A rigid semi-circular hoop rolls without slipping as shown in the figure. Its radius is \( R \), and \( \rho \) is its mass per unit length. Using the principle of stationary action, find the equation for \( \theta(t) \).

Hamilton’s principle holds in an inertial reference frame. We choose the Cartesian frame whose origin lies at the equilibrium position of the hoop. Then points on the hoop have the coordinate locations

\[
(1) \quad x_\alpha(\theta) = -R\theta + R\sin(\alpha + \theta), \quad z_\alpha(\theta) = R - R\cos(\alpha + \theta)
\]

where \( \alpha \) is an angular label that identifies points on the hoop: \( \alpha = -\pi / 2 \) for the point farthest left and \( \alpha = +\pi / 2 \) for the point at the top in the figure. Differentiating (1) we obtain the velocity components \( \dot{x}_\alpha \) and \( \dot{z}_\alpha \). From these we compute the kinetic energy

\[
(2) \quad T = \int_{-\pi/2}^{+\pi/2} d\alpha \ R\rho \frac{1}{2} \left( \dot{x}_\alpha^2 + \dot{z}_\alpha^2 \right) = \int_{-\pi/2}^{+\pi/2} d\alpha \ R\rho \dot{\theta}^2 R^2 \left( 1 - \cos(\alpha + \theta) \right) = R^3 \rho \dot{\theta}^2 (\pi - 2 \cos \theta)
\]

The potential energy is

\[
(3) \quad V = \int_{-\pi/2}^{+\pi/2} d\alpha \ R\rho g \ z_\alpha = -2R^2 \rho g \cos \theta
\]

Hamilton’s principle states that \( \delta \int dt (T - V) = 0 \). That is,
\begin{align*}
0 &= \delta \int dt \left[ R \dot{\theta}^2 (\pi - 2 \cos \theta) + 2g \cos \theta \right] \\
&= \int dt \left[ 2 \dot{\theta} \frac{d\theta}{dt} (\pi - 2 \cos \theta) + 2R \dot{\theta}^2 \sin \theta \delta \theta - 2g \sin \theta \delta \theta \right] \\
&= \int dt \left[ -\delta \theta \frac{d}{dt} \left( 2R \dot{\theta} (\pi - 2 \cos \theta) \right) + 2R \dot{\theta}^2 \sin \theta \delta \theta - 2g \sin \theta \delta \theta \right] \\
&= \int dt \delta \theta \left[ \frac{d}{dt} \left( -2R \dot{\theta} (\pi - 2 \cos \theta) \right) + 2R \dot{\theta}^2 \sin \theta - 2g \sin \theta \right]
\end{align*}

Equating the square-bracket terms to zero, we obtain the equation of motion,

\[ (\pi - 2 \cos \theta) \ddot{\theta} = -\sin \theta \left( \dot{\theta} \right)^2 - \frac{g}{R} \sin \theta \]

One can obtain (4) much more easily by simply taking the time derivative of

\[ T + V = \text{const} \]

where \( T \) and \( V \) are given by (2) and (3). In fact (5) is a first integral of (4) and it is by far the best way of solving this particular system. However, the energy method—complete solution by using energy conservation alone—fails if there is more than one degree of freedom, i.e. if, for example, we had added another pendulum to our hoop. In that case Hamilton’s principle is the most effective means of deriving the complete set of governing equations.

These facts are worth reflecting upon. In general, energy conservation provides only a single equation for a system with arbitrarily many degrees of freedom, and is therefore generally only marginally helpful. In contrast, Hamilton’s principle, which only requires us to know how to write down the energy, is always capable of yielding the full set of equations.

To appreciate the difficulties and pitfalls of solving this same problem using Newtonian mechanics, see