SIO 212B Spring 2007
Problem set #1.

ANSWERS.

1. Multiply the first 3 given equations by \( u, v, w \) respectively and add to get

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) = -\mathbf{v} \cdot \nabla \phi + w \theta
\]

where \( \mathbf{v} = (u,v,w) \) and \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \). Since \( \nabla \cdot \mathbf{v} = 0 \), this can be written

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) + \nabla \cdot \left( \mathbf{v} \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) \right) = -\nabla \cdot (\mathbf{v} \phi) + w \theta
\]

Next multiply the \( \theta \)-equation by \( -z \) to get

\[
-z \frac{D \theta}{D t} = \frac{D}{D t} (-z \theta) + \theta \frac{D z}{D t} = \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (-z \theta) + \theta w = 0
\]

which may also be written

\[
\frac{\partial}{\partial t} (-z \theta) + \nabla \cdot (-z \mathbf{v} \theta) = -\theta w
\]

Adding equations together gives

\[
\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{v} E) = -\nabla \cdot (\mathbf{v} \phi)
\]

Thus

\[
\mathbf{F} = \mathbf{v} E + \mathbf{v} \phi
\]

If the hydrostatic approximation is relaxed, the energy density generalizes to

\[
E = \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 - z \theta
\]
2. This is proved in much the same way, except that the $\theta'$-equation is multiplied by $\theta'/N^2$. The result is

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{v}\phi) = 0$$

where

$$E = \frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{1}{2} \left(\frac{\theta'}{N^2}\right)^2$$

The linear equations contain no advective flux.

3. The paradox is that the equations in problem 2 are an approximation to those in problem 1, but the conserved energies are not approximately the same. The resolution to this paradox is that the energies differ by another conserved quantity. Since

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \theta^2 \right) + \nabla \cdot \left( \frac{1}{2} \theta^2 \mathbf{v} \right) = 0$$

it follows that

$$\iiint d\mathbf{x} \left( \frac{1}{2} \theta^2 \right) = \text{const}$$

Letting $\theta = \overline{\theta}(z) + \theta'(x,y,z,t) = N^2 z + \theta'(x,y,z,t)$, this is

$$\iiint d\mathbf{x} \left( \frac{1}{2} (N^2 z + \theta')^2 \right) = \text{const}$$

which implies that

$$\iiint d\mathbf{x} \ z \theta' + \iiint d\mathbf{x} \left( \frac{1}{2N^2} \theta' \right)^2 = \text{const}$$

On the other hand,

$$\iiint d\mathbf{x} \ (-z \theta) = \text{const} + \iiint d\mathbf{x} \ (-z \theta')$$

Thus the two forms of potential energy differ by a constant.
4. Substituting the modal expansion into the linearized primitive equations and using the orthogonality property of the sines and cosines, we get

\[
\begin{align*}
\frac{\partial u_m}{\partial t} - f v_m &= -\frac{\partial \phi_m}{\partial x} \\
\frac{\partial v_m}{\partial t} + f u_m &= -\frac{\partial \phi_m}{\partial y} \\
0 &= \frac{m \pi}{H} \phi_m + \theta_m \\
\frac{\partial u_m}{\partial x} + \frac{\partial v_m}{\partial y} + \frac{m \pi}{H} w_m &= 0 \\
\frac{\partial \theta_m}{\partial t} + w_m N^2 &= 0
\end{align*}
\]

The definition \( h_m = \phi_m / g \) brings the first two equations into the desired form. The last 3 equations combine to give

\[
\frac{m \pi}{H} \frac{\partial \phi_m}{\partial t} + \frac{N^2 H}{m \pi} \left( \frac{\partial u_m}{\partial x} + \frac{\partial v_m}{\partial y} \right) = 0
\]

This takes the form

\[
\frac{\partial h_m}{\partial t} + H_m \left( \frac{\partial u_m}{\partial x} + \frac{\partial v_m}{\partial y} \right) = 0
\]

if the “reduced depth” is defined as

\[
H_m = \frac{N^2 H^2}{gm^2 \pi^2}
\]

5. If the top and bottom boundaries are flat, then the boundary condition there is \( w=0 \), and the buoyancy equation becomes

(1) \( \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial z} \right) + J \left( \psi, \frac{\partial \psi}{\partial z} \right) = 0 \)

which is obviously satisfied by setting \( \partial \psi / \partial z = 0 \) (uniform buoyancy on the boundaries).

