Lectures on the Statistical Theory of Turbulence

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PUBLISHERS NOTE: In the typeface chosen to set this book, the italic lower-case letter "*v*" looks very similar to the small Greek letter "*ν*" (nu). The publishers apologize for any ambiguity resulting from this style of type.

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1.1 Introduction

A turbulent flow is a fluid motion that possesses complex and seemingly random structure at some macroscopic scale of dynamical importance. Turbulent flows abound in nature. The motions of the atmosphere and ocean are, to a large extent, turbulent. In fact, turbulence plays a vital part in the dynamics of the widest variety of fluid motions on scales from millimeters to light years.

The most important physical consequence of turbulence is the enhancement of transport processes. In a turbulent flow, momentum, energy, and particle transport rates greatly exceed the corresponding molecular transport rates. Practical effects include the efficient turbulent mixing of vermouth with gin to make martinis, as well as the efficient turbulent mixing of pollutants with air to make smog. (Molecular diffusion would require at least a day to mix a martini at room temperature!)

Turbulent flows exhibit much more small-scale structure than their nonturbulent counterparts. In fact, this small-scale structure is correlated with enhanced turbulent transport phenomena. Small-scale structure itself is evidence of enhanced transport in the sense that small scales develop from the degradation of large-scale excitations and are maintained by energy transport from one scale to another.

Another important characteristic of turbulent flows is their apparent randomness and instability to small perturbations. Two turbulent flows that are at some time nearly identical in detail do not remain nearly identical on the time scales of dynamical interest. This property of turbulent flows may be used to give a quantitative definition of turbulence. Also, instability of turbulent motion is related to the limited "predictability" of atmospheric motions.

While the details of fully developed turbulent motions are extremely sensitive to triggering disturbances, average properties are not. Otherwise there would be little significance in the averages. On the other hand, transition flows (which occur naturally at Reynolds numbers several times critical) have statistical properties which are sensitive to the nature of perturbations. The idea that fully-developed turbulent flows are extremely sensitive to small perturbations but have statistical properties that are insensitive to perturbations is of central importance throughout these Lectures.

In the remainder of this section, we illustrate the above by simple examples in order to set out the main trend of ideas in turbulence theory.

1.2 Line Stretching by Isotropic Turbulence

An illustration of the enhanced transport property of turbulence is given by an elementary, but rigorous, proof that infinitesimal line elements are stretched (on the average) by homogeneous isotropic incompressible turbulence (Cocke, 1969. Orszag, 1970a). Here the turbulence is said to be *homogeneous* if all points in space are statistically equivalent; it is said to be *isotropic* if all directions are statistically equivalent. Consider the infinitesimal material line element $\delta\mathbf{x}(t)$ between two fluid particles, say A and B, separated by the (nonrandom) distance $\delta x(0)$ at $t = 0$. We show that

$$\langle \delta x(t)^2 \rangle \gg \delta x(0)^2 \tag{1.1}$$

for all $t \neq 0$, where $\langle\ \rangle$ denotes an appropriate average over the turbulence statistics (cf. §. 1.3). Since the infinitesimal separation vector $\delta\mathbf{x}(t)$ must be linearly related to $\delta\mathbf{x}(0)$, it follows that there exists a matrix U_{ij} [which is a random function of time and the *initial* spatial position of particle A but not of the direction or magnitude of $\delta\mathbf{x}(0)$] such that

$$\delta x_j(t) = U_{ij}\delta x_j(0). \tag{1.2}$$

Therefore,

$$\delta x(t)^2 = W_{jk}\delta x_j(0)\delta x_k(0),\ W_{jk} = U_{ij}U_{ik}, \tag{1.3}$$

where W_{jk} is a real symmetric matrix. Since $\delta\mathbf{x}(0)$ is not random, it follows that

$$\langle \delta x(t)^2 \rangle = \langle W_{jk} \rangle \delta x_j(0)\delta x_k(0). \tag{1.4}$$

Isotropy and homogeneity imply that $\langle W_{jk} \rangle = \lambda(t)\delta_{jk}$ since no point $\mathbf{x}$ or direction can be preferred. Hence,

$$\langle \delta x(t)^2 \rangle = \lambda(t)\delta x(0)^2. \tag{1.5}$$

The proof is completed by showing that $\lambda(t) \gg 1$.

For fixed t, denote the (real) eigenvalues of the symmetric matrix W by w_1, w_2, w_3 and the corresponding (real) eigenvectors by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Since $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ form an orthogonal triad and the volume of the infinitesimal element about particle A spanned by $\epsilon\mathbf{a}_1$, $\epsilon\mathbf{a}_2$, $\epsilon\mathbf{a}_3$ ($\epsilon \ll 1$) is conserved in evolution from 0 to t (by incompressibility), it follows that $w_i > 0$ ($i = 1, 2, 3$) and

$$w_1 w_2 w_3 = 1. \tag{1.6}$$

[After time evolution from 0 to t the vectors about A resulting from $\epsilon\mathbf{a}_1$, $\epsilon\mathbf{a}_2$, $\epsilon\mathbf{a}_3$ at $t = 0$ have lengths ϵw_1, ϵw_2, ϵw_3, respectively, and are still orthogonal (since $\delta\mathbf{x}(t) \cdot \delta\mathbf{y}(t) = W_{jk}\delta x_j(0)\,\delta y_k(0)$).] By the arithmetic-geometric-

mean inequality

$$\lambda = \tfrac{1}{3}\langle TrW\rangle = \tfrac{1}{3}\langle w_1 + w_2 + w_3\rangle \gg \langle (w_1 w_2 w_3)^{1/3}\rangle = 1,$$

proving that line elements are stretched.

The same proof applies to show that surface elements are stretched, to show line stretching in an arbitrary number of dimensions, and to show that (1.1) holds for both $t > 0$ and $t < 0$. The latter result is understood by observing that $t = 0$ is singled as the instant at which $\delta\mathbf{x}$ is nonrandom.

It may appear that a similar proof applies to show that vortex lines are stretched by homogeneous isotropic turbulence. However, it does not. The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ bears a very special relation to the velocity field $\mathbf{v}$, so that although vortex lines move with the fluid (in the inviscid limit), their special phase relation with the velocity precludes the statistical independence of W with $\delta\mathbf{x}(0)$ required for (1.4) to be correct. In fact, the growth of vorticity is constrained by the conservation of kinetic energy so that enhancement of vorticity on small scales must have as a consequence its diminution on large scales.† Thus, the growth rate of vorticity should be expected to be less than that of line elements. This inference has a physical consequence: eddy viscosity ν_e (which governs the enhanced turbulent dissipation of kinetic energy and is related to vortex stretching) should be expected to be less than eddy diffusivity κ_e (which governs the turbulent dissipation of heat and is related to line stretching). In fact, it has been argued (e.g. Kraichnan, 1962) that the eddy Prandtl number ν_e/κ_e is at most 0.05 in order to correlate eddy transport ideas with the large-scale energetics of convection.

1.3 Methods of Taking Averages

There are at least three different kinds of averaging procedures that may be used to obtain statistically-averaged properties from a turbulent flow. They are space averages, time averages, and ensemble averages. The usefulness of space averaging is limited to flows that are statistically homogeneous or at least approximately homogeneous over scales larger than those of the turbulent fluctuations. Similarly, time averages are useful only if the turbulence is, in effect, statistically stationary over time scales much larger than the time scale of the turbulent fluctuations. In practice, because of the convenience afforded by locating a probe at a fixed point in space and integrating in time, experimental results are usually obtained as time averages.

The third type of average, ensemble averages, are most versatile. Here the average is taken over an ensemble of turbulent flows prepared under nearly

†It follows from (2.36) that $\mathrm{d}/\mathrm{d}t \int_0^\infty k^{-2}\,\Omega(k)\mathrm{d}k < 0$, where $\Omega(k)$ is the mean-square vorticity spectrum.

identical external conditions. Of course, these flows are not nearly identical because of the large fluctuations present in turbulence. Each member of the ensemble is called a *realization*. To make the notion of ensemble average more precise (but no more understandable), define a parameter α, $-\infty < \alpha < \infty$, to distinguish between realizations and weight each realization according to the probability density $P(\alpha)$. The ensemble average of a quantity that takes the value $F(\alpha)$ in realization α, e.g. the velocity field $\mathbf{v}(\mathbf{x}, t; \alpha)$, is defined to be

$$\langle F \rangle = \int F(\alpha) P(\alpha) \mathrm{d}\alpha. \tag{1.7}$$

The weight factor $P(\alpha)$ is related to the n-point joint probability densities of the turbulence. Define

$$f_n(\mathbf{v}_1, \mathbf{x}_1, t_1; \mathbf{v}_2, \mathbf{x}_2, t_2; \ldots; \mathbf{v}_n, \mathbf{x}_n, t_n)$$

as the probability density that the velocity is $\mathbf{v}_i$ at each of the space-time points $\mathbf{x}_i$, $t_i (i = 1, \ldots, n)$. It follows that

$$f_n(\mathbf{v}_1, \mathbf{x}_1, t_1; \ldots; \mathbf{v}_n, \mathbf{x}_n, t_n) = \langle \delta(\mathbf{v}(\mathbf{x}_1, t_1) - \mathbf{v}_1) \ldots \delta(\mathbf{v}(\mathbf{x}_n, t_n) - \mathbf{v}_n) \rangle \tag{1.8}$$

The advantage of ensemble averaging over space-time averaging is that it can be applied to inhomogeneous, nonstationary turbulence. The disadvantage of ensemble averaging is that it does not correspond in a simple way to a physical space-time averaging procedure. This jeopardizes the physical interpretation of ensemble averages. There is no way to decide *a priori* on the distribution function $P(\alpha)$ and the dilemma arises of deciding which ensemble corresponds most closely to physical reality. No such criterion has yet been found.

Fortunately, the physics of turbulence suggests a way out of this problem of interpretation. Since fully developed turbulence involves a large number of interacting degrees of freedom, there should be an *asymptotic statistical state* of turbulence that is independent of fine-grained details of the flows and of $P(\alpha)$. Hopefully, this asymptotic state depends critically only on such simple statistical properties as energy spectra, energy-transfer spectra, etc., much as in statistical mechanical equilibrium where the statistical state is determined by the energy spectrum.

The asymptotic statistical state should not be expected to determine uniquely details of individual realizations, because realizations need not be given the same weight in different ensembles with the same low-order statistical properties. Otherwise said, although fully developed turbulence is unstable, there should be a "statistically stable" asymptotic state.

It should be strongly emphasized that the plausibility argument given

above has been given no firm mathematical basis to date. The existence of an asymptotic statistical state is strongly suggested experimentally, in the sense that reproducible statistical results are obtained. However, physical plausibility aside, it is embarassing that such an important feature of turbulence as its statistical stability should remain mathematically unresolved, but such is the nature of the subject.

Even if approach to an asymptotic state be accepted, there remains a problem of interpretation of ensemble averages. Average quantities determined experimentally are usually time (or space-time) averages. Experiments are not usually repeated a sufficiently large number of times for averages over large ensembles to be made. Comparison of the predictions of turbulence theory with these experiments is impossible unless ensemble averages can be computed as space-time averages over almost every realization of the ensemble. We now establish a simple result of this type.

Suppose that $v(t)$ is a signal measured at a fixed point of a field of statistically stationary turbulence, and that the average value of $f(v(t))$ is desired where $f(\cdot)$ is some given nonrandom function. We wish to determine the conditions under which the ensemble average $\langle f(v)\rangle$ (which is independent of t by statistical stationarity) equals the time average $\lim_{T\to\infty} \overline{f(v)}^T$, where

$$\overline{f(v)}^T = \frac{1}{T}\int_0^T f(v(t))\,dt. \tag{1.9}$$

Since the time average is *a priori* a random variable distributed over members of the ensemble, it is clearly only reasonable to demand equality of averages in the sense of probability, i.e. for almost every realization. It follows from the definitions that

$$\langle(\overline{f(v)}^T - \langle f(v)\rangle)^2\rangle = \frac{1}{T^2}\int_0^T\int_0^T R_f(t-s)\,dt\,ds,$$

$$= \frac{2}{T}\int_0^T R_f(s)\left(1-\frac{s}{T}\right)ds, \tag{1.10}$$

where the *covariance*, defined by

$$R_f(t-s) = \langle(f(v(t)) - \langle f(v)\rangle)(f(v(s)) - \langle f(v)\rangle)\rangle \tag{1.11}$$

is an even function of $t-s$ alone by statistical stationarity. It follows that if the covariance and mean value of $f(v(t))$ exist and if the right-hand side of (1.10) approaches zero as $T\to\infty$, then $\lim_{T\to\infty}\overline{f(v)}^T = \langle f(v)\rangle$ for almost every realization [Lumley, 1970; Monin and Yaglom, 1971 (Chap. 2)].

On physical grounds, a considerably stronger assumption is often made,

viz. that the *integral scales* defined by

$$\mathscr{T}_f = \lim_{T\to\infty} \int_0^T R_f(s)\left(1 - \frac{s}{T}\right) \mathrm{d}s / R_f(0) \tag{1.12}$$

are finite. In this case, (1.10) can be used to estimate the mean-square error made by time-averaging for only a finite time T as

$$\langle (\overline{f(v)}^T - \langle f(v)\rangle)^2\rangle \approx 2R_f(0)\ \mathscr{T}_f/T. \tag{1.13}$$

Eq. (1.13) is useful for estimating the averaging time required to get a good estimate of $\langle f(v)\rangle$. It is frequently true that that integral scale $\mathscr{T}_f$ depends only very weakly on the function f for a given random process $v(t)$. Assume $v(t)$ is Gaussian with zero mean, i.e., $\langle v(t)\rangle = 0$ and

$$\langle v(t_1)v(t_2)\ldots v(t_n)\rangle = \sum_p \langle v(t_{p1})v(t_{p2})\rangle \ldots \langle v(t_{p(n-1)})v(t_{pn})\rangle \tag{1.14}$$

where the sum is over all distinct pairings of $1, \ldots, n$; for example,

$$\langle v_1v_2v_3v_4\rangle = \langle v_1v_2\rangle\langle v_3v_4\rangle + \langle v_1v_3\rangle\langle v_2v_4\rangle + \langle v_1v_4\rangle\langle v_2v_3\rangle.$$

In this case, the theorem of Sarmanov and Zakharov (Lumley, 1970; Sec. 3.11) implies that the largest integral scale of any function of $v(t)$ is the integral scale formed with the absolute value of the covariance function $\langle v(t)v(s)\rangle$. Consequently, the principal dependence of $f(\cdot)$ in the rms error of a finite time average is due to $R_f(0) = \langle (f(v(t)) - \langle f(v)\rangle)^2\rangle$. In fact, if $v(t)$ is Gaussian, then $R_{v^n}(0)/\langle v^n(t)\rangle^2$ equals 2 if $n = 2$, 32/3 if $n = 4$, 226/5 if $n = 6$, and 1280/7 if $n = 8$. Hence, neglecting variations in the integral scales, it requires roughly 16/3 times as long to get similar relative accuracy in $\langle v^4\rangle$ as in $\langle v^2\rangle$, roughly 113/5 times as long for $\langle v^6\rangle$ as $\langle v^2\rangle$, and roughly $640/7 \simeq 91$ times as long for $\langle v^8\rangle$ as $\langle v^2\rangle$. This behavior, due to the great dependence of high powers on infrequent large amplitude excursions of $v(t)$, limits accurate experimental results to averages of low powers of the measured signal (Tennekes and Wyngaard, 1972).

The discussion just given of the equality of ensemble and time averages extends easily to space averages in homogeneous turbulence. The existence of integral scales is reasonable, but should not be accepted too uncritically. It will be shown in Section II that the covariance function $\langle v_1(x_1 + r, x_2, x_3)v_1(x_1, x_2, x_3)\rangle$ behaves either as r^{-3}, r^{-4}, r^{-5}, r^{-6} (!) as $r \to \infty$ depending on the nature of the homogeneous turbulence, but does not decay exponentially as $r \to \infty$. Here v_1 is the x_1 component of the velocity field.

1.4 Finite-Mode Model

In order to understand the nature of "random" solutions to differential equations, and thus get insight into the random nature of turbulence, we

consider the following five-mode dynamical system:

$$\frac{dx_i}{dt} = x_{i+1}x_{i+2} + x_{i-1}x_{i-2} - 2x_{i+1}x_{i-1} \quad (i = 1, \ldots, 5), \tag{1.15}$$

where $x_i = x_{i+5}$ is assumed. The system (1.15) is analogous to the inviscid Navier–Stokes (Euler) equations in the following repsects: (a) the system involves only quadratic interaction among degrees of freedom; (b) energy is conserved in the sense that

$$\frac{d}{dt}\frac{1}{2}\sum_{i=1}^{5} x_i^2 = 0. \tag{1.16}$$

(c) Liouville's theorem is valid in the sense that (cf. Sec. 5.2)

$$\sum_{i=1}^{5} \frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right) = 0 \tag{1.17}$$

As in equilibrium statistical mechanics, it is convenient to introduce the five-dimensional phase space $(x_1, x_2, x_3, x_4, x_5)$ of the system. The state of the system (1.15) is represented by a point in this phase space, while the time history of a particular solution is represented by a continuous curve. It follows from (1.16) that all orbits $\mathbf{x}(t)$ lie on the spheres $\Sigma x_i^2 = \text{const.}$ in phase space. Furthermore, it follows from (1.17) that the time evolution of an ensemble (represented by a "cloud" of points in phase space) conserves measure in phase space (so that the volume of the "cloud" is constant in time).

The energy (1.16) is said to be an "isolating" integral of motion of the system (1.15). An isolating integral of motion is one that provides useful information to restrict an orbit to a non-dense subset of phase space. If energy were not the only isolating integral which did not depend *explicitly* on time then typical orbits of (1.15) would be confined (isolated) to a non-dense subset of the energy surfaces. For example, if (1.15) is reduced to a four-mode system $i = 1, \ldots, 4$ by imposition of $x_{i+4} = x_i$ then it follows that, in addition to the energy integral (1.16),

$$\begin{aligned} \frac{d}{dt}(x_1 - x_3) &= (x_2 + x_4)(x_3 - x_1) \\ \frac{d}{dt}(x_2 - x_4) &= (x_1 + x_3)(x_4 - x_2) \end{aligned} \tag{1.18}$$

so that the sign of $x_1 - x_3$ and the sign of $x_2 - x_4$ are isolating integrals (since they are conserved in time). If $x_1(0) > x_3(0)$ and $x_2(0) > x_4(0)$ then the orbit $\mathbf{x}(t)$ is confined to the region $x_1 \geq x_3, x_2 \geq x_4$ of phase space.

