These 2 problems give you practice in deducing equations from variational principles. In the first problem, the variations are like \( \delta x(a,b,c,\tau) \), as in the form of Hamilton’s principle discussed in lecture. In the second problem, the variations are \( \delta \phi(x,y,z,t) \) and \( \delta \eta(x,y,t) \), and the relation to a Hamilton’s principle is (at this point) obscure.

1. In an elastic solid, the particle labels \((a,b,c) = (a_1,a_2,a_3)\) are usually taken to be the Cartesian locations of the particles in the equilibrium (relaxed) state. Then

\[
\xi_i(a,\tau) \equiv x_i(a,\tau) - a_i
\]

is the displacement of particles from their equilibrium locations. Assuming that the density of the equilibrium state is uniform, the kinetic energy of the system is

\[
T = \iiint da \left( \frac{1}{2} \sum_{i,j} \frac{\partial \xi_i}{\partial \tau} \frac{\partial \xi_j}{\partial \tau} \right).
\]

Taking the internal energy to be

\[
V = \iiint da \left\{ \frac{\lambda}{2} \left( e_{ij} \right)^2 + \mu e_{ij} e_{ij} \right\}
\]

where

\[
e_{ij} = \frac{1}{2} \left( \frac{\partial \xi_i}{\partial a_j} + \frac{\partial \xi_j}{\partial a_i} \right)
\]

and repeated indices are summed, show that Hamilton’s principle leads to the equations

\[
\frac{\partial^2 \xi_i}{\partial \tau^2} = (\lambda + \mu) \left( \frac{\partial^2 \xi_i}{\partial a_j \partial a_j} \right) + \mu \frac{\partial}{\partial a_i \partial a_j} \xi_j
\]

Note the curious analogy between these terms and the form of the viscous terms in the general Navier-Stokes equation for a fluid. See, e.g., Batchelor pp. 142-150.

2. Luke’s variational principle. [JFM 27, p. 395 (1967)]. Show that the equations (including boundary conditions) for an incompressible, irrotational fluid with a free surface at \( z = \eta(x,y,t) \) and a rigid bottom at \( z = -H(x,y) \) result from the requirement that

\[
\int_{t_1}^{t_2} dt \int \int dx \ dy \int_{-H(x,y)}^{\eta(x,y,t)} dz \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + g \phi \right)
\]

be stationary with respect to arbitrary, independent variations \( \delta \phi(x,y,z,t) \) and \( \delta \eta(x,y,t) \) that vanish at \( t_1 \) and \( t_2 \). Hint: You will need to use Leibniz’s rule to deal with the boundary terms.