Linearizing the quasigeostrophic equations about a state of rest with constant \( N \), we get

(2) \( \frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, f) = f_0 \frac{\partial w}{\partial z} \)
where \( \nabla = (\partial_x, \partial_y) \), and

\[
(3) \quad \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial z} \right) = -\frac{N^2}{f_0} w
\]

Multiplying (2) by \(-\psi\), we get

\[
(4) \quad -\nabla \cdot \left( \psi \nabla \frac{\partial \psi}{\partial t} \right) + \frac{\partial}{\partial t} \left( \frac{1}{2} \nabla \psi \cdot \nabla \psi \right) - \frac{1}{2} \beta \frac{\partial}{\partial x} (\psi^2) = -f_0 \frac{\partial}{\partial z} (\psi w) + f_0 w \frac{\partial \psi}{\partial z}
\]

Multiplying (3) by \(f_0^2/N^2\ \partial \psi/\partial z\), we get

\[
(5) \quad \frac{\partial}{\partial t} \left( \frac{1}{2} \frac{f_0^2}{N^2} \left( \frac{\partial \psi}{\partial z} \right)^2 \right) = -f_0 \frac{\partial \psi}{\partial z} w
\]

Adding (4) and (5) gives

\[
(6) \quad \frac{\partial E}{\partial t} + \nabla \cdot F = 0
\]

where \( \nabla_3 = (\partial_x, \partial_y, \partial_z) \),

\[
(7) \quad E = \frac{1}{2} \nabla \psi \cdot \nabla \psi + \frac{1}{2} \frac{f_0^2}{N^2} \left( \frac{\partial \psi}{\partial z} \right)^2
\]

and

\[
(8) \quad F = (-\psi \psi_x - \frac{1}{2} \beta \psi^2, -\psi \psi_y, f_3 \psi w)
\]

We can obtain an equivalent result from the potential vorticity equation. We use the latter method to derive the energy equation for the nonlinear case.

Multiplying the general, nonlinear potential vorticity equation by \(-\psi\) gives

\[
(9) \quad -\psi \frac{\partial \psi}{\partial t} - J(\frac{1}{2} \psi^2, q) = 0
\]

But
\[-\psi \frac{\partial q}{\partial t} = -\psi \frac{\partial}{\partial t} \left( \nabla^2 \psi + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \psi \frac{\partial \psi}{\partial z} \right) \right) \]

\[
\frac{\partial E}{\partial t} - \nabla \cdot (\psi \nabla \psi) = \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} \psi \frac{\partial \psi}{\partial z} \right)
\]

where \( E \) is given by (7). Since

\[
-J \left( \frac{1}{2} \psi^2, q \right) \right) = -\frac{\partial}{\partial x} \left( \frac{1}{2} \psi^2 q_x \right) + \frac{\partial}{\partial y} \left( \frac{1}{2} \psi^2 q_y \right)
\]

the general energy equation takes the form (6) and (7), but with

\[
F = \left( -\psi \psi u - \frac{1}{2} q_x \psi^2, -\psi \psi v, + \frac{1}{2} q_y \psi^2, -\frac{f_0^2}{N^2} \psi \frac{\partial \psi}{\partial z} \right)
\]

instead of (8). The connection between (7) and the energy in problem 2 is obvious; in quasigeostrophic theory \((u,v) = (-\psi_x, \psi_y)\) and \(\theta' = f_0 \frac{\partial \psi}{\partial z}\). The connection between the energy fluxes in this problem and those in the first two problems is much harder to demonstrate.

The Fourier representation

\[
\psi(x,y,z,t) = \sum_{m=0}^{\infty} \psi_m(x,y,t) \cos \left( \frac{m \pi z}{H} \right)
\]

\[
w(x,y,z,t) = \sum_{m=1}^{\infty} w_m(x,y,t) \sin \left( \frac{m \pi z}{H} \right)
\]

transforms the linear equations to

\[
\frac{\partial}{\partial t} \nabla^2 \psi_m + \beta \frac{\partial \psi_m}{\partial x} = f_0 \frac{m \pi}{H} w_m
\]

and

\[
\frac{m \pi}{H} \frac{\partial \psi_m}{\partial t} = \frac{N^2}{f_0} w_m
\]

For the barotropic mode \((m=0)\) this is

\[
\frac{\partial}{\partial t} \nabla^2 \psi_0 + \beta \frac{\partial \psi_0}{\partial x} = 0
\]

with dispersion relation
\[ \omega = \frac{-\beta k}{k^2 + l^2} \]

(assuming solutions proportional to \( e^{i(kx + ly - \omega t)} \)). For an internal mode \(( m \neq 0)\) we have

\[
\frac{\partial}{\partial t} \left( \nabla^2 \psi_m - \frac{f_0^2}{N^2} \frac{m^2 \pi^2}{H^2} \psi \right) + \beta \frac{\partial \psi_m}{\partial x} = 0
\]

with dispersion relation

\[
\omega = \frac{-\beta k}{k^2 + l^2 + \lambda_m^2}
\]

where

\[
\lambda_m = \frac{f_0 \cdot m \pi}{N \cdot H}
\]

The \( m \)-th deformation radius is \( \lambda_m^{-1} \). The baroclinic mode of the two-layer model corresponds to \( m = 1 \).