The five mode system (1.15) apparently has no isolating time-independent

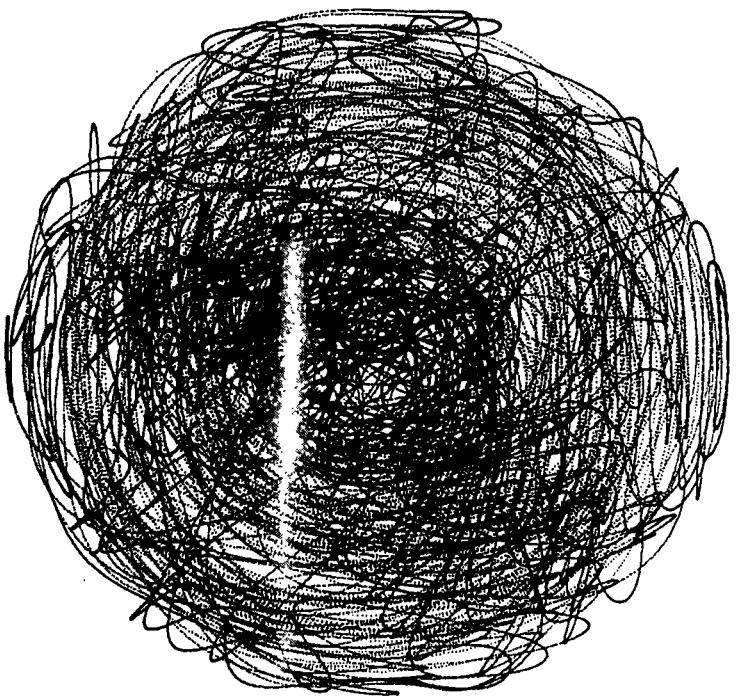

Figure 1.1 Plot of $x_1(t)$ vs $x_2(t)$ for $0 \leqslant t \leqslant 3000$ for that solution of (1.15) with $x_1(0) = \cdot 540323$, $x_2(0) = -1 \cdot 543569$, $x_3(0) = -\cdot 680421$, $x_4(0) = -1 \cdot 185361$, $x_5(0) = -\cdot 676307$. This orbit was determined by numerical solution of (1.15) using a ninth-order predictor-corrector scheme.

integrals of motion in addition to the energy (1.16).† In particular, all equilibrium points and periodic orbits of (1.15) are unstable. In this case, the system is strongly mixing (Halmos, 1956) and *a fortiori* ergodic, with a typical orbit spending equal amounts of time on the average, in all regions of phase space. It also follows that time averages over a single orbit equal space averages over the appropriate energy surface, for almost every orbit. That

†The system (1.15) clearly has a complete set of 5 time-dependent isolating integrals of motion, viz. the initial conditions $x_i(0)$. These integrals isolate orbits since $x_i(t)$ is an analytic function of t for t real. However, they are time dependent since $x_i(0) = C_i(x(t), t)$. When the explicit time dependence is removed to get 4 time-independent integrals $\hat{C}(\mathbf{x}(t))$, the isolating property is generally lost.

TABLE 1.1 Time-average properties of the five-mode system (1.15)†

Property that is averaged	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Average of modes 1–5	Space average
$x_i(t)$	− ·0082	− ·0049	− ·0092	·0012	·0142	− ·0014	0
$[x_i(t)]^2$	·9843	·9766	·9841	1·0218	1·0323	1·0000	1
$[x_i(t)]^3$	− ·0222	− ·0270	·0112	− ·0033	·0546	− ·0005	0
$[x_i(t)]^4$	2·0959	2·0556	2·0595	2·2305	2·2768	2·1437	$\frac{15}{7} \approx 2{\cdot}1429$
$[x_i(t)]^6$	5·8116	5·6041	5·5515	6·2513	6·5272	5·9491	$\frac{125}{21} \approx 5{\cdot}9524$
$[x_i(t)]^8$	18·469	17·500	17·096	19·964	21.484	18.902	$\frac{625}{33} \approx 18{\cdot}939$
$x_1(t)x_i(t)$	·9843	− ·0231	− ·0048	− ·0061	− ·0205	—	0

†Time averages are calculated over the interval $t = 0$ to $t = 3000$ for the orbit with initial conditions $x_1(0) = {\cdot}540323$, $x_2(0) = -1{\cdot}543569$, $x_3(0) = -{\cdot}680421$, $x_4(0) = -1{\cdot}185361$, $x_5(0) = -{\cdot}676307$, with $\Sigma x_i^2 = 5$.

is,

$$\lim_{T\to\infty} \overline{f(\mathbf{x})}^T = \frac{1}{\sigma(S)} \int_S f(\mathbf{x}) \mathrm{d}\sigma. \tag{1.19}$$

for almost every orbit $\mathbf{x}(t)$, where $f(\cdot)$ is an arbitrary integrable function, S is the energy surface $\Sigma x_i^2 = \Sigma x_i^2(0)$ and σ is the area of S.

The properties of (1.15) discussed above have been tested by numerical calculation. In Fig. 1.1, we show an orbit $\mathbf{x}(t)$ projected onto the $x_1 - x_2$ plane. The initial conditions given in the caption to the figure are chosen so that the energy (1.16) is $2 \cdot 5$. If the orbit filled the energy surface $\Sigma x_i^2 = 2E(= 5)$ uniformly, as required by mixing, the density of points in Fig. 1.1 would be proportional to $(2E - x_1^2 - x_2^2)^{1/2}$. Comparison of time averages of various low order moments of $x(t)$ with the corresponding space averages is given in Table 1.1 for the same orbit depicted in Fig. 1.1. These results are consistent with (1.19). The evolution of the time-average energy in modes 2 and 3, $\overline{x_2^2}^T$ and $\overline{x_3^2}^T$, is plotted in Fig. 1.2. Also, the expected number of zeroes of x_i per unit time may be shown on the basis of ergodicity

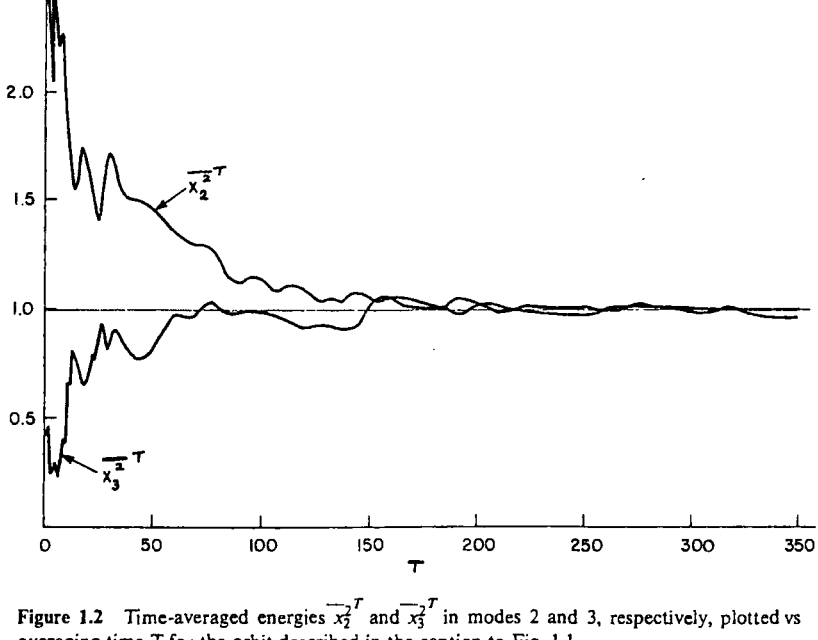

Figure 1.2 Time-averaged energies $\overline{x_2^2}^T$ and $\overline{x_3^2}^T$ in modes 2 and 3, respectively, plotted vs averaging time T for the orbit described in the caption to Fig. 1.1.

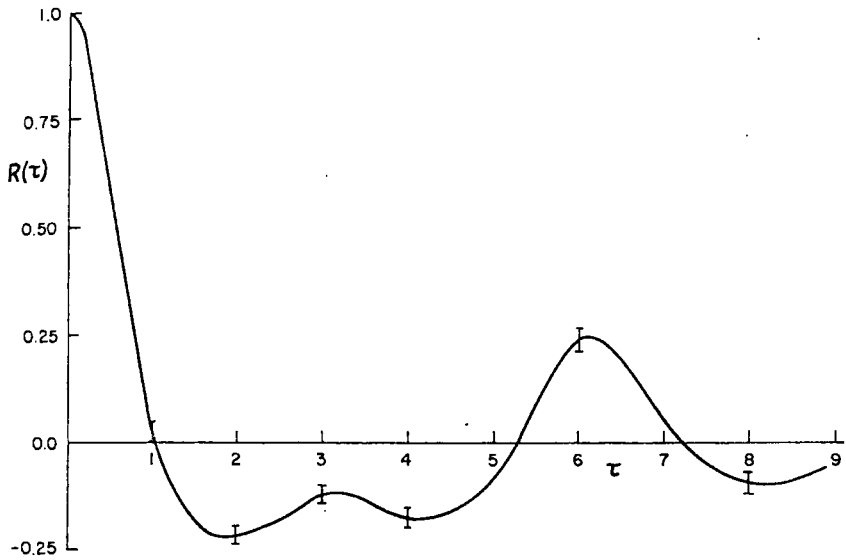

Figure 1.3 Time-correlation function (1.20) for mode 1 using the orbit described in Fig. 1.1. The error bars indicate fluctuations in R obtained as a function of choice of mode, starting time, and random initial conditions with $\sum x_i^2(0) = 1$.

to be roughly $0.64\ (2E/5)^{1/2}$, so that the number of zeroes for $0 \leq t \leq 3000$ for the orbit with $E = 2{\cdot}5$ should be roughly 1920. The calculated number of zeroes of x_3 for the orbit shown in Fig. 1.1 is 2044, while the average number of zeroes of modes 1–5 is 1960. Finally, in Fig. 1.3 we plot the time correlation function

$$R(t - t') = \lim_{T\to\infty} \int_0^T x_i(t + s)x_i(t' + s)\mathrm{d}s/\langle x_i^2\rangle \tag{1.20}$$

determined for $i = 1$ and the same orbit as used in the other figures and Table 1.1. The error bars indicate the fluctuations in R as a function of i, choice of starting time t', *and* random choice of initial conditions $\mathbf{x}(0)$ with $\Sigma x_i^2(0) = 1$. According to the mixing property of (1.15), $R(\tau) \to 0$ as $|\tau| \to \infty$. The slow and tortuous decrease of R to zero is presumably related to the very few (5) degrees of freedom of (1.15); it is expected that systems with large numbers of degrees of freedom and lacking isolating integrals "forget" initial conditions rapidly as initial excitations are mixed into a "sea" of excitable modes.

In summary, the model example (1.15) demonstrates the nature of "ran-

dom" solutions to (deceivingly simple) differential equations. A conservative dynamical system which is mixing (in the dynamical sense) tends to distribute modal excitations as equally as possible consistent with its isolating integrals. In the case of (1.15), the result is a tendency towards equipartition of energy amongst all modes as $t \to \infty$ for almost every set of initial conditions. Notice that (1.15) is time reversible under the transformation $\mathbf{x}(t) \leftarrow -\mathbf{x}(-t)$, so that nonequilibrium statistics at $t = 0$ should be interpreted in the usual statistical mechanical way as a fluctuation that is damped as $|t| \to \infty$ (cf. § 5.5).

1.5 Mathematical Theory of the Navier–Stokes Equations

The Navier–Stokes equations for incompressible flow are

$$\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + \mathbf{v}(\mathbf{x}, t) \cdot \nabla \mathbf{v}(\mathbf{x}, t) = -\nabla p(\mathbf{x}, t) + \nu \nabla^2 \mathbf{v}(\mathbf{x}, t) \tag{1.21}$$

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0, \tag{1.22}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the three-dimensional velocity field, $p(\mathbf{x}, t)$ is the pressure, the (constant) density is assumed unity, and ν is the kinematic viscosity. Application of the incompressible Navier–Stokes equation to turbulence, is justified in nearly all circumstances by the small Mach numbers of most turbulent flows. For example, typical fluctuation velocities in the atmosphere are of order several meters per second while the sound velocity is of order several hundred meters per second. Incompressibility is *not* an appropriate approximation in the turbulent motion of interstellar gas clouds where the Mach number is of order 10 and random strong shocks play an important role. The latter situation is excluded from discussion here.

Eqs. (1.21), (1.22) must be supplemented by boundary conditions on material interfaces. On rigid stationary walls, the appropriate conditions are $\mathbf{v} = 0$.

The pressure field does not satisfy a prognostic equation, but rather only serves the dynamical function of maintaining the incompressibility condition (1.22) for all t. By taking the divergence of (1.21) and applying (1.22), it follows that

$$\nabla^2 p = -\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] \tag{1.23}$$

Thus, the pressure is determined at each instant by the global velocity field as the solution of the Poisson equation (1.23); the instantaneous effect of distant regions on the local flow through the pressure is best understood by recalling that incompressibility corresponds to the limit of zero Mach number or infinite sound propagation speed.

It follows by (scalar) multiplying (1.21) by $\mathbf{v}$ and using (1.22) and the divergence theorem that with stationary rigid boundaries and in the absence of

external forces

$$\frac{d}{dt}\frac{1}{2}\int_V |\mathbf{v}|^2 dx = -\nu \int_V \frac{\partial v_\alpha}{\partial x_\beta}\frac{\partial v_\alpha}{\partial x_\beta} dx, \tag{1.24}$$

where V is the volume occupied by the fluid. This relation is a statement of the overall energy balance in the fluid. In particular, the convective term $(\mathbf{v}\cdot\nabla\mathbf{v})$ and the pressure term $(-\nabla p)$ separately conserve total energy according to (1.21), while the viscous term $(\nu\nabla^2\mathbf{v})$ dissipates it when $\nu > 0$. Equation (1.24) shows that *all motions in a finite domain must cease as* $t \to \infty$ if there are no external forces, the boundaries are rigid and stationary, and $\nu > 0$. In a viscous fluid, steady motion or turbulent motion which does not decay with time requires that the fluid be subject to external forces of some kind (including those provided by moving boundaries).

The mathematical theory of the Navier–Stokes equations is incomplete. There is as yet no general existence and uniqueness theorem that shows that (1.21) with (1.22) is well posed. However, a number of partial results due to Leray, Hopf, and others are available (Ladyzhenskaya, 1969; Ebin and Marsden, 1970). Briefly stated the results established to date include the existence and uniqueness of solutions in three dimensions for short time with arbitrary ν and for all time if ν is large enough. The mathematical difficulties experienced in these attempts to establish general existence and uniqueness theorems for three-dimensional flows are avoidable for plane-parallel two-dimensional or annular axisymmetric flows (Ladyzhenskaya, 1969), in accord with the well-known property that the mathematical regularity of solutions to a partial differential equation goes up with decreasing dimension.

The difficulty with existence and uniqueness theorems for the three-dimensional case has led Ladyzhenskaya (1969) and others to suggest the abandonment of the Navier–Stokes equations, especially for the study of turbulence. Ladyzhenskaya (1963, p. 159) points out that if a biharmonic damping $-\lambda\nabla^4\mathbf{v}$ is added to the right-hand side of the Navier–Stokes equations (1.21), then existence and uniqueness of solutions is ensured for all $\lambda > 0$. In the second edition of Ladyzhenskaya's book (1969, p. 193), it is suggested that the constant viscosity appearing in (1.21) be replaced by a viscosity coefficient that depends on the local deformation rate of the flow. Under suitable hypotheses on this deformation-rate dependence, satisfactory existence and uniqueness theorems can be proven. It is curious that the deformation-rate dependence of the viscosity invoked by Ladyzhenskaya for mathematical reasons is quite similar to the deformation-rate dependence of the viscosity invoked by Smagorinsky (1963; also Deardorff, 1970) to describe the sub-grid scale turbulent motions in numerical flow simulations. The theory of turbulence developed later (and based implicitly on the existence of smooth solutions to the equations) suggests

that the phenomenon of turbulence and many of its properties are independent of the precise form of the viscous damping term. Nevertheless, we must conservatively conclude that the mathematical (and hence physical) basis of the Navier–Stokes equations requires much further elucidation, and that the consequences of future mathematical work may undermine contemporary turbulence theory.

II The General Theory of Homogenous Turbulence

2.1 The Description of Homogeneous Turbulence

Turbulence is homogeneous if the flow has identical statistical properties at all points in space; it is isotropic if, in addition, the statistical properties of the flow are the same in all directions. Isotropic turbulence is necessarily homogeneous, but not vice versa. Homogeneous turbulence is an idealization as there is no known phenomenon in which exact homogeneity is obtained. However, there are circumstances where a very close approximation to homogeneous turbulence exists. Simmons and Salter (1934) were apparently the first to observe that if a regular grid of bars is placed at right angles to the uniform stream of a wind tunnel, the motion far downstream consists of a closely isotropic homogeneous random velocity field superposed on the uniform stream velocity (see Fig. 2.1). The turbulence dies away downstream so that the homogeneity cannot be exact, but the rate of decay is so small that the turbulence may be considered approximately homogeneous in the streamwise direction. In addition, beyond 40 mesh lengths or so downstream from the grid and away from the walls of the wind tunnel, the turbulence is observed to be homogeneous in planes normal to the stream direction almost to within the accuracy of the measurements. At any given instant, the spatial dependence of the turbulence downstream from the grid provides a panoramic view of the decay of homogeneous turbulence with decay time identified with downstream distance divided by the uniform stream velocity. For further details on the nature of experimental techniques involved in the study of grid turbulence, the interested reader should consult the review article by Corrsin (1963).

Turbulence that occurs in nature is usually not even approximately homogeneous. There is frequently important variation of mean velocity with position. In this case, gradients of the average properties of the fluctuating velocity field,

$$\mathbf{v}'(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) - \langle \mathbf{v}(\mathbf{x}, t) \rangle,$$

have a dominant effect on the evolution of the mean field through the action of Reynolds stresses. Despite the importance of these effects of in-

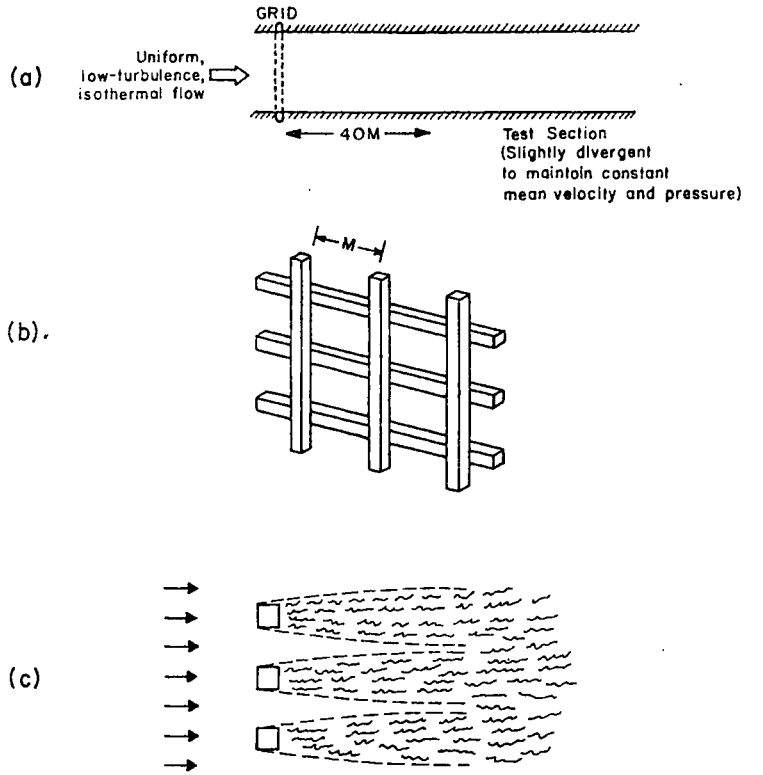

Figure 2.1 (a) Schematic diagram of the arrangement for generating approximately isotropic turbulence in a wind tunnel. (b) Small section of a typical grid. (c) Qualitative view of the turbulence generated in the wake of the grid of bars shown in (b). (After Corrsin, 1963).

homogeneity, it is thought that the small-scale properties of general turbulent flows should be locally homogeneous and, therefore, that the study of homogeneous turbulence should not be entirely without practical consequence. However, the real reason for limiting attention in these lectures to homogeneous turbulence is analytical simplicity (or, rather, minimum analytical complexity). This limitation may be unfortunate as shear turbulence may be basically very much simpler than homogeneous turbulence. Homogeneous turbulence isolates the problem of the self-interaction of of fluctuating components from the problem of the interaction of fluctuating components with the mean field, which is readily treated (Herring, 1963). If the latter effect is dominant, as it is for some kinds of turbulence, we have

unnecessarily complicated the physics by neglecting the mean flow interaction. Furthermore, inhomogeneous turbulence may be studied by the use of rigorous upper bounds on flow quantities, like Reynolds stresses (Busse, 1970; Howard, 1972). The latter methods have not yet been applied to homogeneous turbulence, and are not discussed in the lectures.

