SIO 214A Homework 2 Answers

1.) The eqns for a perfect barotropic fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{2}$$

$$p = F(\rho) \tag{3}$$

To get an energy equation, you could take the dot-product of (2) with $\mathbf{u}$ to get

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) = -\mathbf{u}\cdot\nabla p \tag{4}$$

Using (1), this can be written

$$\frac{\partial K}{\partial t} + \nabla \cdot (\mathbf{u}K) = -\nabla \cdot (\mathbf{u}p) + p\nabla \cdot \mathbf{u} \tag{5}$$

where

$$K = \frac{1}{2}\rho \mathbf{u}\cdot\mathbf{u} \tag{6}$$

is the kinetic energy per unit volume. Using (3) and the relation

$$F(\rho) = \rho^2 \frac{d}{d\rho} E(\rho) \tag{7}$$

(5) becomes

$$\frac{\partial K}{\partial t} + \nabla \cdot (\mathbf{u}K) = -\nabla \cdot (\mathbf{u}p) + \rho^2 \frac{d}{d\rho}E(\rho)\nabla \cdot \mathbf{u} \tag{8}$$

Substituting (1) into the last term gives

$$\frac{\partial K}{\partial t} + \nabla \cdot (\mathbf{u}K) = -\nabla \cdot (\mathbf{u}p) - \rho^2 \frac{d}{d\rho}E(\rho)\frac{1}{\rho}\frac{D\rho}{Dt} \tag{9}$$

$$= -\nabla \cdot (\mathbf{u}p) - \rho\frac{D}{Dt}E(\rho) \tag{10}$$

which can also be written

$$\frac{\partial}{\partial t}(K + \rho E(\rho)) + \nabla \cdot (\mathbf{u}(K + \rho E(\rho)) = -\nabla \cdot (\mathbf{u}p) \tag{11}$$

Integrating this over a rigid container on whose boundary the normal velocity vanishes ($\mathbf{u} \cdot \hat{\mathbf{n}} = 0$) gives the desired result,

$$\frac{d}{dt} \iiint dxdydz \left(\frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho E(\rho) \right) = 0 \tag{12}$$

These manipulations could be performed in many different ways; Kundu (p. 121-125) uses summation-convention notation and includes viscosity, which we are still omitting. However you do it, it is very handy to remember that

$$\rho \frac{DA}{Dt} = \frac{\partial}{\partial t}(\rho A) + \nabla \cdot (\mathbf{u} \rho A) \tag{13}$$

where $A(x, y, z, t)$ is anything.

For adiabatic flow

$$dE = -PdV + TdS \tag{14}$$

reminds us that

$$P = -\frac{dE}{dV} \tag{15}$$

If E is the internal energy per unit mass, and $\alpha = \frac{1}{\rho}$ is the volume per unit mass, this becomes

$$p = -\frac{\partial E}{\partial \alpha} \tag{16}$$

which is equivalent to (7).

The linear equations

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \tag{17}$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' \tag{18}$$

$$p' = c^2 \rho' \tag{19}$$

are easier to handle:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \mathbf{u} \cdot \mathbf{u} \right) = -\mathbf{u} \cdot \nabla p' \tag{20}$$

$$= -\nabla \cdot (\mathbf{u} p') + p' \nabla \cdot \mathbf{u} \tag{21}$$

$$= -\nabla \cdot (\mathbf{u} p') - c^2 \rho' \frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} \tag{22}$$

which implies

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \mathbf{u}\cdot\mathbf{u} + \frac{1}{2}c^2\frac{(\rho')^2}{\rho_0}\right) = -\nabla\cdot(\mathbf{u}p') \tag{23}$$

Compare (23) to (11). The advection terms in (11) are missing from (23). Also, the internal energies look very different. By Taylor expansion, we have

$$\rho E(\rho) = \rho_0 E(\rho_0) + \frac{d}{d\rho}[\rho E(\rho)]|_{\rho=\rho_0}(\rho'-\rho) + \frac{1}{2}\frac{d^2}{d\rho^2}[\rho E(\rho)]|_{\rho=\rho_0}(\rho'-\rho)^2 + \cdots \tag{24}$$

