SIO 212B Spring 2007
Problem set #4.

ANSWERS

1. Consider a particular \( i,j \). The coefficient of \( \alpha_{ij} \) in (1) is

\[
\frac{d\xi_{ij}}{dt} + \frac{1}{3}(J_{1ij} + J_{2ij} + J_{3ij})
\]

where \( J_{1ij} \) is the coefficient of \( \alpha_{ij} \) in

\[
\sum_{\text{gridboxes}} \frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \frac{1}{2\Delta^2} \left[(\psi_2 - \psi_4)(\xi_3 - \xi_1) - (\xi_2 - \xi_4)(\psi_3 - \psi_1)\right];
\]

\( J_{2ij} \) is the coefficient of \( \alpha_{ij} \) in

\[
\sum_{\text{gridboxes}} \frac{1}{4}(\psi_1 + \psi_2 + \psi_3 + \psi_4) \frac{1}{2\Delta^2} \left[(\xi_2 - \xi_4)(\alpha_3 - \alpha_1) - (\alpha_2 - \alpha_4)(\xi_3 - \xi_1)\right];
\]

and \( J_{3ij} \) is the coefficient of \( \alpha_{ij} \) in

\[
\sum_{\text{gridboxes}} \frac{1}{4}(\xi_1 + \xi_2 + \xi_3 + \xi_4) \frac{1}{2\Delta^2} \left[(\alpha_2 - \alpha_4)(\psi_3 - \psi_1) - (\psi_2 - \psi_4)(\alpha_3 - \alpha_1)\right]
\]

Each of (6-8) is expected to be a logical finite-difference approximation to \( J(\psi,\xi) \) at gridpoint \( ij \). Each of (6-8) receives a contribution from the 4 gridboxes surrounding gridpoint \( ij \). Let this gridpoint be point 0 in the following sketch:
Then, for example, $\psi_{i+1,j+1} = \psi_2$. The coefficient of $\alpha_0$ in (6) receives the contribution

$$\frac{1}{8\Delta^2}[(\psi_1 - \psi_3)(\zeta_2 - \zeta_0) - (\zeta_1 - \zeta_3)(\psi_2 - \psi_0)] \quad \text{from gridbox 0123} \quad (9)$$

$$\frac{1}{8\Delta^2}[(\psi_0 - \psi_4)(\zeta_3 - \zeta_5) - (\zeta_0 - \zeta_4)(\psi_3 - \psi_5)] \quad \text{from gridbox 0345} \quad (10)$$

$$\frac{1}{8\Delta^2}[(\psi_7 - \psi_5)(\zeta_0 - \zeta_6) - (\zeta_7 - \zeta_5)(\psi_0 - \psi_6)] \quad \text{from gridbox 0567} \quad (11)$$

$$\frac{1}{8\Delta^2}[(\psi_8 - \psi_0)(\zeta_1 - \zeta_7) - (\zeta_8 - \zeta_0)(\psi_1 - \psi_7)] \quad \text{from gridbox 0781} \quad (12)$$

Adding these together gives the expression

$$J_0 = \frac{1}{8\Delta^2}[(\psi_8 - \psi_2)\zeta_1 + (\psi_1 - \psi_3)\zeta_2 + (\psi_2 - \psi_4)\zeta_3 + (\psi_3 - \psi_5)\zeta_4 + (\psi_4 - \psi_6)\zeta_5 + (\psi_5 - \psi_7)\zeta_6 + (\psi_6 - \psi_8)\zeta_7 + (\psi_7 - \psi_1)\zeta_8] \quad (13)$$

Similarly, the coefficient of $\alpha_0$ in (7) receives the contribution

$$\frac{1}{8\Delta^2}(\psi_0 + \psi_1 + \psi_2 + \psi_3)(\zeta_3 - \zeta_1) \quad \text{from gridbox 0123} \quad (14)$$

$$\frac{1}{8\Delta^2}(\psi_0 + \psi_3 + \psi_4 + \psi_5)(\zeta_5 - \zeta_3) \quad \text{from gridbox 0345} \quad (15)$$

$$\frac{1}{8\Delta^2}(\psi_0 + \psi_5 + \psi_6 + \psi_7)(\zeta_7 - \zeta_5) \quad \text{from gridbox 0567} \quad (16)$$

$$\frac{1}{8\Delta^2}(\psi_0 + \psi_7 + \psi_8 + \psi_1)(\zeta_1 - \zeta_7) \quad \text{from gridbox 0781} \quad (17)$$

Adding these together gives the expression

$$J_2 = \frac{1}{8\Delta^2}[(\psi_7 + \psi_8 - \psi_2 - \psi_3)\zeta_1 + (\psi_1 + \psi_2 - \psi_4 - \psi_5)\zeta_3 + (\psi_3 + \psi_4 - \psi_6 - \psi_7)\zeta_5 + (\psi_5 + \psi_6 - \psi_8 - \psi_1)\zeta_7] \quad (18)$$