If the turbulence is homogeneous, the mean velocity $\langle \mathbf{v}(\mathbf{x}, t) \rangle$ may be assumed zero. For it follows from homogeneity and (1.21) that

$$\frac{\partial}{\partial t} \langle v_\alpha(\mathbf{x}, t) \rangle = 0,$$

since ensemble averaging and space-time differentiation commute. Hence, $\langle \mathbf{v}(\mathbf{x}, t) \rangle$ is space and time independent and may, by suitable choice of reference frame, be chosen zero.

The quantities of most theoretical and experimental interest are averages of the form

$$\langle v_\alpha(\mathbf{x}_1, t) v_\beta(\mathbf{x}_2, t_2) \cdots v_\gamma(\mathbf{x}_n, t_n) \rangle,$$

called a *moment* of order n. If $t_1 = t_2 = \cdots = t_n$, the moment is termed single-time; otherwise, it is called a many-time moment.

Complete specification of the turbulence implies, among other things, knowledge of all moments. As a practical matter, only a small finite number of simple moments may be determined directly either experimentally or theoretically. It is a fundamental, but to date not fully justified, assumption that the asymptotic statistical state of turbulence is described in good approximation by means of a few carefully chosen low-order moments.† Several mathematical subleties are involved, including the possibility that even if the initial values of single-time moments of *all* orders are known, moments may not be determined at later times (Orszag, 1970b).

2.2 Velocity-Correlation Tensor

The simplest (nonzero), and probably most important, moment of homogeneous turbulence is the velocity correlation tensor

$$R_{\alpha\beta}(\mathbf{r}) = \langle v_\alpha(\mathbf{x} + \mathbf{r}) v_\beta(\mathbf{x}) \rangle, \tag{2.1}$$

where the time arguments are omitted for simplicity, it is assumed that $\langle \mathbf{v} \rangle = 0$, and independence of $\mathbf{x}$ is assured by homogeneity. It may easily be shown that

$$R_{\alpha\beta}(\mathbf{r}) = R_{\beta\alpha}(-\mathbf{r}) \tag{2.2}$$

†In Section VI, it will be argued that other kinds of statistical properties (Green's functions) should be introduced in addition to moments.

$$\frac{\partial R_{\alpha\beta}}{\partial r_\alpha} = \frac{\partial R_{\alpha\beta}}{\partial r_\beta} = 0 \tag{2.3}$$

$$|R_{ij}(\mathbf{r})|^2 \leqslant R_{ii}(0)R_{jj}(0), \tag{2.4}$$

where (2.2) is a consequence of homogeneity, (2.3) follows from incompressibility, and (2.4), called a realizability inequality, follows from the Schwarz' inequality; we use the convention of summation on repeated Greek but *not* latin indices.

The vorticity correlation tensor $\langle \omega_\alpha(\mathbf{x} + \mathbf{r})\omega_\beta(\mathbf{x})\rangle$, where the vorticity is defined by $\omega_\alpha = (\nabla \times \mathbf{v})_\alpha = \epsilon_{\alpha\gamma\delta}\partial v_\delta/\partial x_\gamma$, can be easily related to $R_{\alpha\beta}(\mathbf{r})$. Here $\epsilon_{\alpha\gamma\delta} = +1(-1)$ if α, γ, δ is an even (odd) permutation of 1, 2, 3 and is zero otherwise. It may be shown that

$$\langle \omega_\alpha(\mathbf{x} + \mathbf{r})\omega_\beta(\mathbf{x})\rangle = \nabla^2 R_{\alpha\beta}(\mathbf{r}) - \delta_{\alpha\beta}\nabla^2 R_{\gamma\gamma}(\mathbf{r}) + \frac{\partial^2}{\partial r_\alpha \partial r_\beta} R_{\gamma\gamma}(\mathbf{r}) \tag{2.5}$$

upon use of the incompressibility condition (2.3) and some algebra.

If the turbulence is isotropic, further simplification of $R_{\alpha\beta}(\mathbf{r})$ can be made. The standard technique to reduce tensors to their isotropic form is the method of invariants (Robertson, 1940; Batchelor, 1953). If **a** and **b** are arbitrary vectors then

$$F(\mathbf{a}, \mathbf{b}, \mathbf{r}) = a_\alpha b_\beta R_{\alpha\beta}(\mathbf{r}) = \langle \mathbf{a} \cdot \mathbf{v}(\mathbf{x} + \mathbf{r})\mathbf{b} \cdot \mathbf{v}(\mathbf{x})\rangle$$

is a scalar under rotations if the turbulence is isotropic. Consequently, F can depend only on the scalars (invariants) that can be formed out of its arguments; these invariants are all combinations of a^2, b^2, r^2, $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\cdot\mathbf{r}$, $\mathbf{b}\cdot\mathbf{r}$, and $\mathbf{a}\cdot\mathbf{b} \times \mathbf{r}$. Since F is linear in **a** and in **b**, it follows that

$$F = \mathbf{a}\cdot\mathbf{r}\,\mathbf{b}\cdot\mathbf{r}\,A(r) + \mathbf{a}\cdot\mathbf{b}\,B(r) + \mathbf{a}\cdot\mathbf{b} \times \mathbf{r}\,C(r)$$

so that noting the arbitrariness of **a** and **b**,

$$R_{\alpha\beta}(\mathbf{r}) = r_\alpha r_\beta A(r) + \delta_{\alpha\beta}B(r) + \epsilon_{\alpha\beta\gamma}r_\gamma C(r). \tag{2.6}$$

If the turbulence is invariant to reflections in space in addition to being invariant to rotations then the pseudoscalar $a \cdot b \times r$ is not invariant and the term involving $C(r)$ cannot appear. In these lectures, we assume reflection-invariant isotropic turbulence, but the recent theory of turbulent hydromagnetic dynamos discussed elsewhere in this volume (Moffatt, 1976) requires reflection noninvariance.

The incompressibility constraint (2.3) implies a relation between the surviving scalar functions $A(r)$ and $B(r)$ in (2.6). It is conventional to rewrite (2.6) as

$$R_{\alpha\beta}(\mathbf{r}) = v_{\text{rms}}^2\left[\frac{f(r) - g(r)}{r^2} r_\alpha r_\beta + g(r)\delta_{\alpha\beta}\right], \tag{2.7}$$

where v_{rms} is the rms velocity of each velocity component

$$R_{ii}(0) = v_{\text{rms}}^2 \tag{2.8}$$

and the incompressibility constraint becomes

$$g(r) = f(r) + \frac{1}{2} r \frac{df}{dr}. \tag{2.9}$$

It follows from (2.8) that $f(0) = g(0) = 1$. The functions $f(r)$ $[g(r)]$ are called the longitudinal [latitudinal] velocity correlation functions because

$$v_{\text{rms}}^2 f(r) = \langle v_p(\mathbf{x} + \mathbf{r}) v_p(\mathbf{x}) \rangle \tag{2.10a}$$

$$v_{\text{rms}}^2 g(r) = \langle v_n(\mathbf{x} + \mathbf{r}) v_n(\mathbf{x}) \rangle, \tag{2.10b}$$

where v_p is the velocity component parallel to $\mathbf{r}$ and v_n is one of the components perpendicular to $\mathbf{r}$. Homogeneity implies $f'(0) = g'(0) = 0$ so that expansion about $r = 0$ gives

$$f(r) = 1 - \frac{r^2}{2\lambda^2} + 0(r^4) \tag{2.11a}$$

$$g(r) = 1 - \frac{r^2}{\lambda^2} + 0(r^4), \tag{2.11b}$$

where

$$\frac{1}{\lambda^2} = -f''(0) = \langle (\partial v_1 / \partial x_1)^2 \rangle / v_{\text{rms}}^2; \tag{2.12}$$

λ is called the Taylor microscale. An interpretation of the dynamical significance of λ is given in Section III. For now, we note that λ is related to $\langle \omega^2 \rangle$ by (2.5), (2.7) and (2.11) as

$$\langle \omega^2 \rangle = -\nabla^2 R_{\alpha\alpha}(\mathbf{r})\Big|_{\mathbf{r}=0} = -v_{\text{rms}}^2 \frac{1}{r} \frac{d^2}{dr^2} r(f + 2g)\Big|_{r=0} = 15 v_{\text{rms}}^2 / \lambda^2.$$

Consequently, the rate of energy dissipation in isotropic turbulence is given by

$$\epsilon = \nu \langle \omega^2 \rangle = 15 \nu v_{\text{rms}}^2 / \lambda^2, \tag{2.13}$$

noting that $\langle \mathbf{v} \cdot \nabla^2 \mathbf{v} \rangle = -\langle \omega^2 \rangle$ in homogeneous, incompressible turbulence.

2.3 Fourier Analysis of the Velocity Field

It is convenient to Fourier analyze the velocity field of homogeneous turbulence. This representation permits resolution of the velocity field into components of various sizes. A Fourier component is not a local coordinate but is a collective coordinate which specifies the total excitation in some

scale over the whole flow. In this way, it is possible to give precise meaning to concepts such as the energy of small-scale motions. The disadvantage of Fourier analysis also stems from its collective nature. Local coherent flow structures, such as shear layers, are not easily recognizable except by detailed examination of phase relations among Fourier components.

A typical realization of homogeneous turbulence cannot be represented as either a Fourier series or a Fourier integral. At any particular instant of time, there is no reason why a typical realization $\mathbf{v}(\mathbf{x}, t)$ should be periodic in $\mathbf{x}$ and, hence, expansible in a Fourier series. Neither should $\mathbf{v}(\mathbf{x}, t)$ be expected to be absolutely integrable or square integrable so that a Fourier integral representation would apply. A technique for rendering a stationary random function (of position), such as $\mathbf{v}(\mathbf{x}, t)$, Fourier transformable was first given by Wiener (1930). The Fourier transforms so obtained are distributions (Lighthill, 1958). An equivalent and conceptually simpler technique is to consider only flows which are periodic with the basic period a large cube in which the velocity field is expanded in a Fourier series. In this latter method, the box size is allowed to approach infinity and the asymptotic ordering in terms of box volume of the various statistical averages of interest is used to establish finite dynamical equations (Orszag and Kruskal, 1968). However, in these lectures we avoid all technical niceties by assuming a Fourier transform representation of $\mathbf{v}(\mathbf{x}, t)$ in terms of distributions.

A statistically homogeneous velocity field can be represented as

$$\mathbf{v}(\mathbf{x}, t) = \int \mathbf{u}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{k} , \tag{2.14}$$

where $\mathbf{u}(\mathbf{k}, t)$ is a random generalized function (distribution). It follows that

$$\mathbf{u}(\mathbf{k}, t) = \left(\frac{1}{2\pi}\right)^3 \int \mathbf{v}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{x}.$$

This representation has immediate consequences for moments. For example,

$$\langle u_\alpha(\mathbf{k}) u_\beta(\mathbf{p}) \rangle = \frac{1}{(2\pi)^6} \int d\mathbf{x} \int d\mathbf{y} \, \langle v_\alpha(\mathbf{x}) v_\beta(\mathbf{y}) \rangle e^{-i\mathbf{k}\cdot\mathbf{x} - i\mathbf{p}\cdot\mathbf{y}}$$

which upon the change of variables $\mathbf{z} = \mathbf{y}$, $\mathbf{r} = \mathbf{x} - \mathbf{y}$ gives

$$\langle u_\alpha(\mathbf{k}) u_\beta(\mathbf{p}) \rangle = S_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{p}) \tag{2.15}$$

$$S_{\alpha\beta}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int R_{\alpha\beta}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{r} \tag{2.16}$$

using (2.1) and the integral representation

$$\delta(\mathbf{p}) = \frac{1}{(2\pi)^3} \int e^{-i\mathbf{p}\cdot\mathbf{z}} \, d\mathbf{z}.$$

Similarly, it follows that

$$\langle u_\alpha(\mathbf{k})u_\beta(\mathbf{p})u_\gamma(\mathbf{q})\rangle = T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p})\delta(\mathbf{k} + \mathbf{p} + \mathbf{q}), \tag{2.17}$$

where

$$T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}) = \frac{1}{(2\pi)^6}\int d\mathbf{y}\int d\mathbf{z}\,\langle v_\alpha(\mathbf{x} + \mathbf{y})v_\beta(\mathbf{x} + \mathbf{z})v_\gamma(\mathbf{x})\rangle e^{-i\mathbf{k}\cdot\mathbf{y} - i\mathbf{p}\cdot\mathbf{z}}. \tag{2.18}$$

Calculation of higher-order Fourier space moments is facilitated by the following *factorization* (cluster) property of physical space moments:

$$\langle v_\alpha(\mathbf{x}_1, s_1)\ldots v_\beta(\mathbf{x}_m, s_m)v_\gamma(\mathbf{y}_1 + \mathbf{x}, t_1)\ldots v_\delta(\mathbf{y}_n + \mathbf{x}, t_n)\rangle$$
$$\to \langle v_\alpha(\mathbf{x}_1, s_1)\ldots v_\beta(\mathbf{x}_m, s_m)\rangle\langle v_\gamma(\mathbf{y}_1 + \mathbf{x}, t_1)\ldots v_\delta(\mathbf{y}_n + \mathbf{x}, t_n)\rangle \tag{2.19}$$

as $X \to \infty$ for any set of fixed $\mathbf{x}_i, s_i, \mathbf{y}_j, t_j (i = 1, \ldots, m, j = 1, \ldots, n)$. This factorization is plausible since the statistical properties of widely separated sets of points should be independent. One consequence of (2.19) is the approach of $R_{\alpha\beta}(\mathbf{r})$ to zero as $r \to \infty$ (since $\langle \mathbf{v}\rangle = 0$). A measure of the length scale over which $R_{\alpha\beta}(\mathbf{r})$ goes to 0 is given, for isotropic turbulence by the *longitudinal integral scale* L_p:

$$L_p = \int_0^\infty f(r)\,dr, = 2\int_0^\infty g(r)\,dr. \tag{2.20}$$

It follows from (2.19) that

$$\langle u_\alpha(\mathbf{k})u_\beta(\mathbf{p})u_\gamma(\mathbf{q})u_\delta(\mathbf{m})\rangle = S_{\alpha\beta}(\mathbf{k})S_{\gamma\delta}(\mathbf{q})\delta(\mathbf{k} + \mathbf{p})\delta(\mathbf{q} + \mathbf{m})$$
$$+ S_{\alpha\gamma}(\mathbf{k})S_{\beta\delta}(\mathbf{p})\delta(\mathbf{k} + \mathbf{q})\delta(\mathbf{p} + \mathbf{m}) + S_{\alpha\delta}(\mathbf{k})S_{\beta\gamma}(\mathbf{p})\delta(\mathbf{k} + \mathbf{m})\delta(\mathbf{p} + \mathbf{q})$$
$$+ U_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{p}, \mathbf{q})\delta(\mathbf{k} + \mathbf{p} + \mathbf{q} + \mathbf{m}), \tag{2.21}$$

where

$$U_{\alpha\beta\gamma\delta}(\mathbf{k}, \mathbf{p}, \mathbf{q}) = \frac{1}{(2\pi)^9}\int d\mathbf{y}\int d\mathbf{z}\int d\mathbf{w}$$
$$[\langle v_\alpha(\mathbf{x} + \mathbf{y})v_\beta(\mathbf{x} + \mathbf{z})v_\gamma(\mathbf{x} + \mathbf{w})v_\delta(\mathbf{x})\rangle - R_{\alpha\beta}(\mathbf{y} - \mathbf{z})R_{\gamma\delta}(\mathbf{w}) -$$
$$R_{\alpha\gamma}(\mathbf{y} - \mathbf{w})R_{\beta\delta}(\mathbf{z}) - R_{\alpha\delta}(\mathbf{y})R_{\beta\gamma}(\mathbf{z} - \mathbf{w})]\cdot e^{-i\mathbf{k}\cdot\mathbf{y} - i\mathbf{p}\cdot\mathbf{z} - i\mathbf{q}\cdot\mathbf{w}}. \tag{2.22}$$

U and its Fourier transform [in square brackets in (2.22)] are called fourth-order *cumulants*, as opposed to the fourth-order moment (2.21). Similarly, it is possible to use the factorization property to obtain expressions for the general moment of order n in terms of cumulants of order n or less (Orszag and Kruskal, 1968). Second and third-order moments are also cumulants (since $\langle \mathbf{v}\rangle = 0$).

Note that the existence of cumulants requires a somewhat stronger factorization property than is provided by (2.19). In order for the Fourier trans-

forms (2.16), (2.18), (2.22) to exist, the approach of moments to their factored forms must be sufficiently fast as $X \to \infty$. Thus, the existence of cumulants is an additional assumption, but it is one that is consistent with the dynamical equations, in the sense that if cumulants exist at some time they continue to exist at later times. On the other hand, it is known that the approach of moments to their factored forms is generally only algebraic in the separation distance X (Batchelor and Proudman, 1956; Saffman, 1967a), so that the question of the existence of cumulants should not be taken too lightly (cf. Sec. 2.7).

The cumulant tensors S, T, U, ... satisfy a number of kinematical constraints that follow from their definitions. First, there are realizability inequalities that must be satisfied because the cumulants are obtained by averaging over a probability (non-negative) distribution of realizations. It follows from (2.15) that

$$a_\alpha a_\beta^* S_{\alpha\beta}(\mathbf{k}) \geqslant 0 \tag{2.23}$$

for any complex vector $\mathbf{a}$. Similarly, it may be shown (Orszag and Kruskal, 1968) that

$$\left| \int T_{ijk}(\mathbf{k}, \mathbf{p})\, d\mathbf{p} \right|^2 \leqslant S_{ii}(\mathbf{k}) \int d\mathbf{p} \left[\int U_{jkjk}(\mathbf{p}, \mathbf{k} - \mathbf{p}, \mathbf{q})\, d\mathbf{q} + S_{jj}(\mathbf{p}) S_{kk}(\mathbf{k} - \mathbf{p}) + S_{jk}(\mathbf{p}) S_{kj}(\mathbf{k} - \mathbf{p}) \right], \tag{(2.24)}$$

where i, j, k are not summed. Second, reality of the velocity field $\mathbf{v}(\mathbf{x})$ implies

$$S_{\alpha\beta}(\mathbf{k}) = [S_{\alpha\beta}(-\mathbf{k})]^*, \quad T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}) = [T_{\alpha\beta\gamma}(-\mathbf{k}, -\mathbf{p})]^* \tag{2.25}$$

etc., while the incompressibility constraint (1.22) becomes

$$k_\alpha S_{\alpha\beta}(\mathbf{k}) = k_\beta S_{\alpha\beta}(\mathbf{k}) = 0, \tag{2.26}$$

etc.