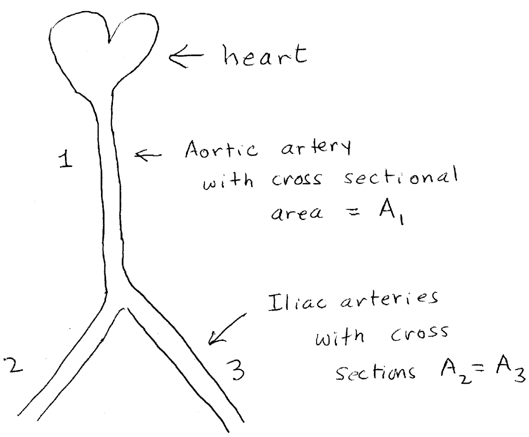
The first term is an irrelevant constant; by (17), the spatial integral of the second term is separately conserved; and the coefficient of the third term simplifies as follows:

$$\frac{d^2}{d\rho^2}(\rho E(\rho)) = \frac{d}{d\rho}\left(\frac{p}{\rho} + E(\rho)\right) = \left(-\frac{p}{\rho^2} + \frac{dE}{d\rho} + \frac{1}{\rho}\frac{dp}{d\rho}\right) = \left(0 + \frac{c^2}{\rho}\right) \tag{25}$$

and thus the last term in (24) becomes

$$\frac{1}{2}\frac{c^2}{\rho_0}(\rho'-\rho)^2 \tag{26}$$

This represents the *available internal energy*, the internal energy that could be wholly converted into kinetic energy if the fluid were to arrange itself into a state in which $\rho' = 0$. It is entirely analogous to the *available potential energy* that you will hear about in other courses.



2.) The pressure associated with the waves in the aorta is

$$p_1(x,t) = F(t - x/c) + G(t + x/c) \tag{27}$$

where F is set by the heart, and G represents the wave reflected by the iliac bifurcation. The pressure in the two iliac arteries is

$$p_2(x,t) = p_3(x,t) = H(t - x/c) \tag{28}$$

where H represents the transmitted waves. The corresponding blood velocities are given by

$$\rho_0 c u_1(x,t) = F(t - x/c) - G(t + x/c) \tag{29}$$

and

$$\rho_0 c u_2(x,t) = \rho_0 c u_3(x,t) = H(t - x/c) \tag{30}$$

We are given the function F and must find G and H. (None of this includes the steady background blood velocity.)

Let $x = 0$ at the bifurcation. The matching conditions there are

$$p_1(0,t) = p_2(0,t) = p_3(0,t) \tag{31}$$

(continuous momentum flux) and

$$A_1 \rho_0 u_1(0,t) = A_2 \rho_0 u_2(0,t) + A_3 \rho_0 u_3(0,t) = 2A_2 \rho_0 u_2(0,t) \tag{32}$$

(continuous mass flux). Applying the matching conditions we obtain

$$F(t) + G(t) = H(t) \tag{33}$$

$$F(t) - G(t) = \frac{2A_2}{A_1} H(t) \tag{34}$$

with solution

$$H(t) = TF(t), \quad G(t) = RF(t) \tag{35}$$

where

$$T = \frac{2A_1}{A_1 + 2A_2}, \quad R = \frac{A_1 - 2A_2}{A_1 + 2A_2} \tag{36}$$

Thus

$$p_1(x,t) = F(t - x/c) + RF(t + x/c) \tag{37}$$

$$\rho_0 c u_1(x,t) = F(t - x/c) - RF(t + x/c) \tag{38}$$

$$p_2(x,t) = p_3(x,t) = TF(t - x/c) \tag{39}$$

$$\rho_0 c u_2(x,t) = \rho_0 c u_3(x,t) = TF(t - x/c) \tag{40}$$

Using the results of the first problem, the energy flux, evaluated at $x = 0$, in the aorta is

$$A_1 p_1(0,t) u_1(0,t) = \frac{A_1}{\rho_0 c}(1 + R)(1 - R)F(t)^2 = \frac{A_1}{\rho_0 c}(1 - R^2)F(t)^2 \tag{41}$$

Note that the reflected wave contributes a negative flux. The energy fluxes in the two iliac arteries is

$$2A_2 p_2(0,t) u_2(0,t) = 2\frac{A_2}{\rho_0 c} T^2 F(t)^2 \tag{42}$$

Since

$$A_1(1 - R^2) = 2A_2 T^2 \tag{43}$$

the energy fluxes balance: The energy flux in the wave emitted by the heart equals the sum of the energy fluxes in the reflected and transmitted waves. It is interesting that the positive pressure pulse reflects as a positive pressure pulse if $A_1 > 2A_2$, which is the situation in the human body. (If $A_1 < 2A_2$, it would reflect as a negative pulse; see (35b).) Since there is a small 'Stokes drift' of blood associated with the pulses, this may be an evolutionary adjustment that prevents blood in the aorta from being depleted.