Finally, the coefficient of $\alpha_0$ in (8) receives the contribution
\[
\frac{1}{8\Delta^2} (\zeta_0 + \zeta_1 + \zeta_2 + \zeta_3)(\psi_1 - \psi_3) \quad \text{from gridbox 0123} \quad (19)
\]

\[
\frac{1}{8\Delta^2} (\zeta_0 + \zeta_3 + \zeta_4 + \zeta_5)(\psi_3 - \psi_5) \quad \text{from gridbox 0345} \quad (20)
\]

\[
\frac{1}{8\Delta^2} (\zeta_0 + \zeta_5 + \zeta_6 + \zeta_7)(\psi_5 - \psi_7) \quad \text{from gridbox 0567} \quad (21)
\]

\[
\frac{1}{8\Delta^2} (\zeta_0 + \zeta_7 + \zeta_8 + \zeta_1)(\psi_7 - \psi_1) \quad \text{from gridbox 0781} \quad (22)
\]

Adding these together gives the expression

\[
J3_0 = \frac{1}{8\Delta^2} \left[ (\psi_7 - \psi_3)\zeta_1 + (\psi_1 - \psi_3)\zeta_2 + (\psi_1 - \psi_3)\zeta_3 + (\psi_3 - \psi_5)\zeta_4 \\
+ (\psi_3 - \psi_7)\zeta_5 + (\psi_5 - \psi_7)\zeta_6 + (\psi_5 - \psi_7)\zeta_7 + (\psi_7 - \psi_1)\zeta_8 \right] \quad \text{(23)}
\]

Each of (13), (18) and (23) is a logical approximation to \( J(\psi, \zeta) \) at the point 0. To see this, substitute the Taylor expansions

\[
\psi_i = \psi_0 + \Delta \psi_i + \cdots \quad \text{(24)}
\]

etc. for the \( \psi_i \) and \( \zeta_i \). For example, (23) becomes

\[
J3_0 = \frac{1}{8} \left[ (-2\psi_y) \left( \frac{\zeta_0}{\Delta} + \zeta_x \right) + (\psi_y - \psi_x) \left( \frac{\zeta_0}{\Delta} + \zeta_x + \zeta_y \right) + (2\psi_y) \left( \frac{\zeta_0}{\Delta} + \zeta_y \right) + (\psi_y + \psi_x) \left( \frac{\zeta_0}{\Delta} - \zeta_x + \zeta_y \right) \\
+ (2\psi_y) \left( \frac{\zeta_0}{\Delta} - \zeta_x \right) + (-\psi_x + \psi_y) \left( \frac{\zeta_0}{\Delta} - \zeta_x - \zeta_y \right) + (-2\psi_y) \left( \frac{\zeta_0}{\Delta} - \zeta_y \right) + (-\psi_y - \psi_x) \left( \frac{\zeta_0}{\Delta} + \zeta_x - \zeta_y \right) \right] \\
+ O(\Delta^2)
\]

\[
= \psi_y \zeta_x - \psi_x \zeta_y + O(\Delta^2) \quad \text{(25)}
\]

Of course, the same must be true of the average that appears in (5). To obtain the explicit form of (5), we substitute from (13), (18) and (23) to obtain
\[
\frac{d\zeta_0}{dt} + \frac{1}{24\Delta^2} \left[ (2\psi_2 + 2\psi_3 - 2\psi_2 - 2\psi_3)\zeta_1 + (2\psi_1 - 2\psi_3)\zeta_2 \\
+ (2\psi_1 + 2\psi_3 - 2\psi_2 - 2\psi_3)\zeta_3 + (2\psi_3 - 2\psi_3)\zeta_4 \\
+ (2\psi_3 + 2\psi_4 - 2\psi_2 - 2\psi_4)\zeta_5 + (2\psi_3 - 2\psi_3)\zeta_6 \\
+ (2\psi_3 + 2\psi_6 - 2\psi_3 - 2\psi_6)\zeta_7 + (2\psi_3 - 2\psi_3)\zeta_8 \right] = 0
\]  

(26)

Expressing this in \textit{ij}-notation, we have

\[
\frac{d\zeta_{ij}}{dt} + \frac{1}{12\Delta^2} \left[ (\psi_{i+1,j} - \psi_{i,j-1})\zeta_{i,j} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{i,j+1} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{i+1,j} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{i+1,j+1} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{i,j} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{i,j+1} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{i+1,j} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{i+1,j+1} \right] = 0
\]

(27)

which is equivalent to the formula (2).

Note that, although the general procedure gives a formula analogous to (27) at boundary points, the specific formula (27) holds only at interior points. We can use (27) everywhere if we assume that the fluid extends infinitely in both directions. A convenient way to do this is to assume that the flow is periodic in both directions. We therefore make the assumption of periodic flow.