Isotropy leads to much simplification in the structure of the cumulants. The theory of isotropic invariants implies $a_\alpha b_\beta S_{\alpha\beta}(\mathbf{k})$ can be a function of $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{k}$, $\mathbf{b} \cdot \mathbf{k}$, and $\mathbf{k} \cdot \mathbf{k}$ alone, so that linearity in $\mathbf{a}$ and $\mathbf{b}$ implies

$$S_{\alpha\beta}(\mathbf{k}) = A(k) k_\alpha k_\beta + B(k) \delta_{\alpha\beta}.$$

The incompressibility constraint (2.26) then implies that $S_{\alpha\beta}(\mathbf{k})$ can be expressed as

$$S_{\alpha\beta}(\mathbf{k}) = \frac{E(k)}{4\pi k^2} P_{\alpha\beta}(\mathbf{k}), \tag{2.27}$$

where

$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2 \tag{2.28}$$

Here the scalar function $E(k)$ is interpreted as the (kinetic) energy density

in modes of wavenumber k so that

$$\frac{3}{2}v_{\text{rms}}^2 = \int_0^\infty E(k)\,dk,$$

which is a special case of

$$\frac{1}{2}v_{\text{rms}}^2[f(r) + 2g(r)] = \int_0^\infty E(k)\frac{\sin kr}{kr}\,dk. \tag{2.29}$$

The realizability inequality (2.23) becomes simply

$$E(k) \geq 0.$$

Similar applications of the isotropy requirement show that the number of scalar functions of wavenumber required to represent the general cumulant of order n is $1(n = 2)$, $2(n = 3)$, $4(n = 4)$, $4(n = 5)$, $9(n = 6)$, and so on (Orszag, 1969). The explicit results for $T(\mathbf{k}, \mathbf{p})$ is (Proudman and Reid, 1954)

$$T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}) = P_{\alpha\mu}(\mathbf{k})P_{\beta\rho}(\mathbf{p})P_{\gamma\sigma}(\mathbf{q})[k_\alpha\delta_{\mu\rho}\Phi(k, p, q) + p_\mu\delta_{\rho\sigma}\Phi(p, q, k) -$$
$$- k_\rho\delta_{\mu\sigma}\Phi(q, k, p) + p_\mu k_\alpha k_\rho\Psi(k, p, q)], \tag{2.30}$$

where $\mathbf{q} = -\mathbf{k} - \mathbf{p}$ and the scalar functions Φ, Ψ satisfy

$$\Phi(k, p, q) = -\Phi(p, k, q)$$

$$\Psi(k, p, q) = \Psi(p, q, k) = -\Psi(p, k, q).$$

It is of some interest to construct examples of statistically homogeneous, isotropic, incompressible velocity fields, e.g. for use as initial conditions in numerical simulations of homogeneous isotropic turbulence. This is most easily done using a Gaussian white noise process $\mathbf{a}(\mathbf{k})$ such that

$$\langle a_\alpha(\mathbf{k})a_\beta(p)\rangle = \delta_{\alpha\beta}\delta(\mathbf{k} + \mathbf{p})$$

as constructed by Wiener. The velocity field whose Fourier transform is

$$u_\alpha(\mathbf{k}) = \frac{E(k)}{4\pi k^2}P_{\alpha\beta}(\mathbf{k})a_\beta(\mathbf{k})$$

satisfies the required kinematical conditions and has the (isotropic) energy spectrum $E(k)$. The resulting multivariate Gaussian velocity field has nth-order cumulants which are identically zero for $n \geq 3$.

2.4 Dynamical Equations for Cumulants

It follows from (2.14) and a similar Fourier representation of the pressure that the Navier–Stokes equations for $\mathbf{u}(\mathbf{k}, t)$ are

$$\left[\frac{\partial}{\partial t} + \nu k^2\right]u_\alpha(\mathbf{k}, t) = -i\int p_\beta u_\beta(\mathbf{k} - \mathbf{p}, t)u_\alpha(\mathbf{p}, t)\,d\mathbf{p} - ik_\alpha p(\mathbf{k}, t).$$

Application of the incompressibility constraint $k_\alpha u_\alpha(\mathbf{k}, t) = 0$ allows elimination of the pressure [essentially by the Fourier transform of (1.23)] giving

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k}, t) = -\frac{i}{2} P_{\alpha\beta\gamma}(\mathbf{k}) \int u_\beta(\mathbf{p}, t) u_\gamma(\mathbf{k} - \mathbf{p}, t)\, d\mathbf{p}. \tag{2.31}$$

The summation convention over repeated Greek indices is implied and

$$P_{\alpha\beta\gamma}(\mathbf{k}) = k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k}),$$

where $P_{\alpha\beta}(\mathbf{k})$ is given by (2.28). Notice that (2.31) gives $[\partial/\partial t + \nu k^2] k_\alpha u_\alpha(\mathbf{k}, t) = 0$, so the flow remains incompressible if it is so initially. Notice also that the terms in (2.31) proportional to the part $\delta_{\alpha\beta}$ of $P_{\alpha\beta}(\mathbf{k})$ originates from the convective term $(\mathbf{v} \cdot \nabla \mathbf{v})$ in (1.21), while the terms proportional to the part $-k_\alpha k_\beta / k^2$ of $P_{\alpha\beta}(\mathbf{k})$ are due to the pressure gradient.

Dynamical equations for the cumulants follow directly from their definitions (2.15), etc. by application of (2.31). It follows from (2.15) and (2.17) that

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] S_{\alpha\beta}(\mathbf{k}) = -\frac{i}{2} P_{\alpha\rho\sigma}(\mathbf{k}) \int T_{\beta\rho\sigma}(-\mathbf{k}, \mathbf{p})\, d\mathbf{p}$$

$$-\frac{i}{2} P_{\beta\rho\sigma}(-\mathbf{k}) \int T_{\alpha\rho\sigma}(\mathbf{k}, \mathbf{p})\, d\mathbf{p} \tag{2.32}$$

where it is assumed that the cumulants are all single-time. Similarly, the equation for T is of the form

$$\left[\frac{\partial}{\partial t} + \nu(k^2 + p^2 + |\mathbf{k} + \mathbf{p}|^2)\right] T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}) =$$

$$-\frac{i}{2} P_{\alpha\rho\sigma}(\mathbf{k}) \int U_{\beta\gamma\rho\sigma}(\mathbf{p}, -\mathbf{k} - \mathbf{p}, \mathbf{q})\, d\mathbf{q}$$

$$+ \cdots - iP_{\alpha\rho\sigma}(\mathbf{k}) S_{\beta\rho}(\mathbf{p}) S_{\gamma\sigma}(-\mathbf{k} - \mathbf{p}) + \ldots, \tag{2.33}$$

where the dots denote similar terms obtained by interchanging $\mathbf{k}$, $\mathbf{p}$, and $-\mathbf{k} - \mathbf{p}$.

It is apparent that the equations obtained in this way do not form a determinate set. The equation for nth-order cumulants involves $(n + 1)$-order cumulants, as well as those of order n and lower, so a closed set of equations for low-order cumulants is not obtained. *Any finite subsystem of this infinite set of equations possesses more unknown functions than are determined by the subsystem.* This "closure problem" is perhaps the most fundamental difficulty of turbulence theory.

2.5 Energy Balance

As discussed in Sec. 1.5, the nonlinear terms of the Navier–Stokes equations conserve kinetic energy while only the viscous term dissipates it. In fact,

the convective term $\mathbf{v}\cdot\nabla\mathbf{v}$ in (1.21) conserves energy componentwise, in the sense that

$$\int_V v_i(\mathbf{v}\cdot\nabla)v_i\,d\mathbf{x}$$

(no sum on i) can be integrated to a surface integral of $(1/2)\mathbf{v}\cdot\mathbf{n}v_i^2$, where $\mathbf{n}$ is an outward normal. On the other hand, the pressure term ∇p in (1.21) conserves kinetic energy in each Fourier mode, since $u_\alpha(-\mathbf{k})[-ik_\alpha p(k)]=0$.

If the turbulence is isotropic, the cumulant equation (2.32) reduces to a single equation for $E(k,t)$:

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]E(k,t)=T(k,t), \tag{2.34}$$

where

$$T(k,t)=-4\pi k^2 k_\rho Im\int T_{\alpha\rho\alpha}(\mathbf{k},\mathbf{p})\,d\mathbf{p}$$

and $\mathbf{k}$ is any vector with $k=|\mathbf{k}|$. Conservation of energy by nonlinear interaction implies

$$\int_0^\infty T(k,t)\,dk=0. \tag{2.35}$$

It follows that

$$\frac{d}{dt}\frac{3}{2}v_{rms}^2=-\epsilon \tag{2.36}$$

where

$$\epsilon(t)=2\nu\int_0^\infty k^2E(k,t)\,dk \tag{2.37}$$

is the rate of kinetic energy dissipation (per unit volume) [cf. (2.13)]. The result (2.37) shows that high wavenumber Fourier components are dissipated more effectively by viscosity than low wavenumber components.

Since inertia forces conserve each directional component of the integrated energy-spectrum tensor, it is plausible to assert that, in the absence of other constraints, the net tendency of inertia is to spread energy over all wavevectors. In particular, the tendency should be to transfer energy to those parts of wavevector space where the energy density is lowest. This can not be more than a tendency for surely initial conditions can be envisaged in which the phases of the various Fourier components are chosen so that there is initially flow of energy into any part of wavevector space. However, since viscosity tends to deplete more rapidly the energy of small eddies (high wavenumbers), the typical effect of inertia should be to transfer energy from large to small eddies. This suggests that the energy spectrum evolves towards

a state where the rate of energy input to small scales due to inertia balances the rate of dissipation due to viscosity. If it were not for viscosity, inertia forces should succeed in spreading energy over all Fourier space, giving a flow of energy to infinite wavenumber. In physical space, the corresponding effect is the formation of infinitely sharp shear layers.

Similarly, since pressure forces conserve the total energy in each Fourier component, they may be expected to tend to equalize the energy in any wavevector $\mathbf{k}$ uniformly over directions normal to $\mathbf{k}$, the latter restriction due to incompressibility. In x-space, pressure should tend to distribute the energy uniformly over all directions, i.e., to "isotropize". However, pressure cannot isotropize completely by itself. A uniform distribution of energy in k-space over directions perpendicular to each wavevector $\mathbf{k}$ requires $S_{\alpha\beta}(\mathbf{k}) = F(\mathbf{k})P_{\alpha\beta}(\mathbf{k})$, while isotropy requires in addition that $F(\mathbf{k})$ be an isotropic function of $\mathbf{k}$.

2.6 The Spectrum of Homogeneons Turbulence

In homogeneous turbulence, the energy-spectrum tensor $S_{\alpha\beta}(\mathbf{k})$ is a quantity of fundamental interest. Studies of this spectrum tensor are classified according to whether the wavenumber range concerned satisfies $kL_p \ll 1$, $kL_p \sim 1$, $kL_p \gg 1$, where L_p is given by (2.20). Identifying the size of an eddy of a turbulent flow with the wavelength $2\pi/k$, these ranges are called the large eddies, the energy-containing eddies, and the small eddies.

The large-eddy region of the energy spectrum was for some time thought to be most accessible to analysis. However, it has since been learned that the original analyses were not correct, and the actual nature of this range of eddies is still unsettled. Some aspects of the theory of the large eddies are reported in Sec. 2.7. In any case, the structure of the large eddies is without much practical significance. The spectral region $kL_p \ll 1$ contains negligible energy and, as seen in the next chapter, interacts only very weakly with the rest of the spectrum. At high Reynolds number, there would appear to be little consequence if the large-eddy region of the spectrum were entirely neglected.

The dynamical importance of the energy-containing range of eddies cannot be denied. The characteristic velocity of these eddies is v_{rms} and their characteristic length is L_p. They make the dominant contribution to $\int_0^\infty E(k)\,dk$. As will be discussed in §2.9, the time scale for decay of the turbulence is L_p/v_{rms} which is also the circulation time of an eddy in the energy-containing range. This means that the turbulence decays on approximately the same time scale as eddies in this range execute their motion. This makes it unlikely that the energy-containing eddies are in any simple statistical state before the turbulence has decayed appreciably. There seems no compelling reason why there should be universally valid results concern-

ing such eddies which do not have time to reach some form of approximate equilibrium (but see the discussion of quasi-equilibrium states given by Batchelor, 1953, Chap. 7). The evolution of the energy-containing eddies is studied later by numerical solution of the equations of an analytical turbulence theory.

The small-eddy wavenumber range is most suitable for statistical analysis. In §2.5, it was mentioned that the inertial terms of the Navier–Stokes equations tend to spread energy in wavevector space while viscosity dissipates energy with increasing effect at high wavenumbers. If the Reynolds number is large, inertial spreading is expected to dominate viscous damping for an appreciable range of wavenumbers, resulting in the excitation of high-wavenumber Fourier components. If the intuitively plausible assumption (to be checked later) is made that the characteristic time for dynamical evolution of an eddy is a monotonically increasing function of eddy size, then small eddies have a characteristic time much shorter than the decay time of the turbulence (which gives the characteristic time of the energy-containing eddies). In this case, small eddies have sufficient time to reach some form of statistical equilibrium. The picture that emerges is that the high-wavenumber components approach a universal statistical equilibrium in a time much shorter than the overall decay time of the turbulence with the excitation in these small eddies adjusting itself rapidly to a level consistent with the rate at which energy is fed in from the energy-containing range.

In support of this picture, there is the property to be discussed in §2.9 that ϵ is independent of ν for large R. This means that the rate of energy dissipation is primarily determined by the mutual interaction of energy-containing eddies. The effect of decreasing ν is not to decrease ϵ but rather to modify the structure of the small eddies in order that they may dissipate at a rate independent of ν. Increasing R merely excites higher wavenumbers so that (2.37) remains valid. This is consistent with an *equilibrium range* of small eddies which accommodates itself to the energy-containing range that feeds it.

2.7 The Structure of the Large Eddies

The large eddies, $kL_p \ll 1$, are studied by expansion of the cumulants about $\mathbf{k} = 0$. In early studies, cumulants were assumed analytic functions of $\mathbf{k}$ near $\mathbf{k} = 0$. For example, $S_{\alpha\beta}(\mathbf{k})$ was assumed expansible in the power series

$$S_{\alpha\beta}(\mathbf{k}) = C_{\alpha\beta} + C_{\alpha\beta\gamma}k_\gamma + C_{\alpha\beta\gamma\delta}k_\gamma k_\delta + O(k^3). \tag{2.38}$$

The analyticity of cumulants follows if spatial correlations decay exponentially at infinity. In fact, (2.38) follows by expansion of (2.16) in powers of $\mathbf{k}$, which is justifiable if all the integrals

$$\int r_\gamma r_\delta \ldots r_\rho R_{\alpha\beta}(\mathbf{r})\, d\mathbf{r} \tag{2.39}$$

exist.

Assuming (2.38), (2.26) implies that for small k

$$k_\alpha C_{\alpha\beta} + O(k^2) = 0.$$

Consequently, since $\mathbf{k}/k$ is arbitrary, $C_{\alpha\beta} = 0$. Similarly, (2.23) requires that

$$a_\alpha a_\beta^* k_\gamma C_{\alpha\beta\gamma} + O(k^2) \geqslant 0,$$

which implies $C_{\alpha\beta\gamma} = 0$. Furthermore, if third-order cumulants are also analytic, it can be shown that $(\mathrm{d}/\mathrm{d}t)C_{\alpha\beta\gamma\delta} = 0$ for all $\alpha, \beta, \gamma, \delta$ (Batchelor, 1953, §5.3). If the turbulence is isotropic, then

$$E(k) = Ck^4 + o(k^4) \tag{2.40}$$

for $k \to 0$, where $C = (2\pi/3)C_{\alpha\alpha\beta\beta}$ is time independent. This result is known as the "permanence of the large eddies". In the special case of isotropic turbulence where $R_{\alpha\beta}(\mathbf{r})$ has the form (2.7), permanence of the large eddies is equivalent to $\mathrm{d}J/\mathrm{d}t = 0$, where

$$J = v_{\mathrm{rms}}^2 \int_0^\infty r^4 f(r)\, \mathrm{d}r \tag{2.41}$$

is called the Loitsiansky invariant (Loitsiansky, 1939).

More recent work (Proudman and Reid, 1954; Batchelor and Proudman, 1956) has shown that analyticity assumptions such as (2.38) are untenable, that the large eddies are not permanent, that J is not invariant, and that J may not even exist. These difficulties are traced to the long-range nature of the pressure field: in an unbounded fluid, the relevant solution to (1.23) is

$$p(\mathbf{x}) = \frac{1}{4\pi} \int \frac{A(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\, \mathrm{d}\mathbf{y},$$

where $A(\mathbf{x}) = \nabla \cdot (\mathbf{v} \cdot \nabla)\mathbf{v}$.

Batchelor and Proudman (1956) argued that even if all the moment integrals of velocity cumulants, like (2.39), exist initially, the long-range pressure force induces correlations such that they cannot all exist at later instants. They found that $R_{\alpha\beta}(\mathbf{r})$ was generally of order $1/r^5$ as $r \to \infty$, except for isotropic turbulence where it is at most of order $1/r^6$.

The hypothesis of convergent integral moments of the cumulants of the initial velocity field leads to self-consistent results for the behavior of cumulants at large distances. However, this hypothesis of Batchelor and Proudman is not free from controversy. It *is* physically unreasonable to assert that there is much long-range initial correlation, as it is hard to see why the random fluctuations triggering the instabilities that lead to turbulence should be correlated over large distances. However, Saffman (1967a) considers the physically plausible alternative to Batchelor and Proudman's hypothesis that all integral moments of cumulants of the *vorticity* field converge at the initial instant. This assumption is somewhat weaker than

Batchelor and Proudman's in th' t it allows nonanalytic Fourier-space cumulants of the velocity field even at the initial instant. Saffman finds that near $\mathbf{k} = 0$

$$S_{\alpha\beta}(\mathbf{k}) = \frac{1}{4\pi} C' P_{\alpha\beta}(\mathbf{k}) + O(k^2)$$

so that for isotropic turbulence

$$E(k) = C'k^2 + o(k^2).$$

The latter spectral form follows from the requirement that the energy per mode be finite at $\mathbf{k} = 0$. Further, Saffman shows that C' is constant during decay and that cumulants decay as $1/r^3$ except for isotropic turbulence where they decay as $1/r^4$. It has not yet been decided which of the two hypotheses concerning initial integral moments is actually realized, though Saffman presents an argument that grid turbulence more nearly satisfies the Batchelor and Proudman hypothesis.

2.8 The Probability Distribution of Velocity

Instead of using cumulants to describe the ensemble, it may be described by n-point joint-probability distribution functions. Knowledge of all the joint-probability distributions up to order n gives considerably more information about the turbulence than knowledge of all cumulants up to order n. Indeed, the distribution functions determine *all* moments involving at most n distinct points. The price paid for this further detail is the dependence of f_n on a large number of independent variables. Single-time joint-probability distributions of order n depend on $6n - 3$ independent variables ($3n$ velocity variables and $3n - 3$ space variables) while nth-order cumulants depend on $3n - 3$ variables. This disadvantage of joint-probability distributions is amplified by the considerable simplification of low-order cumulants if the turbulence is isotropic. For this reason, joint-probability distributions have been used mainly for kinematical analysis rather than dynamical studies. However, a coupled hierarchy of dynamical equations for the various f_n has been derived by Lundgren (1967). This hierarchy is most easily derived from (1.8).

Experimental measurements in grid turbulence indicate that the time-average one-point velocity distribution [defined by (1.8) with $n = 1$] is approximately Gaussian. For isotropic turbulence, a Gaussian one-point distribution takes the form

$$f_1(\mathbf{v}, t) = (2\pi v_{\text{rms}}^2)^{-3/2} \exp\left[-v^2/(2v_{\text{rms}}^2)\right] \tag{2.42}$$

which is independent of $\mathbf{x}$ because of homogeneity. A fairly sensitive test of the closeness of fit for large v is given by measuring $\langle v_1^4 \rangle / \langle v_1^2 \rangle^2$, where v_1

is the component of v parallel to the axis of the wind tunnel. For a Gaussian distribution, this ratio is exactly 3, while the experimental value lies between 2.9 and 3.0 (Batchelor, 1953, §8.1; Frenkiel and Klebanoff, 1967).

