SIO 214A Homework 3

Please hand in on paper, in class, on Tuesday, November 2

1.) In lecture we followed the path of first considering the limit of incompressible flow, and then specializing to the case of irrotational flow. However, it is possible to reverse these steps. That is, you can *first* assume that the flow is irrotational, and *then* (if you want) assume that it is incompressible.

For a perfect, barotropic fluid governed by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p \tag{2}$$

$$p = F(\rho) \tag{3}$$

show that: If the velocity is irrotational,

$$\mathbf{u} = \nabla \phi \tag{4}$$

then the flow is governed by the following two coupled equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \phi) = 0 \tag{5}$$

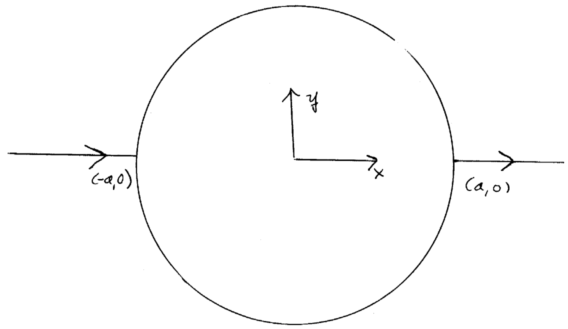
$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla \phi \cdot \nabla \phi + \frac{p}{\rho} + E(\rho) = 0 \tag{6}$$

for $\phi(x, y, z, t)$ and $\rho(x, y, z, t$. Eqns (5-6) are closed set of equations for inviscid, barotropic, compressible, irrotational flow.

Hint: Once again you will need the relation

$$p = F(\rho) = \rho^2 \frac{dE}{d\rho} = -\frac{dE}{d\alpha} \tag{7}$$

where $\alpha = 1/\rho$ is the specific volume.



2.) A circular lake of radius a and constant depth h is fed by river inflow from the left through the river mouth at $(x, y) = (-a, 0)$. Water exits the lake through the river mouth at $(x, y) = (a, 0)$. The rate at which water enters the lake is Q (volume per unit time). The width of both rivers is $w << a$.

The flow within the lake has been determined to be nondivergent and irrotational, and hence governed by the equations

$$\mathbf{u} = (\phi_x, \phi_y) = (-\psi_y, \psi_x), \quad \nabla^2\phi = \nabla^2\psi = 0 \tag{8}$$

What are the appropriate boundary conditions for $\phi(x, y)$ and $\psi(x, y)$?

Show, using the divergence theorem, that the outflow at $(x, y) = (a, 0)$ must equal the inflow at $(x, y) = (-a, 0)$.

Determine the velocity potential $\phi(x, y)$ within the lake. Verify that your solution satisfies the boundary conditions.

At what location is the flow speed a minimum? What is its value at this point?

Hint: An appropriately placed source and sink may be key.