By construction, (27) conserves energy and enstrophy. To explicitly test enstrophy conservation, we use (27) to show that

\[
\sum_{ij} \zeta_{ij} \frac{d\zeta_{ij}}{dt} = 0
\]

(28)

That is, we must show that

\[
\sum_{ij} \left[ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{ij} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{ij} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{ij} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{ij} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{ij} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{ij} \\
+ (\psi_{i+1,j} - \psi_{i,j+1})\zeta_{ij} + (\psi_{i,j+1} - \psi_{i,j+1})\zeta_{ij} \right] = 0
\]

(29)

Since the flow is infinitely periodic, each summand in (29) can be shifted so that its psi factor is \(\psi_{ij}\). Then, since (29) must vanish for arbitrary \(\psi\) and \(\zeta\), the quadratic in \(\zeta\) must vanish. That is, expressing (29) as
we find that

\[
F_{ij} = \left( \xi_{ij+1} \xi_{i+1,j+1} + \xi_{i+1,j} \xi_{i+1,j+1} - \xi_{i+1,j-1} \xi_{i+1,j} - \xi_{i,j+1} \xi_{i+1,j+1} \right) \\
+ \left( \xi_{i+1,j} \xi_{i+1,j+1} - \xi_{i+1,j-1} \xi_{i+1,j} \right) \\
+ \left( \xi_{i+1,j} \xi_{i+1,j+1} - \xi_{i+1,j-1} \xi_{i+1,j} \right) \\
+ \left( \xi_{i+1,j+1} \xi_{i+1,j+1} - \xi_{i+1,j-1} \xi_{i+1,j} \right)
\]

(31)

As expected, the terms in (31) sum to zero.

2. Write the given momentum equations as

\[
\frac{D\mathbf{u}}{Dt} - (\xi + f) v + w \frac{\partial \mathbf{u}}{\partial z} = -\frac{\partial}{\partial x} \left( \phi + \frac{1}{2} u^2 + \frac{1}{2} v^2 \right)
\]

\[
\frac{D\mathbf{v}}{Dt} + (\xi + f) v + w \frac{\partial \mathbf{v}}{\partial z} = -\frac{\partial}{\partial y} \left( \phi + \frac{1}{2} u^2 + \frac{1}{2} v^2 \right)
\]

\[
-u \frac{\partial \mathbf{u}}{\partial z} - v \frac{\partial \mathbf{v}}{\partial z} = -\frac{\partial}{\partial z} \left( \phi + \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) + \theta
\]

where

\[
\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\]

The motivation for this is to push as many terms as possible into the gradient term. The above momentum equations are equivalent to

\[
\frac{\partial \mathbf{u}}{\partial t} + \omega_p \times \mathbf{v} = -\nabla \Phi + \Theta \mathbf{k}
\]

where
\[ u = (u,v,0), \quad v = (u,v,w), \quad \Phi = \phi + \frac{1}{2} u^2 + \frac{1}{2} v^2 \]

and

\[ \omega_p = (-v_z, u_z, \xi + f) = \nabla \times u + f k \]

Taking the curl of the momentum equation gives

\[ \frac{\partial}{\partial t} \omega_p + (v \cdot \nabla) \omega_p - (\omega_p \cdot \nabla)v + 0 + 0 = 0 + (-\theta_y, \theta_x, 0) \]

where we have used \( \nabla \cdot v = \nabla \cdot \omega_p = 0 \). Thus

\[ \frac{D}{Dt} \omega_p = (\omega_p \cdot \nabla)v + (-\theta_y, \theta_x, 0) \]

and hence

\[ \nabla \theta \cdot \frac{D}{Dt} \omega_p = \nabla \theta \cdot ((\omega_p \cdot \nabla)v) \]

The final step is the same as in Ertel’s theorem. Adding the equation directly above to

\[ \omega_p \cdot \nabla \frac{D\theta}{Dt} = 0 \]

gives

\[ \frac{D}{Dt} (\omega_p \cdot \nabla \theta) = 0 \]

The primitive-equation potential vorticity

\[ Q = \omega_p \cdot \nabla \theta = \left( f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial \theta}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial \theta}{\partial z} \]

differs from

\[ Q_{Ertel} = \frac{(\nabla \times v + 2\Omega) \cdot \nabla \eta}{\rho} \]

in the replacement of \( 2\Omega \) by its vertical component \( f \) (the traditional approximation); the replacement of the entropy \( \eta \) by the buoyancy \( \theta \); the replacement of \( \rho \) by a constant (the Boussinesq approximation); and the replacement of

\[ \nabla \times v = \left( \begin{array}{c} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial z} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} - \frac{\partial w}{\partial z} \frac{\partial u}{\partial y} - \frac{\partial w}{\partial z} \frac{\partial u}{\partial x} \end{array} \right) \]

by

\[ \left( \begin{array}{c} \frac{\partial v}{\partial z} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial x} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \end{array} \right) \]

(related to the hydrostatic approximation).