There appears to be no convincing theoretical reason yet suggested why $f_1(\mathbf{v}, t)$ should be so closely Gaussian. An obvious suggestion is that some form of the central limit theorem (see, e.g., Lumley, 1972) applies to ensure that (2.42) is realized. However, there is no obvious way to write the velocity at a point as the sum of many *independent* random variables (except in the final stages of decay when the Reynolds number is low). A full investigation requires consideration of the effect of pressure forces on a representation of the velocity field at a point as a sum of "weakly dependent" random variables. However, it may be that f_1 Gaussian is not a general property but rather is a result of the particular mechanism of generation of grid turbulence.

The two-point joint-probability distribution $f_2(\mathbf{v}, \mathbf{v}', \mathbf{r}, t) = f_2(\mathbf{v}, \mathbf{x}, t; \mathbf{v}', \mathbf{x} + \mathbf{r}, t)$ is not approximately Gaussian unless $r > L_p$. The skewness factor is defined by

$$S(r) = -\langle[v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^3\rangle / \langle[v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^2\rangle^{3/2} \tag{2.43}$$

and the flatness factor by

$$F(r) = \langle[v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^4\rangle / \langle[v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^2\rangle^2, \tag{2.44}$$

where $\mathbf{r} = (r, 0, 0)$ is parallel to the axis of the wind tunnel and all velocities are measured simultaneously. Further, define $S_0 = S(r)|_{r=0+}$, $F_0 = F(r)|_{r=0+}$ so that

$$S_0 = -\langle(\partial v_1/\partial x_1)^3\rangle / \langle(\partial v_1/\partial x_1)^2\rangle^{3/2} \tag{2.45}$$

$$F_0 = \langle(\partial v_1/\partial x_1)^4\rangle / \langle(\partial v_1/\partial x_1)^2\rangle^2. \tag{2.46}$$

A Gaussian two-point velocity distribution requires that $S_0 = S(r) = 0$, $F_0 = F(r) = 3$, for all r. Experiments indicate that $S(r) \to 0$ and $F(r) \to 3$ as $r \to \infty$, as follows from the definitions assuming the independence of widely separated fluid elements and the Gaussian single-point distribution (2.42). However, $S(r)$ and $F(r)$ differ appreciably from their Gaussian values for $r < L_p$. A variety of grid turbulence experiments give values of $S_0 \cong 0{\cdot}4$ and $F_0 \simeq 4$.†

If the flatness factor of some zero-mean random variable is greater than 3, it indicates that the probability of very small and very large values of the variable is larger than for a Gaussian distribution of the same standard deviation. This can be explained by considering a six-valued zero-mean random variable chosen so that the probability of taking the values $\pm a_i$ is

†Other experiments indicating somewhat larger values of F_0 are described in Section III.

$p_i/2$ for $i = 1, 2, 3$ and where $0 < a_1 < a_2 < a_3$. Then $p_1 + p_2 + p_3 = 1$ and fixed standard deviation implies fixed $\langle a^2 \rangle = p_1 a_1^2 + p_2 a_2^2 + p_3 a_3^2$. The maximum flatness $\langle a^4 \rangle / \langle a^2 \rangle^2$ is attained when the distribution is $p_1 = (a_3^2 - \langle a^2 \rangle)/(a_3^2 - a_1^2) = 1 - p_3, p_2 = 0$.

A random variable having a large probability of taking values both very large and very small compared to its standard deviation is called intermittent. Intermittency in homogeneous turbulence was first noted by Batchelor and Townsend (1949) who observed that high-order derivatives of the velocity field alternated between regions of quiescence where the derivative is small and regions of activity where it is large. Such behavior is not unexpected since the viscous term in (1.21) contains the highest-order derivative while high Reynolds number turbulence involves the limit $\nu \to 0$. This limit is a singular perturbation problem and localized regions in which gradients are large should be expected to form. These regions should have the character of vortex sheets and lines.

From the Schwarz inequality, it follows that $|S(r)| \ll [F(r)]^{1/2}$ and $|S_0| \ll F_0^{1/2}$. Improvement on the latter inequality is possible using the kinematical constraints of incompressibility and isotropy. The result, due to Betchov (1956) and Batchelor and Townsend (1956), is

$$|S_0| \ll \frac{2}{\sqrt{21}} F_0^{1/2}. \tag{2.47}$$

For a Gaussian two-point velocity distribution $F_0 = 3$ and Betchov's inequality requires $|S_0| \ll 0{\cdot}756$. With $F_0 = 4$, the inequality requires $|S_0| \ll 0{\cdot}872$. The experimental value $S_0 \simeq 0{\cdot}4$ is well within this bound.

2.9 The Decay of Total Energy

If the flow were not turbulent, we could estimate $|\omega| = O(v_{\text{rms}}/L_p)$ (where we use v_{rms} and L_p to characterize the large-scale features of the flow), so that (2.13) would give

$$\epsilon = O(\nu v_{\text{rms}}^2 / L_p^2). \tag{2.48}$$

If this estimate were correct, the rate of energy dissipation would tend to zero with increasing Reynolds number. In fact, it is found experimentally (cf. Batchelor, 1953, Fig. 6.1) that ϵ does not depend so strongly on Reynolds number and that

$$A = \epsilon L_p / v_{\text{rms}}^3 \tag{2.49}$$

is of order unity even for large Reynolds numbers, in contrast to $A = O[1/R(= \nu/v_{\text{rms}} L_p)]$ which follows from the estimate (2.48). This result marks a fundamental difference between laminar and turbulent flows.

Evidently fine-grained velocity fluctuations give rise to values of $|\nabla \mathbf{v}|$

far in excess of v_{rms}/L_p. The value of ϵ given by (2.49) with $A = 0(1)$ may be reconciled with (2.48) by replacing ν with an "eddy viscosity"

$$\nu_e = A v_{rms} L_p \gg \nu,$$

when $R \gg 1$. The ratio ν_e/ν gives the enhancement of energy dissipation by the turbulence.

Equation (2.36) may be rewritten as

$$\frac{d}{dt}\frac{3}{2} v_{rms}^2 = -A v_{rms}^2/(L_p/v_{rms}). \tag{2.50}$$

The quantity L_p/v_{rms} is the turnover time of an eddy of size L_p and typical velocity v_{rms}. Hence the interpretation of (2.50) is that the total energy in the flow decays in about one turnover time of an eddy of size L_p. It may be said that the energy-containing range is approximately critically damped.

There has been considerable interest in the literature in establishing simple features of the overall energy decay process. One possibility examined in detail is that, during decay of high-Reynolds-number turbulence, the energy obeys the similarity law (Karman and Howarth, 1938)

$$E(k, t) = \frac{3}{2} v_{rms}^2(t) L(t) F(kL(t)),$$

where $\int_0^\infty F(x)\, dx = 1$. It follows that

$$\epsilon \propto v_{rms}^2(t)/L(t)^2.$$

Hence, if v_{rms} decays according to a power-law, $v_{rms}^2(t) \propto (t - t_0)^{-n}$ where t_0 is a virtual origin of time, then $L(t) \propto (t - t_0)^{1/2}$. But the longitudinal integral scale L_p is given by

$$L_p = \int_0^\infty f(r)\, dr, = \frac{3\pi}{4} \frac{\int_0^\infty k^{-1} E(k)\, dk}{\int_0^\infty E(k)\, dk} \tag{2.51}$$

(which follows from isotropy), so that $L_p(t) \propto L(t)$. Consequently, if (2.50) is valid with $A = 0(1)$, it follows that $v_{rms}^2 \propto (t - t_0)^{-1}$. Since similarity over all wavenumbers is a rather strong and unsubstantiated assumption, this result should be regarded suspiciously. In fact, $v_{rms}^2 \propto (t - t_0)^{-1}$ is not in agreement with experiments.

Lin (1948) found that the weaker assumption that similarity extends overall wavenumbers except the very lowest implies that $v_{rms}^2 = a(t - t_0)^{-1} + b$, where a and b are constant during decay. The underlying idea is that low wavenumber components do not have time to relax to a similarity state. The constants a, b, t_0 may be chosen to give reasonably good fit with experiments.

Other possible decay laws for $v_{rms}(t)$ may be derived if similarity assump-

tions of the above type are not made. As a crude model of $E(k)$, suppose that

$$E(k) = \begin{cases} Ck^r \ (0 \le kL(t) \le 1) \\ C[L(t)]^{-s-r}k^{-s} \ (kL(t) > 1), \end{cases} \tag{2.52}$$

where $r > 0$, $s > 1$ and $L(t)$ is a time-dependent parameter specifying the scale of the flow (not a similarity parameter). The basis for (2.52) is that $E(k, t)$ is known to be of such form for kL_p very small (Sec. 2.7) and moderately large (Sec. 3.1). In fact, if k refers to a large eddy and Loitsiansky's invariant is invariant, then $r = 4$ and C is constant during decay. On the other hand, Saffman's (1967a) analysis of the large eddies gives $r = 2$ and C constant during decay. It follows from (2.52) that

$$v_{\text{rms}}^2(t) \propto [L(t)]^{-r-1}, \quad L_p(t) \propto L(t),$$

where the constants of proportionality are time independent. Substituting in (2.50) gives

$$v_{\text{rms}}^2 \propto (t - t_0)^{-2(r+1)/(r+3)}, \quad L_p \propto (t - t_0)^{2/(r+3)} \tag{2.53}$$

as obtained by Comte-Bellot and Corrsin (1966). Here t_0 is a constant of integration. With $r = 4$ and C constant, it follows that

$$v_{\text{rms}}^2 \propto (t - t_0)^{-10/7}, \quad L_p \propto (t - t_0)^{2/7} \tag{2.54}$$

as found by Kolmogorov (1941b). However, if $r = 2$

$$v_{\text{rms}}^2 \propto (t - t_0)^{-6/5}, \quad L_p \propto (t - t_0)^{2/5} \tag{2.55}$$

as found by Saffman (1967b). The laws (2.54) and (2.55) are both in satisfactory agreement with experimental results, though the general nonexistence of the Loitsiansky invariant would appear to give greater theoretical support to (2.55).

Further insight into the decay of total energy is gained by considering the mechanism of decay. Since the vorticity is related to ϵ by (2.13), it follows that production of vorticity is equivalent to the enhancement of dissipation by turbulence (Taylor, 1938). For this reason, it is often said that turbulence is a field of random vorticity.

The stretching of vortex lines by the velocity field induced by the vorticity distribution is the fundamental process involved in the decay of turbulence. If vortex lines tend to stretch, vorticity is generated and ϵ is enhanced. This process may be studied using the dynamical equation for vorticity

$$\frac{\partial}{\partial t}\,\boldsymbol{\omega}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t)\cdot\nabla\boldsymbol{\omega}(\mathbf{x}, t) = \boldsymbol{\omega}(\mathbf{x}, t)\cdot\nabla\mathbf{v}(\mathbf{x}, t) + \nu\,\nabla^2\boldsymbol{\omega}(\mathbf{x}, t). \tag{2.56}$$

The left-hand side of (2.56) describes the convection of vorticity while

the first term on the right describes the generation of vorticity by stretching of vortex lines and the second accounts for diffusion of vorticity by viscosity. It follows that

$$\frac{d\Omega}{dt} = \langle \omega_\alpha \omega_\beta \frac{\partial v_\alpha}{\partial x_\beta} \rangle - \nu \langle \frac{\partial \omega_\alpha}{\partial x_\beta} \frac{\partial \omega_\alpha}{\partial x_\beta} \rangle, \tag{2.57}$$

where the "enstrophy" $\Omega = 1/2\langle |\omega|^2 \rangle$. Since the viscous term is non-positive, production of enstrophy can only be accomplished by the first term on the right-hand side of (2.57) which describes the mean stretching of vortex lines.

The stretching term in (2.57) had a particularly simple form in isotropic turbulence (Batchelor and Townsend, 1947):

$$\langle \omega_\alpha \omega_\beta \frac{\partial v_\alpha}{\partial x_\beta} \rangle = \frac{35}{2} S_0 \left(\frac{\epsilon}{15\nu} \right)^{3/2}, \tag{2.58}$$

where S_0 is the skewness (2.45) or, equivalently,

$$S_0 = \left(\frac{135}{98} \right)^{1/2} \frac{\int_0^\infty k^2 T(k)\, dk}{\left[\int_0^\infty k^2 E(k)\, dk \right]^{3/2}}. \tag{2.59}$$

The latter form shows that S_0 is a nondimensional measure of the rate of enstrophy production by the nonlinear terms; on the other hand, (2.58) shows that in order for enstrophy to be produced by mean stretching of vortex lines, it is necessary that $S_0 > 0$.

At high Reynolds numbers, (2.58) implies that the time rate of change of enstrophy is much smaller than either the rate of production of enstrophy by vortex stretching or the rate of dissipation of enstrophy by viscosity. For

$$\frac{d\Omega}{dt} = \frac{1}{2\nu} \frac{d\epsilon}{dt} \ll \frac{35}{2} S_0 \left(\frac{\epsilon}{15\nu} \right)^{3/2},$$

since S_0, ϵ, and $d\epsilon/dt$ are nearly independent of ν as $\nu \to 0$. Therefore, the left-hand side of (2.57) is of order $R^{-1/2}$ compared to the terms on the right taken separately, giving the approximate balance

$$S_0 \approx \frac{2\nu}{35} \left(\frac{15\nu}{\epsilon} \right)^{3/2} \langle |\nabla \omega|^2 \rangle, = \frac{\sqrt{270}}{7} \nu \frac{\int_0^\infty k^4 E(k)\, dk}{\left[\int_0^\infty k^2 E(k)\, dk \right]^{3/2}}. \tag{2.60}$$

This result is not very surprising since the small-eddy range of wavenumbers dominates $\langle \omega^2 \rangle = 2 \int_0^\infty k^2 E(k)\, dk$ and the small eddies are expected to reach a quasi-steady statistical equilibrium long before the turbulence has decayed appreciably.

2.10 The Taylor–Green Vortex

Taylor and Green (1937) initiated the study of a model three-dimensional vortex field in order to clarify the dynamics of turbulence. This vortex motion illustrates the basic turbulence decay mechanisms of the production of small eddies and the enhancement of dissipation by the stretching of vortex lines.

At $t = 0$, the velocity field is assumed to be

$$\begin{aligned} v_1(x_1, x_2, x_3) &= \cos x_1 \sin x_2 \cos x_3 \\ v_2(x_1, x_2, x_3) &= -\sin x_1 \cos x_2 \cos x_3 \\ v_3(x_1, x_2, x_3) &= 0, \end{aligned} \tag{2.61}$$

where we have shifted the origin of x_3 by $\pi/2$ from the initial conditions chosen by Taylor and Green. Although the streamlines of (2.61) are the curves $\cos x_1 \cos x_2 = \text{const}$ in the planes $x_3 = \text{const}$, the flow that develops is three-dimensional. The initial vortex lines are the curves $\sin x_1/\sin x_2 = \text{const}$, $\sin^2 x_1 \cos x_3 = \text{const}$, so they are twisted and may induce a velocity field to stretch themselves. In fact, since $\omega \cdot \nabla v$ is initially nonzero such stretching does take place. Also, v_3 becomes nonzero for $t > 0$. The Taylor–Green vortex is perhaps the simplex example of self-induced vortex stretching by a three-dimensional velocity field.

Taylor and Green (1937) investigated the evolution of their vortex by developing a perturbation solution to the Navier–Stokes equations in powers of the time t. They found that the enstrophy is given by

$$\begin{aligned} \Omega(t) = \frac{3}{8}\Big[1 - \frac{6t}{R} + \left(\frac{5}{48} + \frac{18}{R^2}\right)t^2 - \left(\frac{5}{3} + \frac{36}{R^2}\right)\frac{t^3}{R} \\ + \left(\frac{25}{3168} + \frac{1835}{144R^2} + \frac{54}{R^4}\right)t^4 + \cdots\Big], \end{aligned} \tag{2.62}$$

where $R = 1/\nu$ is a Reynolds number for this flow. It is clear that finite-order truncations of (2.62) cannot remain valid as $t \to \infty$ (since $\Omega \to 0$ as $t \to \infty$). More recently, Orszag and Fateman (1974) extended the series through terms of order t^{14}.

Goldstein (1940) investigated the evolution of the Taylor and Green vortex by developing a perturbation series in powers of the Reynolds number R, finding

$$\begin{aligned} \Omega(t) = \frac{3}{8}\Big[e^{-6t/R} - \frac{R^2}{384}(e^{-6t/R} - 20e^{-12t/R} + 35e^{-14t/R} - 16e^{-16t/R}) \\ + \cdots\Big]. \end{aligned} \tag{2.63}$$

This series is a resummation of (2.62); each term of (2.63) is a partial sum of an infinite number of terms of (2.62). Finite order truncations of

(2.63) do not have the secular behavior exhibited by truncations of (2.62) as $t \to \infty$.

Neither perturbation series in t nor R describes properly the evolution of the flow field for large t or R. Orszag and Fateman (1974) report numerical solutions of the Navier–Stokes equations for the Taylor–Green vortex using a method specially suited to the problem. In Fig. 2.2, we plot the evolution of $\Omega(t)$ at $R = 300$ determined by perturbation series in t truncated at t^5 (curve 1). perturbation series in R truncated at R^4 (curve 2), and numerical simulation (curve 3). Independent tests show that the numerical simulation is accurate to within 5% at this Reynolds number. An indication of the magnitude of the Reynolds number R is given by the relation $R_\lambda = 0.372\ R$ at $t = 0$, where

$$R_\lambda = v_{\text{rms}}\lambda/\nu$$

is the Reynolds number based on the Taylor microscale. Thus, at $R = 300$, $R_\lambda = 112$ at $t = 0$ while it is found that $R_\lambda = 37$ at $t = 6$. These values of R_λ should be compared with laboratory wind-tunnel experiments on grid generated turbulence which are generally performed in the range $R_\lambda = 25$–50. Also, notice that in the absence of the nonlinear terms of the Navier–

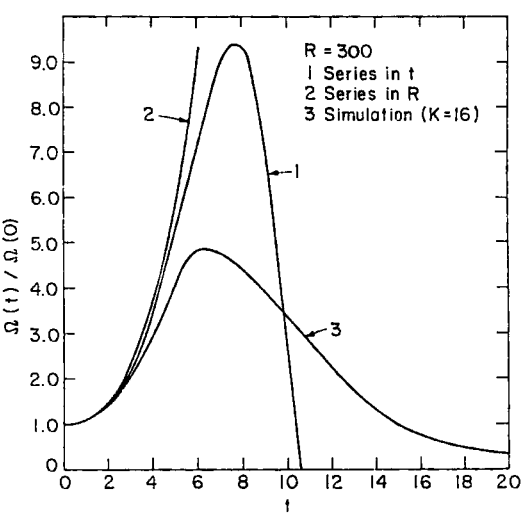

Figure 2.2 Enhancement of mean-square vorticity $\Omega(t)/\Omega(0)$ vs t for the Taylor–Green vortex at $R = 300$. Curve 1: perturbation series in powers of t truncated at order t^5. Curve 2: perturbation series in powers of R truncated at order R^4. Curve 3: numerical solution of the Navier–Stokes equations.

Stokes equations $\Omega(t) \leq \Omega(0)$, so that the enhancement of mean-square vorticity observed on Fig. 2.2 is a measure of the strength of the nonlinearity.

