(a) The Poisson bracket is defined by

\[ \{ A, B \} = \frac{\partial A}{\partial z^i} J^{ij} \frac{\partial B}{\partial z^j} \]  

(1)

where repeated indices are summed. Show that the Jacobi identity

\[ \{ A, \{ B, C \} \} + \{ B, \{ C, A \} \} + \{ C, \{ A, B \} \} = 0 \]  

(2)

is equivalent to

\[ J^{im} \frac{\partial J^{jk}}{\partial z^m} + J^{lm} \frac{\partial J^{ki}}{\partial z^m} + J^{km} \frac{\partial J^{ij}}{\partial z^m} = 0 \]  

(3)

Answer. Substituting (1) into (2) yields

\[ \frac{\partial A}{\partial z^i} J^{ij} \frac{\partial B}{\partial z^j} \left( \frac{\partial B_{l}}{\partial z^l} J^{lm} \frac{\partial C}{\partial z^m} \right) + \text{cyc}(A, B, C) = 0 \]  

(4)

Expanding this out,

\[ \frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^l} \frac{\partial C}{\partial z^j} J^{ij} J^{lm} \frac{\partial C}{\partial z^m} + \text{cyc}(A, B, C) \]  

\[ + \frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^l} \frac{\partial C}{\partial z^j} \frac{\partial B}{\partial z^m} J^{ij} J^{lm} + \text{cyc}(A, B, C) \]  

\[ + \frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^l} \frac{\partial C}{\partial z^j} \frac{\partial^2 B}{\partial z^m \partial z^j} J^{ij} J^{lm} + \text{cyc}(A, B, C) = 0 \]  

(5)

Now we go to work on the last line in (5). It can be rewritten

\[ \frac{\partial C}{\partial z^i} \frac{\partial A}{\partial z^l} \frac{\partial B}{\partial z^m} \frac{\partial^2 B}{\partial z^m \partial z^j} J^{ij} J^{lm} + \text{cyc}(A, B, C) \]  

(6)

Next we note that all the indices in (6) are repeated; they are all dummy indices. Thus we may permute the indices as follows:

\[ m \to l \]
\[ l \to i \]
\[ i \to m \]  

(7)

The result is

\[ \frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^l} \frac{\partial C}{\partial z^j} J^{mj} J^{il} + \text{cyc}(A, B, C) \]  

(8)

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The strategy here is to make (8) look as much like the second line in (5) as possible. We are almost there. The final step is to interchange \( j \) and \( l \) in (8). This gives

\[
\frac{\partial A}{\partial z^i} \frac{\partial^2 B}{\partial z^j \partial z^l} \frac{\partial C}{\partial z^m} J^{ml} J^{ij} + \text{cyc}(A, B, C)
\]

(9)

By the antisymmetry of \( J^{ml} \), (9) cancels the second line in (5). Thus (5) reduces to

\[
\frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^j} \frac{\partial C}{\partial z^m} J^{ij} \frac{\partial J^{lm}}{\partial z^j} + \text{cyc}(A, B, C) = 0
\]

(10)

Since \( A, B, C \) are arbitrary functionals, (10) implies (3). QED.

(b) Verify that

\[
\frac{dF}{dt} = \{F, H\} = \int \int d\mathbf{x} \left\{ \frac{\delta(F, H)}{\delta(u, v)} - \frac{\delta F}{\delta \mathbf{u}} \cdot \nabla \frac{\delta H}{\delta h} + \frac{\delta H}{\delta \mathbf{u}} \cdot \nabla \frac{\delta F}{\delta h} \right\}
\]

(11)

implies the shallow water equations, where \( F[u, v, h] \) is an arbitrary functional, \( H[u, v, h] \) is the shallow-water Hamiltonian, and

\[
q = \frac{\zeta + f}{h}
\]

(12)

is the potential vorticity. Then show that

\[
\{A, C\} = 0
\]

(13)

for any \( A \) whatsoever, and any \( C \) of the form

\[
C = \int \int d\mathbf{x} \ hG(q)
\]

(14)

where \( G(q) \) is an arbitrary function of the potential vorticity \( q \). Answer. It is relatively easy to verify the equations. To show that \( C \) is a Casimir, first note that

\[
\delta C = \int \int d\mathbf{x} \ \{\delta h \ G(q) + hG'(q)\delta q\}
\]

\[
= \int \int d\mathbf{x} \ \{\delta h \ G(q) + hG'(q)(-\frac{\zeta + f}{h^2} \delta h + \frac{(\delta v)_x - (\delta u)_y}{h})\}
\]

\[
= \int \int d\mathbf{x} \ \{(G - qG')\delta h + (G')_y \delta u - (G')_x \delta v\}
\]

(15)
implies that

\[
\frac{\delta C}{\delta h} = G - qG' \\
\frac{\delta C}{\delta u} = (G')_y \\
\frac{\delta C}{\delta v} = -(G')_x
\]

Thus

\[
\{A, C\} = \int \int d\mathbf{x} \left\{ q \frac{\delta (A, C)}{\delta (u, v)} - \frac{\delta A}{\delta u} \cdot \nabla \frac{\delta C}{\delta h} + \frac{\delta C}{\delta u} \cdot \nabla \frac{\delta A}{\delta h} \right\} \\
= \int \int d\mathbf{x} \left\{ -q \frac{\delta A}{\delta u} (G')_x - q \frac{\delta A}{\delta v} (G')_y - \frac{\delta A}{\delta u} \cdot \nabla (G - qG') - (\nabla \cdot \frac{\delta C}{\delta u}) \frac{\delta A}{\delta h} \right\}
\]

which vanishes for any \( A \), because the coefficients of \( \delta A/\delta u, \delta A/\delta v, \delta A/\delta h \) vanish separately, for any function \( G(q) \). QED