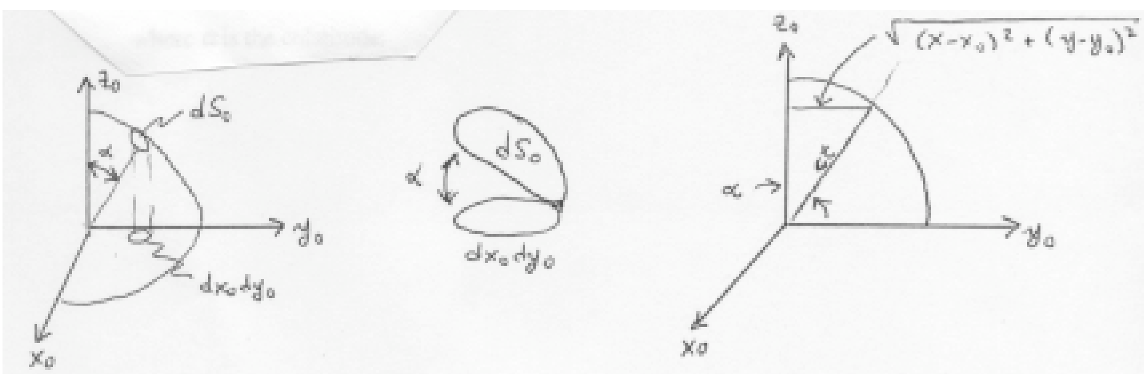
$$(63) \qquad \theta(\mathbf{x},t) = \frac{\partial}{\partial t}\Big(t\,\langle f\rangle_0\Big) + t\,\langle g\rangle_0,$$

an astonishingly simple result. (Use (63) to obtain the solution of the balloon-break problem.)

We obtain the corresponding result in 2 dimensions by specializing (63) to the case in which everything dpends on (x,y) but not on z. This is called the *method of descent*. If $f{=}f(x,y)$ then

$$(64) \qquad \begin{aligned} \langle f\rangle_0 &= \frac{1}{4\pi c^2 t^2} \iint_{|(x,y,z)-(x_0,y_0,z_0)|=ct} f(x_0,y_0)\, dS_0 \\ &= \frac{1}{4\pi c^2 t^2} \iint_{|(x,y)-(x_0,y_0)|=ct} f(x_0,y_0)\, \frac{2}{\cos\alpha}\, dx_0\, dy_0 \end{aligned}$$

where α is the colatitude:



By geometry

$$(65) \qquad \cos^2\alpha = \frac{c^2t^2 - (x-x_0)^2 - (y-y_0)^2}{c^2t^2}$$

Thus

$$(66) \qquad t\langle f\rangle_0 = \frac{1}{2\pi c}\iint dx_0 dy_0\, \frac{f(x_0,y_0)}{\left[c^2t^2 - (x-x_0)^2 - (y-y_0)^2\right]^{1/2}}$$

and similarly for $\langle g\rangle_0$, where the integral in (66) is over the interior of the circle $(x-x_0)^2 - (y-y_0)^2 = c^2t^2$. Thus in 2 space dimensions, the solution of the initial value problem is

(67)
$$\theta(\mathbf{x},t) = \frac{\partial}{\partial t}\left\{ \frac{1}{2\pi c}\iint dx_0 dy_0 \frac{f(x_0, y_0)}{\left[c^2t^2 - (x-x_0)^2 - (y-y_0)^2\right]^{1/2}} \right\} + \frac{1}{2\pi c}\iint dx_0\, dy_0 \frac{g(x_0, y_0)}{\left[c^2t^2 - (x-x_0)^2 - (y-y_0)^2\right]^{1/2}}$$

This has a completely different character than the 3d solution. A balloon breaking in 2 dimensions would be heard forever!

Reference. Whitham chapter 7.