The variation of $\Omega(t)$ with Reynolds number relates indirectly to the effect of Reynolds number on large-scale structures in the flow since

$$\epsilon(t) = \frac{2}{R}\Omega(t).$$

If $\Omega(t)$ is asymptotically proportional to R (for t beyond some initial adjustment period) then $\epsilon(t)$ is Reynolds number independent, i.e. A in (2.49) is $0(1)$ as $R \to \infty$. Some support for this behavior is given by the results plotted in Fig. 2.3. Here $\epsilon(t)$ is plotted as a function of t for $R = 100 - 400$, the simulation with $R = 400$ being only moderately accurate.

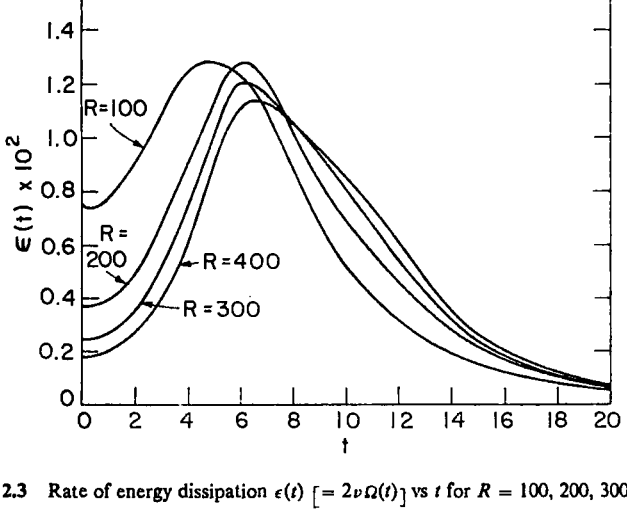

Figure 2.3 Rate of energy dissipation $\epsilon(t)$ $[= 2\nu\Omega(t)]$ vs t for $R = 100, 200, 300, 400$.

It may be suggested from these results that, as $R \to \infty$, $\epsilon(t)$ approaches a finite limiting function $\epsilon_\infty(t)$ [with the property that $\epsilon_\infty(t) = 0$ for $t < t_\infty$]. In order that $\epsilon_\infty(t) > 0$ for $t > t_\infty$, it is necessary that $\Omega(t) \to \infty$ as $R \to \infty$ for $t > t_\infty$. In other words, typical solutions of the Euler equations (inviscid Navier–Stokes equations) should develop infinite vorticity after a finite time even in the absence of solid boundaries. It is not now known whether the generation of infinite vorticity in a finite time is consistent or not with the

three-dimensional Euler equations,† though there are several indications that it is from analytical turbulence theory.

The asymptotic Reynolds number independence of $\epsilon(t)$ has important consequences. In fact, if all large scale features of turbulent flows are similarly Reynolds number independent then it is possible to simulate (numerically or otherwise) large-scale flow features of very high Reynolds number turbulence when the Reynolds number of the simulation (and, hence, the required resolution) is quite modest (cf. §6.6), so long as the Reynolds number of the simulation is large enough for the flow to be "turbulent".

In Fig. 2.4, we plot the evolution at $R = 200$ of

$$D3/D1 = \langle \nabla v_3 \cdot \nabla v_3 \rangle / \langle \nabla v_1 \cdot \nabla v_1 \rangle$$

$$E3/E1 = \langle v_3^2 \rangle / \langle v_1^2 \rangle$$

where $\langle\ \rangle$ denotes space average, which are, respectively, measures of the anisotropy of energy dissipation and energy. It is apparent that energy dissipation approaches a state of near isotropy for $t = 4$–16, while the kinetic energy itself is always far from isotropy. This result is consistent with the qualitative ideas on local equilibrium outlined in §2.6. Small eddies dominate dissipation but energy-containing eddies dominate the energy.

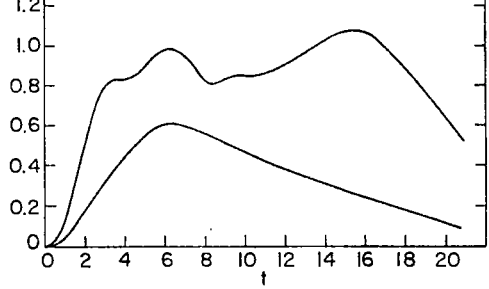

Figure 2.4 Anisotropy ratios of kinetic energy ($E3/E1$) and dissipation ($D3/D1$) vs t for the Taylor–Green vortex at $R = 200$.

†In order that vorticity, become infinite in a finite length of time, Kelvin's and Helmholtz' vorticity theorems require that vortex lines be stretched to infinite length in a finite time. Since the velocity is bounded (by energy considerations), this can only be done in the interior of the inviscid fluid by small-scale "crinkling" of vortex lines. The author knows of no analytical example of this complex process.

Finally, we remark that, as $t \to \infty$, the flow reverts to a form proportional to the initial conditions (2.61), which is very anisotropic. This result may be demonstrated directly from a modal representation of the velocity field (Orszag and Fateman, 1974).

III Small-Scale Structure of Turbuleuce

3.1 The Equilibrium Range

According to the discussion of §2.6 and §2.9, the principal dynamical effect of small eddies back on larger scales is through their contribution to the energy dissipation rate ϵ. In order for ϵ to be asymptotically independent of Reynolds number, it is necessary that the wavenumber range contributing to enstrophy grow with Reynolds numbers. The excitation of small scales is accomplished by the nonlinear terms in the Navier–Stokes equations. If the small scales are indeed generated solely to dissipate at the rate ϵ, then we may argue that the energy spectrum $E(k)$ may be a function of only ϵ, k, and ν, the molecular viscosity. Since $[E(k)] = \mathrm{cm}^3\mathrm{s}^{-2}$, $[\epsilon] = \mathrm{cm}^2\mathrm{s}^{-2}$, $[k] = \mathrm{cm}^{-1}$, and $[\nu] = \mathrm{cm}^2\mathrm{s}^{-1}$, it follows by dimensional analysis that the energy spectrum for $k \gg L_p^{-1}$ must be of the form

$$E(k) = \epsilon^{2/3}k^{-5/3}\,F(k/k_d) \tag{3.1}$$

$$k_d = (\epsilon/\nu^3)^{1/4}, \tag{3.2}$$

where $F(x)$ is an arbitrary function satisfying $\int_0^\infty x^{1/3}\,F(x)\,dx = 1/2$ in order that ϵ be given by (2.37).

The result (3.1) is precisely that given by the theories of Kolmogorov (1941 a,c), Obukhov (1941), Onsager (1945, 1949), and von Weizsäcker (1948). We now examine the relationship of the theories to the above argument in more detail. The principal assumption of these theories is that high-wavenumber components are not influenced directly by the external conditions that give rise to the turbulence. To see the plausibility of this assumption, consider the mechanism of excitation of small eddies. Only the energy containing eddies are directly excited by whatever drives the turbulence. Small eddies are excited when the energy-containing eddies become unstable and nonlinear interaction acts to transfer energy to high wavenumbers. If, at some time, only Fourier components up to wavenumber K are excited appreciably and if R is so large that viscosity is unimportant for $k < 2K$ then nonlinear interaction in (2.31) induces excitation in wavenumbers up to $2K$. Once these latter wavenumbers become appreciably excited and if viscosity is still negligible, a further doubling of wavenumber up to $4K$ will be possible, and so on. If the characteristic external scale is $L_p \equiv 2\pi/k_L$, then modes with $k \gg k_L$ are excited only after this wave-

number-doubling process acts many times [of order $\log_2 (k/k_L)$]. It is plausible to expect that while the external conditions do have an important effect on wavenumbers of order k_L, all such effects are "forgotten" by very small eddies for which the wavenumber-doubling process intervenes a large number of times.†

It is not *a priori* obvious that Kolmogorov's assumption is dynamically consistent. Surely the energy-containing eddies have an important *kinematical* effect on small eddies—small eddies are convected from point to point by the large-scale flow. The essence of the assumption is that this convection is *dynamically* unimportant. In fact, a small eddy superimposed on a uniform velocity field is convected without any change of shape. If the large scales of turbulence can be approximated by uniform convection, then large scales should have only a weak effect on the structure of small eddies. In this case, interactions between large and small scales should be dynamically insignificant and small eddies should be statistically independent of large eddies.

Another aspect of Kolmogorov's assumption is worthwhile pointing out. Consider turbulence maintained in a fully-developed state by random forces acting on large scales. At high Reynolds number, the energy fed in at low wavenumbers is lost to viscous dissipation at high wavenumbers. In order for Kolmogorov's assumption to be justified, the process in which energy is transferred in many steps of size k_L ($k_L \ll k$) to maintain the excitation at $\mathbf{k}$ must be dynamically insignificant compared to the direct transfer of excitation from wavenumbers of order $k/2$, say, to wavenumber k. However, it is clear from (2.31) that the former interaction involves $\mathbf{u}(\mathbf{k}_L)\mathbf{u}(\mathbf{k})$ while the latter involves $\mathbf{u}(\mathbf{p})\mathbf{u}(\mathbf{q})$ with $p = 0(k/2)$ and $q = 0(k/2)$. Now, since $k \gg k_L$, $|\mathbf{u}(\mathbf{p})\mathbf{u}(\mathbf{q})| \ll |\mathbf{u}(\mathbf{k}_L)\mathbf{u}(\mathbf{k})|$ if $|\mathbf{u}(\mathbf{k})|$ decreases with k as, say, a power of $1/k$. It would seem that interactions (k_L, k) are dominant relative to $(k/2, k/2)$ and that Kolmogorov's assumption is unjustified. This difficulty is resolved by noting that the principal effect of (k_L, k) interactions is a phase change of $\mathbf{u}(\mathbf{k})$ corresponding to the uniform convection of small scales by large scales. This phase change will be seen to have no dynamical effect on the structure of small eddies (cf. Sec. 6.5).

Kolmogorov's assumption requires that energy transfer among small eddies not depend on large-scale peculiarities of the flow, so that energy transfer is *local* in $\mathbf{k}$-space. That is, direct effects of wavenumbers $k_L \ll k$ through (k_L, k) interactions are not effective in changing the energy in

†This argument indicates that energy flow to wavenumbers much smaller than those of the energy-containing range need not be accompanied by loss of information. Such flow can proceed in one step when two energy-containing wavevectors nearly the negatives of each other interact to excite a very low wavenumber.

mode k. Heuristically, it is helpful to think that energy from a large eddy passes through many intervening wavenumbers in the process of reaching a small eddy. Consequently, if Σ is a surface in k-space not intersecting wavenumbers of order k_L and if the Reynolds number is sufficiently large, it is possible to associate an energy flux $\epsilon(\Sigma)$ to Σ which depends only on the parameters of the flow determined by modes "near" the surface Σ. Similarly, all dynamical characteristics of small eddies must be transferred locally in k-space. Another consequence of Kolmogorov's assumption is that the energy spectrum must be approximately isotropic at large k, since dependence on any direction preferred by imposed external conditions is ruled out.

The picture emerges that high-Reynolds-number k-space flow consists of an energy source (low k) and an energy sink (high k) with the properties of the sink statistically independent of those of the source. If R is sufficiently large, energy dissipation is principally confined to wavenumbers $k > k_d \gg k_L$ where k_d, given by (3.2), is a dissipation wavenumber at which viscous dissipation first becomes significant. The principal contribution to $\int_0^\infty E(k)\,dk$ comes from $k \sim k_L$, while the principal contribution to $\int_0^\infty k^2E(k)\,dk$ comes from $k \sim k_d$. In effect, high-Reynolds-number energetics involves two distinct "black boxes", one containing the energy source $k \sim k_L$ and the other, the energy sink $k \sim k_d$, which are connected by a transmission line $k_L \ll k \ll k_d$. The part of the spectrum corresponding to the transmission line is called the inertial range since inertial forces, not external conditions or friction, dominate the dynamics. The wavenumber range $k > k_d$ is called the dissipation range, while $k \gg k_d$ is the far dissipation range. The complete range $k \gg k_L$ is called the equilibrium range.

The properties of the inertial range are particularly simple. Since $k \gg k_K$, statistical properties are approximately isotropic in space and quasi-stationary in time. Further, since $k \ll k_d$, effects of viscosity are negligible. Since the inertial range is completely local, $E(k)$ can depend only on the local properties of the transmission line at k. Two of the simplest properties at k are k itself and $\epsilon(k)$, the rate of energy transfer through the sphere $|\mathbf{k}| = k$. Notice that although $\epsilon(k)$ may be defined locally at k, it has a global significance. The rate of energy transfer $\epsilon(k)$ must be constant throughout the inertial range and must equal the rate of energy input ϵ from the energy-containing range which must itself equal the rate of energy dissipation given by (2.37). In other words, the transmission line is loss-free. It is natural to assume, following Kolmogorov, that the properties of the transmission line at k are completely determined by k and ϵ. In particular, $E(k)$ can depend only on k and ϵ. It follows by dimensional analysis that

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \quad (k_L \ll k \ll k_d), \tag{3.3}$$

where C_K is a universal constant independent of viscosity and large-scale conditions. This result is consistent with (3.1) if $F(0) = C_K$.

As a simple consistency check, it may be verified that the Kolmogorov spectrum (3.3) is such that most of the energy of the turbulence lies in wavenumbers much smaller than those of the inertial range, while most of the dissipation occurs in much larger wavenumbers. This follows since $\int_0^\infty k^{-5/3}\,dk$ diverges at $k = 0$ and $\int_0^\infty k^2 k^{-5/3}\,dk$ diverges at $k = \infty$. Consistency by this criterion requires only that if $E(k) \propto k^{-n}$ in the inertial range then $1 < n < 3$.

A further check involves the characteristic time of inertial-range eddies, which will be denoted by $\tau_i(k)$. Since $\tau_i(k)$ can depend only on k and ϵ, dimensional analysis shows that

$$\tau_i(k) = A_K \epsilon^{-1/3} k^{-2/3}, \tag{3.4}$$

where A_K is a universal constant. There are at least three reasons why $\tau_i(k)$ may be called a characteristic time of an inertial-range mode. First, the total energy in inertial-range eddies whose physical size is of order $2\pi/k$ may be taken to be

$$\frac{1}{2}[\Delta v(k)]^2 \equiv \int_{1/2k}^{2k} E(p)\,dp, \propto \epsilon^{2/3} k^{-2/3},$$

which is clearly a definition of some arbitrariness. The quantity $\Delta v(k)$ is a measure of velocity differences across distances of order $2\pi/k$. Therefore, the time it takes an eddy of wavenumber k to circulate *relative* to itself, or turn over, is of order $[k\Delta v(k)]^{-1} \propto \tau_i(k)$. Another interpretation of $\tau_i(k)$ obtains by considering the time it would take energy transfer occurring at rate ϵ to deplete the energy $\frac{1}{2}[\Delta v(k)]^2$, if the input to $\Delta v(k)$ from lower modes were turned off. This time is clearly $\frac{1}{2}[\Delta v(k)]^2/\epsilon \propto \tau_i(k)$. These two alternative interpretations of $\tau_i(k)$ show that inertial-range modes in Kolmogorov's theory share the property of energy-containing modes of being critically damped: an inertial-range eddy is damped, when the energy input is turned off, in a time of order its turnover time. Notice that $\tau_i(k) \gg \tau_s(k) = (kv_{\text{rms}})^{-1}$ when $k \gg k_L$. Here $\tau_s(k)$ is the time it takes an energy-containing eddy to *sweep* an inertial-range eddy past a fixed observation point. If sweeping were dynamically significant, inertial-range energetics would be governed by $\tau_s(k)$ and the inertial-range energy spectrum would be proportional to $k^{-3/2}$ (cf. Sec. 4.9). Since weak shearing does not effectively mix small eddies, $\tau_s(k)$ is not dynamically significant.

A third interpretation of $\tau_i(k)$ is the time it takes to decorrelate the relative phase of the *simultaneous* Fourier amplitudes of distinct wavevectors of magnitude of order k. With regard to this third interpretation, it is important to observe that $\tau_i(k)$ is *not* the time it takes nonsimultaneous phases to become uncorrelated. In fact, we will show in § VI that the time for $\mathbf{u}(\mathbf{k}, t)$ to become uncorrelated with $\mathbf{u}(\mathbf{k}, t')$ is $t - t' \triangleq 0(\tau_s(k))$. Essentially, the distinction is that sweeping by a large eddy is effective in decorrelating the

phase of a mode at different times because the position in space of the eddies corresponding to the mode is changing randomly, but sweeping is not effective in decorrelating the simultaneous relative phase of correlated modes which is not sensitive to change of position by large-scale convection.

Equation (3.4) may be used to test the consistency of (3.3). In order for k to reach approximate equilibrium, it is necessary that $\tau_l(k) \ll \tau_L$ where τ_L is the time for appreciable decay of the turbulence. According to (2.49), $\epsilon = 0(v_{\text{rms}}^3 k_L)$ so that $\tau_L = 0(k_L^{-1} v_{\text{rms}}^{-1})$. Therefore, $\tau_l(k) \ll \tau_L$ is equivalent to $k^{2/3} \gg k_L^{2/3}$, which is satisfied in the inertial range.

In the dissipation range $k \gtrsim k_d$, the spectrum takes the general form (3.1). The dissipation wavenumber k_d measures the wavenumber at which viscous effects become important. The latter identification follows in at least three other ways. First, viscous effects become important when the viscous dissipation time for mode k, $\tau_d(k) = (\nu k^2)^{-1}$, is shorter than $\tau_l(k)$, which requires $k \gtrsim k_d$. Notice that $\tau_l(k_d) = \tau_d(k_d) \doteq (\nu/\epsilon)^{1/2}$.

Second, viscosity may be expected to damp rapidly the energy spectrum at high k, so that the function $F(x)$ should approach zero rapidly as x increases. In this case, the dissipation integral (2.37) may be approximated using the inertial-range spectrum (3.3) up to a cutoff k_d, so that

$$\epsilon \approx 2\nu \int_0^{k_d} k^2 (C_K \epsilon^{2/3} k^{-5/3})\, dk.$$

Equation (3.2) follows to within a numerical factor.†

A third interpretation of k_d is obtained by considering the "local" Reynolds number $R_l(k)$ defined by

$$R_l(k) = \Delta v(k)/(k\nu), \quad \propto \epsilon^{1/3}/(k^{4/3}\nu), \quad = (k_d/k)^{4/3}$$

the latter two relations holding for k in the inertial range. For $k \ll k_d$, $R_l(k) \gg 1$ and the corresponding inertial-range eddy should be strongly unstable. However, when $k = 0(k_d)$, $R_l(k) = 0(1)$ and, therefore, k_d specifies the wavenumber beyond which eddies are stable and the nonlinear transfer mechanism ceases to be effective.

It is interesting to compare the relative sizes of the advective $(\mathbf{v} \cdot \nabla \mathbf{v})$ and viscous $(\nu \nabla^2 \mathbf{v})$ terms of the Navier–Stokes equations. Naively, the estimates $|\mathbf{v} \cdot \nabla \mathbf{v}| = 0(v_{\text{rms}}^2/L_p)$ and $|\nu \nabla^2 \mathbf{v}| = 0(\nu v_{\text{rms}}/L_p^2)$ give

$$\frac{|\mathbf{v} \cdot \nabla \mathbf{v}|}{|\nu \nabla^2 \mathbf{v}|} = 0(R).$$

†Similarly, if k_L is defined as the low-wavenumber cutoff of the inertial range and if it is required that $\frac{1}{2} v_{\text{rms}}^2 = \int_{k_L}^{\infty} C_K \epsilon^{2/3} k^{-5/3}\, dk$, then $\epsilon = C_K^{-3/2} v_{\text{rms}}^3 k_L$. This shows that ϵ independent of ν as $\nu \to 0$ is an essential ingredient of the Kolmogorov theory which requires C_K to be universal.

Actually,

$$\langle|\mathbf{v}\cdot\nabla\mathbf{v}|^2\rangle = 0(v_{\text{rms}}^2\Omega), \quad = 0(\epsilon v_{\text{rms}}^2/\nu)$$

while

$$\langle\nu^2|\nabla^2\mathbf{v}|^2\rangle = 0\left(\nu^2\int_0^\infty k^4E(k)\,dk\right), \quad = 0(\nu k_d^2\epsilon).$$

Thus, the correct estimate of the ratio of advection to dissipation is given by

$$\frac{|\mathbf{v}\cdot\nabla\mathbf{v}|}{|\nu\nabla^2\mathbf{v}|} = 0\left(\frac{v_{\text{rms}}}{\nu k_d}\right), \quad = 0(R^{1/4}).$$

Corresponding to these k-space results, there are x-space expressions for the so-called structure-function tensor defined by

$$B_{\alpha\beta}(\mathbf{r}, t) = \langle[v_\alpha(\mathbf{x}+\mathbf{r}, t) - v_\alpha(\mathbf{x}, t)][v_\beta(\mathbf{x}+\mathbf{r}, t) - v_\beta(\mathbf{x}, t)]\rangle. \tag{3.5}$$

If the turbulence is isotropic, it follows that

$$B_{\alpha\alpha}(\mathbf{r}, t) = 4\int_0^\infty E(k, t)\left[1 - \frac{\sin kr}{kr}\right]dk, \tag{3.6}$$

where $r = |\mathbf{r}|$. If $k_L r \ll 1$, the contribution from wavenumbers $k = 0(k_L)$ where $E(k)$ is largest, is suppressed by the factor $[kr - \sin kr]/kr = \frac{1}{6}(kr)^2 + 0[(kr)^4]$. With a spectrum of the form (3.3), this latter factor ensures that the dominant contribution to (3.6) comes from the equilibrium range, in fact $k \lesssim 1/r$. Therefore, using (3.1),

$$B_{\alpha\alpha}(\mathbf{r}) \approx (\epsilon\nu)^{1/2}G(k_d r), \tag{3.7}$$

where

$$G(y) = 4\int_0^\infty F(x)x^{-8/3}y^{-1}[xy - \sin xy]\,dx. \tag{3.8}$$

As $y \to \infty$, it may be verified that

$$G(y) \sim B_K y^{2/3},$$

where

$$B_K = 4F(0)\int_0^\infty z^{-8/3}[z - \sin z]\,dz, \quad = \frac{9}{5}\Gamma\left(\frac{1}{3}\right)C_K, \approx 4{\cdot}82\,C_K. \tag{3.9}$$

Therefore, if $k_L r \ll 1 \ll k_d r$,

$$B_{\alpha\alpha}(\mathbf{r}) \approx B_K\epsilon^{2/3}r^{2/3}. \tag{3.10}$$

These results are precisely what would be derived by dimensional analysis with the assumptions that the probability distribution of $\mathbf{v}(\mathbf{x}+\mathbf{r}, t) - \mathbf{v}(\mathbf{x}, t)$ depends only on ϵ, r, and ν for $k_L r \ll 1$ and that the dependence on ν is

negligible if $k_d r \gg 1$. Indeed, this is the formulation originally used by Kolmogorov. This version of the theory is formulated using velocity differences since it is necessary to remove the kinematical effect of convection of small eddies by large eddies before applying the statistical hypothesis of dynamical independence of large and small eddies. For example, naive dimensional analysis applied directly to $R_{\alpha\alpha}(\mathbf{r})$ indicates that it has the form (3.10)—proper dimensional analysis based on velocity differences not absolute velocity indicates that $B_{\alpha\alpha}(\mathbf{r}) = 2(R_{\alpha\alpha}(0) - R_{\alpha\alpha}(\mathbf{r}))$ is given by (3.10).

Since the skewness factor $S(r)$ and the flatness factor $F(r)$ depend only on velocity differences, dimensional analysis applies, showing that

$$S(r) = \mathscr{S}(k_d r), \quad F(r) = \mathscr{F}(k_d r) \tag{3.11}$$

for $k_L r \ll 1$. Here $\mathscr{S}(x)$ and $\mathscr{F}(x)$ are universal functions. In particular, $S_0 = \mathscr{S}(0)$ and $F_0 = \mathscr{F}(0)$ are universal constants independent of Reynolds number. For $k_L r \ll 1 \ll k_d r$, $S(r) \approx \mathscr{S}(\infty)$ is uniquely expressible in terms of C_K. The result, due to Kolmogorov (1941c), is

$$S(r) \approx 0{\cdot}53\, C_K^{-3/2} \quad (k_L r \ll 1 \ll k_d r) \tag{3.12}$$

which is derived as follows. Let $\mathbf{r} = (r, 0, 0)$. Then, since $B_{11}(\mathbf{r}) = 2v_{\text{rms}}^2[1 - f(r)]$, $\propto r^{2/3}$, it follows from (2.9) that

$$g(r) = \tfrac{1}{4} f(r) - \tfrac{1}{3} \quad (k_L r \ll 1 \ll k_d r). \tag{3.13}$$

But since $B_{\alpha\alpha}(\mathbf{r}) = 2v_{\text{rms}}^2[3 - f(r) - 2g(r)]$, (3.10) implies that

$$\langle [v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^2 \rangle = B_{11}(\mathbf{r}), \quad = \tfrac{3}{11} B_K \epsilon^{2/3} r^{2/3}. \tag{3.14}$$

Also, dimensional analysis shows that $\langle [v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^3 \rangle \propto \epsilon r$ for $k_L r \ll 1 \ll k_d r$. In fact, a dynamical argument not reproduced here but given by Landau and Lifschitz (1959, §33) shows that the latter constant of proportionality is exactly $-4/5$, so that

$$\langle [v_1(\mathbf{x} + \mathbf{r}) - v_1(\mathbf{x})]^3 \rangle \approx -\tfrac{4}{5} \epsilon r \quad (k_L r \ll 1 \ll k_d r). \tag{3.15}$$

Equations (3.9), (3.14) and (3.15) establish (3.12). It is important to emphasize that (3.12) does *not* apply to S_0—evidently, as r decreases from L_p, $S(r)$ first reaches the plateau (3.12) with S_0 approached only when $k_d r \ll 1$.

The consistency of the equilibrium range with the dynamical equations for cumulants developed in Sec. 2.4 has been examined by Orszag and Kruskal (1968). It was found that the inertial range $k^{-5/3}$ spectrum is consistent with the cumulant equations, but that its uniqueness is by no means guaranteed. In the far dissipation range, it was also shown that, if the energy spectrum is of the form

$$E(k) \sim C k^{\beta} e^{-\gamma k^{\alpha}} \quad (k \gg k_d) \tag{3.16}$$

it is necessary that $\alpha \leq 1$. The reason is simply that if $\alpha > 1$, wavevector interactions involving wavevectors of order $\frac{1}{2}\mathbf{k}$ ($\frac{1}{2}\mathbf{k} + \frac{1}{2}\mathbf{k} \to \mathbf{k}$) would drive too large a response in $\mathbf{u}(\mathbf{k})$; in fact, $|\mathbf{u}(\frac{1}{2}\mathbf{k})\mathbf{u}(\frac{1}{2}\mathbf{k})| \gg |\mathbf{u}(\mathbf{k})|$ if $\alpha > 1$. A spectrum of the form (3.16) with $\alpha = 1$, $\beta = 3$ was shown to be consistent with the hierarchy of cumulant equations.

In closing this section, it is fitting to point out the distinctive nature of the equilibrium range. The equilibrium range is not in "equilibrium" in the usual statistical mechanical sense, since there need be no detailed balance. On the other hand, the equilibrium range continually dissipates energy with a stationary state maintained only because characteristic time scales are short compared to large-scale evolutionary times.

3.2 Comparison with Experimeut

The basic assumptions of the Kolmogorov theory are sufficiently general that there is no *a priori* reason why the theory should not apply equally to all types of high-Reynolds-number turbulence. The small-scale structure of such diverse flows as turbulent wakes, jets, and boundary layers should be governed by the theory of §3.1. For this reason, Kolmogorov's theory is sometimes called the universal equilibrium theory. This is a misnomer. Even if the theory is basically sound, which it may not be (cf. §3.3), it cannot justifiably be applied to hydromagnetic turbulence (Kraichnan, 1965a), large-Mach-number compressible turbulence, and a host of other physically interesting turbulent flows which involve high-frequency modifications of small-eddy dynamics (Orszag and Kruskal, 1968).

Over the past decade, there have been many experimental verifications of the $k^{-5/3}$ law (3.3). Recent references that provide fairly exhaustive citations of earlier work include Gibson, Stegun, and Williams (1970), Van Atta and Park (1972), and Wyngaard and Pao (1972). The inertial-range power-law exponent $-5/3$ is accurately verified with measurements ruling out a significantly flatter inertial-range spectrum but perhaps consistent with a slightly steeper spectrum than $k^{-5/3}$. The experimental results are consistent with $C_K \approx 1{\cdot}5$ or perhaps somewhat larger, but there are indications that the value of C_K may depend on the conditions under which the turbulence develops.

The predictions of the Kolmogorov theory for higher-order statistics are not consistent with experiment. According to (3.11), the skewness factor S_0 and the flatness factor F_0 should be universal constants. Moderate Reynolds number grid turbulence experiments give $S_0 \approx 0{\cdot}4$ and $F_0 \approx 4$ at $R_\lambda \approx 50$. However, more recent measurements show an unmistakable increase of F_0 with R_λ. Kuo and Corrsin (1971) present laboratory data that show F_0 increasing to nearly 10 at $R_\lambda \approx 1000$. This is about the limit for laboratory flows, but geophysical flows can be studied to nearly ten times that Reynolds

number. Wyngaard and Tennekes (1970) and Gibson and Masiello (1972) present data from an atmospheric boundary layer giving values of S_0 up to about 1.0 and F_0 up to about 40. The data are consistent with $F_0 \propto R_\lambda^{0.6}$ (Kuo and Corrsin, 1971). The flatness factors show no sign of approaching constants even at $R_\lambda \approx 10^4$, which is generally believed high enough for Kolmogorov's theory to apply. It should be noted that there are a number of technical difficulties in making reliable measurements of high-order statistical properties at high R_λ, as has been recently emphasized by Tennekes and Wyngaard (1972) and Van Atta (1974).

3.3 Intermittency and Models of Turbulent Motion

The Reynolds number dependencies of F_0 and S_0 are not consistent with the theory of §3.1. As noted in §2.8, a statistical quantity with a large flatness has a larger probability of taking on either very small or very large values than a Gaussian variable. The fact that the flatness of $\partial v_1/\partial x_1$ is large implies that high Reynolds number turbulence is intermittent with regions of high turbulence activity separated by regions of very low turbulence.

A related defect in the theory of Sec. 3.1 was pointed out by Landau, who noted that Kolmogorov's theory does not take proper account of spatial fluctuations of local dissipation rates (see the footnote on p. 126 of Landau and Lifschitz, 1959). The average rate of energy dissipation ϵ is the space (or ensemble) average of the fluctuating quantity

$$\hat{\epsilon} = \frac{1}{2}\nu\left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)^2,$$

where summation on α, β is implied. With increase of Reynolds number, Landau argued that the variance of $\hat{\epsilon}$ should increase without limit. On the other hand, the equilibrium theory of § 3.1 gives $\langle(\hat{\epsilon} - \epsilon)^2\rangle = 0(\epsilon^2)$, independent of Reynolds number. There has been a spate of work recently on estimating the Reynolds number dependence of the variance of $\hat{\epsilon}$ and the consequent effects on the Kolmogorov theory.

Obukhov (1962) determined the effect of the variation of $\hat{\epsilon}$ on $B_{\alpha\alpha}(\mathrm{r})$, defined by (3.5), by assuming that (3.10) remains valid if ϵ is replaced by the *spatial average of $\hat{\epsilon}$ throughout a region of linear dimension r* about the point $\mathbf{x}$. Call this spatial average $\epsilon_r(\mathbf{x}, t)$. Since the averaging volume in its definition is finite, $\epsilon_r(\mathbf{x}, t)$ is a random function of $\mathbf{x}$, t, but $\langle\epsilon_r(\mathbf{x}, t)\rangle = \epsilon$. The local structure function obtained in this way can be averaged over the fluctuations of ϵ_r to obtain an "average" structure function

$$B_{\alpha\alpha}(\mathrm{r}, t) = B_K\langle\epsilon_r^{2/3}\rangle r^{2/3}, \quad = B_K b(r)\epsilon^{2/3}r^{2/3}, \tag{3.17}$$

where

$$b(r) = \langle \epsilon_r^{2/3} \rangle / \epsilon^{2/3}.$$

The function $b(r)$, not determined without further hypotheses, gives the effect of the dissipation fluctuations on the structure function. The result so obtained neglects the effect of fluctuations of $k_d(\epsilon_r)$, in the sense that Obukhov assumed that (3.10) applies for $\langle k_d(\epsilon_r) \rangle r \gg 1$, while it may seem that the local criterion $k_d(\epsilon_r) r \gg 1$ is more appropriate. However, more fundamentally, if (3.10) is not generally valid, then there is no evident reason why (3.10) should hold in Obukhov's modified form. In fact, there is no evident reason to expect a direct connection between $\epsilon_r(\mathbf{x}, t)$ and inertial range quantities. While the dissipation rate ϵ must equal the rate of energy cascade through the inertial range, there is no such connection between $\epsilon_r(\mathbf{x}, t)$ and the local inertial-range cascade rate.

Kolmogorov (1962) assumed that the distribution of the random variable $\epsilon_r(\mathbf{x}, t)$ is log-normal, i.e., the distribution of $\ln \epsilon_r$ is Gaussian, and that the variance of $\ln \epsilon_r$ is

$$\sigma_r^2 = \langle [\ln \epsilon_r(\mathbf{x}, t) - m_r(\mathbf{x}, t)]^2 \rangle = \begin{cases} A(\mathbf{x}, t) + 9h \ln(L_p/r), & k_d r \gg 1 \\ A'(\mathbf{x}, t) + 9h \ln(k_d L_p), & k_d r \ll 1, \end{cases} \tag{3.18}$$

where $m_r(\mathbf{x}, t) = \langle \ln \epsilon_r(\mathbf{x}, t) \rangle$, h is a universal constant (with the factor 9 inserted for convenience), and $A(\mathbf{x}, t)$, $A'(\mathbf{x}, t)$ depend on the large-scale structure of the flow. The log-normal distribution of ϵ_r requires that the probability density that $\epsilon_r(\mathbf{x}, t) = a$ be

$$P_r(a) = (2\pi\sigma_r^2 a^2)^{-1/2} \exp[-(\ln a - m_r)^2 / 2\sigma_r^2]. \tag{3.19}$$

These assumptions of Kolmogorov are derived by Yaglom (1966) and Gurvich and Yaglom (1967) from a hypothesis of self-similarity of the spatial structure of $\hat{\epsilon}(\mathbf{x}, t)$ at large Reynolds number. An implicit assumption of independence between scales is also used by Yaglom *et al.*

The assumptions of log-normality and (3.18) suffice to determine the function $b(r)$. The distribution (3.19) implies that

$$\langle \epsilon_r^p \rangle = \exp(pm_r + \tfrac{1}{2}p^2\sigma_r^2) \tag{3.20}$$

for all p. It follows that

$$\epsilon = \exp(m_r + \tfrac{1}{2}\sigma_r^2)$$

$$\langle \epsilon_r^{2/3} \rangle = \exp(\tfrac{2}{3}m_r + \tfrac{2}{9}\sigma_r^2), \quad \propto \epsilon^{2/3}(r/L_p)^h \, (k_d r \gg 1).$$

Consequently,

$$B_{\alpha\alpha}(\mathbf{r}) \propto \epsilon^{2/3} r^{2/3} (r/L_p)^h \, (k_L r \ll 1 \ll k_d r)$$

so that (3.6) gives, in the inertial range,

$$E(k) \propto \epsilon^{2/3} k^{-5/3} (kL_p)^{-h}. \tag{3.21}$$

The log-normal distribution of ϵ_r has not been subject to direct experimental test. However, the probability distribution and spectrum of fluctuations of $D(\mathbf{x}) = (\partial v_1/\partial x_1)^2$ have been measured. The latter spectrum is the Fourier transform of

$$\langle (D(\mathbf{x} + \mathbf{r}) - \langle D \rangle)(D(\mathbf{x}) - \langle D \rangle) \rangle.$$

It has found that the probability distribution of $D(\mathbf{x})$ is approximately log-normal (cf. Stewart, Wilson and Burling, 1970; Gibson and Masiello, 1972; Wyngaard and Pao, 1972, and papers cited therein). The experimental value of $9h$ in (3.18) is determined to be approximately 0·4, so that the induced modification in the $k^{-5/3}$ spectral law is exceedingly small.

It follows from the log-normality of $D(\mathbf{x})$ and (3.20) that

$$F_0 = \exp(\sigma^2)$$

$$S_0 = \exp(\tfrac{1}{8}\sigma^2),$$

where σ^2 is the variance of $\ln D$ and the expression for S_0 requires the (incorrect) *assumption* that $-(\partial v_1/\partial x_1)^3 > 0$ *everywhere*. With this latter assumption, it follows that $S_0 \propto F_0^{3/8}$, which is not inconsistent with experiments (Wyngaard and Tennekes, 1970).

The correctness or not of log-normal models of small-scale fluctuations have recently been given close scrutiny (Novikov, 1971; Kraichnan, 1974a; Mandelbrot, 1974). Kraichnan considers the basic self-consistency of both the original Kolmogorov theory of §3.1 and the log-normal modifications described here. He concludes that the intermittent nature of turbulence is exceedingly subtle, does not reflect general statistical mechanical principles, but rather depends on the detailed structure of the Navier–Stokes equations. Novikov, Kraichnan, and Mandelbrot argue that log-normality cannot be exact, but nevertheless it can be a good model of the phenomenon. Nearly all the experimental results show substantial deviations from log-normality at very large and very small amplitudes.

Additional studies of intermittency of small scales have been based on physical models of turbulent motion. The progenitor of these models is Townsend's (1951) model of turbulence as a random collection of vortex sheets and lines. Several models have been suggested (Corrsin, 1962; Tennekes, 1968; Saffman, 1968, 1970; Kuo and Corrsin, 1972; Kraichnan, 1974a), but unfortunately none is fully satisfactory. Saffman observes that the energy-containing eddies produce a local straining field whose magnitude α is of order v_{rms}/L_p. If the convergence associated with these motions tends to produce vortex sheets and tubes, then the thickness of the sheets or the radius of the tubes is easily shown to be of order $(\nu/\alpha)^{1/2} = (A/15)^{1/2}\lambda$,

according to (2.13) and (2.49). This gives the physical interpretation of the Taylor microscale as the typical thickness of shear layers formed in the flow.

Saffman next assumes that the characteristic velocity within the sheets and tubes is of order v_{rms}. Also, he observes that the model structure thus far obtained can explain neither the empirical result that ϵ is independent of Reynolds number nor the physical meaning of the Kolmogorov length scale $1/k_d$. This leads him to suggest that the curved vortex sheets of thickness λ are themselves unstable to a kind of Taylor–Görtler instability (Taylor, 1923; Görtler, 1940). This instability has the property that a secondary motion is formed with a cellular structure of typical size the thickness of the original vortex sheet. The boundary layers between these Taylor–Görtler cells have a thickness of order $(\nu\lambda/v_{\text{rms}})^{1/2} \propto 1/k_d$, since the strain rate within the sheet is of order v_{rms}/λ. On the basis of this model of $1/k_d$, Saffman argues that the dissipation within the boundary layers between Taylor–Görtler cells is of the right magnitude to explain ϵ independent of Reynolds number. An interesting prediction of the model is that the flatness factor F_0 is proportional to R_λ, a prediction not in complete disagreement with available experiments (which are better fit by $R_\lambda^{0.6}$ as noted in §3.2). However, the model has some shortcomings that await clarification. First, the $k^{-5/3}$ inertial-range law is not obtained, nor is the equivalent constancy of the skewness factor $S(r)$ in the inertial range secured. Second, it is by no means obvious, at least to the author, that Taylor–Görtler instability need occur in the absence of solid boundaries. Third, if the latter instability does occur, it must be explained why the Taylor–Görtler cellular motion is not itself unstable. Other criticisms are given by Kraichnan (1974).

3.4 Two-Dimensional Homogeneous Turbulence

Motion in two dimensions has the property that the vorticity of each fluid element is unchanged, except by viscous diffusion. In fact, since $\boldsymbol{\omega} = (0, 0, \omega)$ for two-dimensional motion in the x_1–x_2 plane, the vortex stretching term on the right-hand side of (2.57) vanishes, so that

$$\frac{\partial}{\partial t}\omega(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t)\cdot\nabla\omega(\mathbf{x}, t) = \nu\nabla^2\omega(\mathbf{x}, t).$$

it follows that there exist an infinite number of isolating inviscid constants of motion

$$\int [\omega(\mathbf{x}, t)]^n \, d\mathbf{x} \tag{3.22}$$

in addition to the energy integral. The presence of these additional constraints has fundamental effects on the nature of two-dimensional turbulence. For one thing, inertial forces alone cannot produce an equilibrium

range with a universal statistical distribution independent of externally applied boundary conditions and random forces. The presence of infinitely many constrains on inviscid motion clouds the validity of ergodicity arguments for the approach to an asymptotic statistical state.† Furthermore, even accepting the existence of an asymptotic state, the properties of two-dimensional turbulence differ fundamentally from three-dimensional turbulence.

In isotropic two-dimensional turbulence, the energy spectrum $E(k)$ is defined in analogy to (2.27) by

$$S_{\alpha\beta}(\mathbf{k}) = \frac{E(k)}{\pi k} P_{\alpha\beta}(\mathbf{k}), \tag{3.23}$$

where α, β now take the values 1, 2. It follows that

$$\frac{1}{2}\langle v_\alpha(\mathbf{x}+\mathbf{r}) v_\alpha(\mathbf{x})\rangle = \int_0^\infty E(k) J_0(kr)\, dk \tag{3.24}$$

$$\frac{1}{2}\langle \omega(\mathbf{x}+\mathbf{r})\omega(\mathbf{x})\rangle = \int_0^\infty k^2 E(k) J_0(kr)\, dk, \tag{3.25}$$

where J_0 is the Bessel function of zero order.

In freely decaying turbulence, the equations of motion imply

$$\frac{d}{dt}\int_0^\infty E(k,t)\, dk = -2\nu \int_0^\infty k^2 E(k,t)\, dk \left[\equiv -\epsilon(t)\right] \tag{3.26}$$

$$\frac{d}{dt}\int_0^\infty k^2 E(k,t)\, dk = -2\nu \int_0^\infty k^4 E(k,t)\, dk \left[\equiv -\eta(t)\right], \tag{3.27}$$

where $\eta(t)$ is the rate of dissipation of enstrophy $\Omega = \frac{1}{2}\langle\omega^2\rangle$. Since $\eta \geqslant 0$, enstrophy is bounded by its initial value. Therefore, $\epsilon \to 0$ as $\nu \to 0$, in marked contrast to three dimensions where (2.49) holds with $A = 0(1)$.

Just as we inferred that the principal dynamical effect of small scales in three dimensions is to ensure that $\epsilon = 0(1)$ as $R \to \infty$, so we may hypothesize that in two dimensions the principal effect of small scales should be to ensure $\eta = 0(1)$ as $R \to \infty$. While vorticity growth is bounded in two dimensions so that $\epsilon \approx 0(1/R)$ as $R \to \infty$, vorticity gradients can grow and thereby maintain $\eta = 0(1)$ as $R \to \infty$. Unfortunately, it is known that $\eta = 0(1)$ is untenable; it would require that the Euler equations generate infinite mean-

†In inviscid three-dimensional flow, the circulation in any circuit composed wholly of fluid elements is conserved. This gives an infinity of integrals in addition to the energy. However, in contrast to the two-dimensional vorticity integrals (3.22), these circulation integrals should not be of consequence to the statistical mechanics of turbulence since they are probably not isolating. A closed curve consisting of fluid elements twists, turns, and tangles in a complicated way, so that the circulation constraint does not effectively "isolate" realizations.

square vorticity gradients after a finite time (cf. § 2.10). However, it is known that, in two dimensions, mean-square vorticity gradients can grow at most exponentially fast, so that at any fixed time $\eta = 0(1/R)$ as $R \to \infty$. Nevertheless, the heuristic argument is still tenable that small scales are generated in response to the need to dissipate enstrophy at the rate η.

Using the argument that led to (3.1), it follows that for small scales, $E(k)$ is a function of η, k, and ν so that [Kraichnan, 1967; Leith, 1968; Batchelor, 1969]

$$E(k) = \eta^{2/3} k^{-3}\, G(k/k_\eta) \tag{3.28}$$

$$k_\eta = (\eta/\nu^3)^{1/6} \tag{3.29}$$

since $[\eta] = s^{-3}$. For $^-k \ll k_\eta$, it follows that

$$E(k) \approx C'\eta^{2/3} k^{-3}, \tag{3.30}$$

where $C' = G(0)$.

In this modified inertial range, the analog of the eddy-circulation time (3.4) is

$$\tau_i'(k) = A'\eta^{-1/3}. \tag{3.31}$$

Since $\tau_i'(k)$ is independent of k in this enstrophy-transfer inertial range, high wavenumbers do not relax in a characteristic time much shorter than the characteristic time of energy-containing eddies. Therefore, the ideas leading to (3.30), in particular, the notion of an equilibrium range, are not strictly correct and (3.30) may be in error. Alternatively, the spectral law (3.30) is inconsistent because it implies that the enstrophy-cascade is not local. The enstrophy spectrum corresponding to (3.30) is proportional to k^{-1} so that each octave contributes equally to the mean-square shear. Since shear on scales larger than the size of an eddy can distort the eddy, in contrast to shear-free convection which gives no distortion, it follows that the large mean-square shear due to small k gives nonlocal effects of large scales on the transfer process. However, it can be shown that logarithmic corrections to the k^{-3} law suffice to maintain self-consistency. Kraichnan (1971b) argues that $E(k) = C'\eta^{2/3} k^{-3} (\ln k/k_1)^{-1/3}$ for some k_1.

There are several length scales that are associated with the enstrophy-transfer inertial-range (3.30) (Lilly 1971). The length L_2 defined by

$$\eta = v_{\text{rms}}^3 / L_2^3 \tag{3.32}$$

is analogous to the integral scale of three-dimensional turbulence, because of the formal similarity with (2.49). It may be shown that, aside from logarithmic factors, L_2 is proportional to the Taylor microscale [cf. (2.13)]. This follows since (3.30) may be used to estimate Ω, with the result proportional to $\eta^{2/3}$ except for a logarithmic factor dependent on the cutoffs of

the enstrophy-transfer range. It is a simple exercise to show that the Taylor microscale is a nondecreasing function of time in two dimensions. Another length scale is given by $2\pi/k_\eta$, which may be interpreted analogously to $2\pi/k_d$ in three dimensions as a dissipation length. A second Taylor microscale may be defined by (2.13), replacing ϵ by η and v_{rms}^2 by Ω, i.e. $\lambda_2 = (v\Omega/\eta)^{1/2}$. However, since enstrophy equals $v^{2/3}$ to within a logarithmic factor, λ_2 is proportional to $2\pi/k_\eta$.

Batchelor (1969) infers the manner of decay of enstrophy from similarity principles. Since v_{rms} is nearly constant during decay of high-Reynolds-number two-dimensional turbulence, and *assuming* that $E(k, t)$ depends only on the parameters v_{rms}, t, k, so that the spectrum is shape preserving, it follows that

$$E(k, t) = v_{rms}^2 t\, g(v_{rms} kt), \tag{3.33}$$

where g is some undetermined function. Therefore, enstrophy decays as

$$\Omega(t) = \int_0^\infty k^2 E(k, t), \quad = \int_0^\infty g(x)\, dx\, t^{-2} \tag{3.34}$$

at large Reynolds number. It follows that

$$\eta(t) = 2\int_0^\infty g(x)\, dx\, t^{-3}, \quad \epsilon(t) = 2v\int_0^\infty g(x)\, dx\, t^{-2}. \tag{3.35}$$

As $t \to \infty$, v_{rms} approaches a nonzero limit while enstrophy decays to zero. This requires that energy be concentrated in small wavenumber, which is consistent with Onsager's (1949) picture of the clumping together and coalescence of similarly signed vortices. In Onsager's picture, an initial motion evolves into a number of strong, isolated vortices. Whatever high-wavenumber excitation survives is produced in the boundary layers between these vortices and, therefore, should be intermittent.

Saffman (1971) has extended this argument to model the inertial range of two-dimensional turbulence as originating in the small scales in the boundary layers between nearly-constant vorticity eddies. This model leads to a k^{-4} inertial range spectrum (since $\omega(x)$ is discontinuous) and an enstrophy dissipation rate $\eta \propto v^{2/3}\Omega^{11/6}$. This model is not inconsistent with presently available numerical and experimental results, but Kraichnan (1974b) presents cogent arguments for believing it to be oversimplified. Kraichnan (1974b) also argues that intermittency does not affect the log-corrected k^{-3} inertial-range spectrum.

Numerical simulations of two-dimensional turbulence have been quite useful for testing turbulence theories and overall energy dynamics. However, as shown by Herring, Orszag, Kraichnan, and Fox (1974), the currently available simulations do not give trustworthy results at Reynolds numbers large enough that there be an inertial range sufficiently extensive to test power law exponents.

Two-dimensional turbulence relates to the planetary-scale dynamics of the atmosphere, since the atmosphere is a thin, stratified fluid layer in which large-scale motions are largely two-dimensional. A more relevant theory of planetary-scale motions is given by the theory of quasi-geostrophic turbulence (Charney, 1971) in which a k^{-3} inertial range is also obtained. Observed spectra [Julian, Washington, Hembree, and Ridley (1970) and references cited therein] are not inconsistent with the k^{-3} spectral law, though the data reduction methods used by Julian et al may have influenced their results.

Shur (1962) and Lumley (1964) deduce a k^{-3} spectrum for a physical situation different from the strictly two-dimensional flows considered above. Shur and Lumley consider turbulence in a stably stratified fluid, i.e. a fluid in a gravitational field with the density of the fluid decreasing upwards, assuming the Boussinesq approximation. The stable stratification has an important effect on the vertical motion of eddies whenever the eddy-circulation time (3.4) is longer than the internal oscillation time of the stratified fluid, viz. the inverse Brunt–Vaisala frequency $N = [-g\partial(\ln \rho)/\partial x_3]^{1/2}$, where g is the gravitational acceleration acting in the negative x_3-direction and ρ is the potential density (Phillips, 1966, 2.2). This condition requires $k \ll k_b$, where $k_b = A_K^{3/2} N^{3/2} \epsilon^{-1/2}$. The same wavenumber criterion for effects of stratification on vertical motions follows from the condition that the kinetic energy of an eddy be small compared to the gravitational potential energy necessary to flip the eddy over. In the so-called buoyancy subrange, $k_L \ll k \ll k_b$, the vertical-velocity spectrum should depend only on N and k, so that dimensional analysis gives the buoyancy-subrange spectrum proportional to $N^2 k^{-3}$. This result is identical to (3.30) with the time $\eta^{-1/3}$ replaced by N^{-1}. For $k \gg k_b$, vertical motions are not constrained by stratification and the $k^{-5/3}$ law (3.3) results for the vertical-velocity spectrum.

IV Introductiou to Analytical Theories of Turbulence

4.1 Introduction

Ideally, the objective of analytical turbulence theory is the exact calculation of all statistical properties of turbulence. At the present time this objective has not been met. Several *formally* exact theories have been proposed, but these have not proved workable. For example, there has been much interest in the application of functional analysis to turbulence. Hopf (1952) devised an elegant single functional-differential equation that is formally equivalent to the complete cumulant hierarchy introduced in § 2.4, Tatarskii (1962) and others have solved Hopf's equation in terms of functional integrals—a lucid exposition is given in the monograph by Monin and Yaglom (1967). However, at present, these formal functional integrals remain intractable

to both exact and approximate analyis. The results obtained to date are just a formal restatement of the Navier–Stokes equations. It even seems that the extraction of useful information from the functional integrals is more difficult than the brute-force study of turbulence by computer simulation of solutions of the Navier–Stokes equations. The point is that while finite-difference methods for the solution of partial differential equations, such as (1.21), are well founded, known approximate methods for the solution of Hopf's equation are not. The problem of approximating the solutions to Hopf's equation does not involve only numerical analysis—essential physics must be understood and accounted for.

A more realistic objective of analytical turbulence theory is the approximate calculation of some statistical properties of turbulence. This goal is consistent with the motive for introducing statistical averages in the first place, viz., to be able to say something about complicated, otherwise unwieldy, solutions of the Navier–Stokes equations. For one thing, it is necessary to show that statistical information is more accessible from the equations of an analytical turbulence theory than from computer simulation of turbulence. In this regard, analytical turbulence theory offers two principal advantages over direct computer simulation. First, the statistical averages employed by an analytical theory exhibit symmetries, such as isotropy, which need not be evident in individual realizations. Second, statistically averaged quantities are smooth functions of their arguments, and do not exhibit the fine-scale intricacy of individual realizations. A comparison of the relative difficulty of solution of the equations of turbulence theory versus solution of the Navier–Stokes equations is given in §§4.5, 5.1, and 6.6.

In summary, we require that a theory be workable, in the sense that approximate results can be obtained from it, at least in principal. For example, the complete unclosed hierarchy of §2.4 is formally exact, but it is not clearly workable, at least until a physically plausible prescription is given for resolving the closure problem.

4.2 Models for Energy Transfer

A simple theory of high-Reynolds-number turbulence is obtained by resolving the closure difficulty at the level of (2.34) with an expression for $T(k)$ in terms of $E(k)$. A number of such relations have been proposed, notably by Obukhov (1941), Heisenberg (1948), Kovasznay (1948), von Kármán (1948), Ellison (1962), Kraichnan and Spiegel (1962), and Leith (1967). These models for energy transfer are not strictly deductive, as they all involve one or more undetermined constants or functions. The spirit of these transfer approximations is exemplified by the theories of Heisenberg and Leith, described below.

The basis of Heisenberg's theory is the assumption that the small eddies play a similar role in nonlinear transfer to that played by molecules in viscous dissipation. This assumption is plausible provided the eddies responsible for the eddy-viscous dissipation have much smaller size and time scale than the eddies which are to be dissipated. The rate at which eddies of wavenumber less than K lose energy by viscous dissipation is $2\nu \int_0^K k^2 E(k)\,dk$. By analogy, Heisenberg proposed that the energy balance equation (2.34) be written

$$-\frac{\partial}{\partial t}\int_0^K E(k, t)\,dk = \left[\nu + \nu_e(K)\right]\int_0^K 2k^2 E(k, t)\,dk, \tag{4.1}$$

where $\nu_e(K)$ is an "eddy viscosity" coefficient used to model transfer to small eddies. Heisenberg observed that $\nu_e(K)$ should depend only on the structure of small eddies, and so he assumed that $\nu_e(K)$ depends only on $E(k, t)$ for $k > K$. Since $[\nu_e(K)] = \mathrm{cm^2\,s^{-1}}$ and $[E(k)] = \mathrm{cm^3 s^{-2}}$, Heisenberg suggested the dimensionally correct expression

$$\nu_e(K) = \gamma_H \int_K^\infty \sqrt{E(k, t)/k^3}\,dk, \tag{4.2}$$

where γ_H is an undertermined constant. By differentiation of (4.1) with respect to K, it follows that

$$\frac{\partial E(K, t)}{\partial t} = -2\left[\nu + \gamma_H \int_K^\infty \sqrt{E(k, t)/k^3}\,dk\right] K^2 E(K, t) + \tag{4.3}$$

$$+ \gamma_H \sqrt{E(K, t)/K^3}\int_0^K 2k^2 E(k, t)\,dk.$$

Detailed numerical solutions to (4.3) are presented in the review article by Lin and Reid (1963).

For K in the equilibrium range, (4.1) with (4.2) takes the form

$$\epsilon = \left[\nu + \gamma_H \int_K^\infty \sqrt{E(k)/k^3}\,dk\right]\int_0^K 2k^2 E(k)\,dk, \tag{4.4}$$

where ϵ is the rate of energy dissipation. Equation (4.4) is most easily solved by using $\nu_e(K)$ instead of $E(k)$ as the dependent variable. The result is (Bass, 1949)

$$E(k) = \left(\frac{8}{9\gamma_H}\right)^{2/3}\left[1 + \frac{8\nu^3 k^4}{3\gamma_H^2 \epsilon}\right]^{-4/3} k^{-5/3}. \tag{4.5}$$

In the inertial range, $k \ll (\gamma_H^2 \epsilon/\nu^3)^{1/4}$ so that the $k^{-5/3}$ law (3.3) is recovered with $C_K = (8/9\gamma_H)^{2/3}$. With $C_K \approx 1{\cdot}5$, it follows that $\gamma_H \approx 0{\cdot}5$.

In the far-dissipation-range, (4.5) reduces to

$$E(k) = \tfrac{1}{4}\gamma_H^2\epsilon^2\nu^{-4}k^{-7} \quad (k \gg (\gamma_H^2\epsilon/\nu^3)^{1/4}). \tag{4.6}$$

This k^{-7} law for the far-dissipation-range follows directly from (4.3) with the approximations

$$\frac{\partial E(K,t)}{\partial t} \ll 2\nu K^2 E(K,t), \quad \nu_e(K) \ll \nu, \quad \int_0^K 2k^2 E(k)\,dk \approx \epsilon/\nu.$$

Since $\int^\infty k^6 E(k)\,dk$ diverges for the spectrum (4.5), it follows that third- and higher-order derivatives of the velocity field do not exist a mean square. This prediction of Heisenberg's theory is contrary to the expectation that viscosity is sufficiently smoothing that all derivatives of the velicity exist in mean square. The origin of the unphysical power-law dissipation-range spectrum may be understood as follows. In the dissipation range, an alternative for $\nu_e(K)$, as plausible as Heisenberg's, is

$$\nu_e(K) = \int_K^\infty g(k/k_d)\sqrt{E(k,t)/k^3}\,dk, \tag{4.7}$$

where $g(0) = \gamma_H$, $g(x) \to 0$ as $x \to \infty$, and $k_d = (\epsilon(t)/\nu^3)^{1/4}$. In the inertial range, this modified $\nu_e(K)$ also yields the law (3.3). However, the far-dissipation-range spectrum (4.6) is modified by (4.7). With $g(x) = \gamma_H e^{-cx}$, $c > 0$, it follows that the far-dissipation-range spectrum is

$$E(k) = \tfrac{1}{4}\gamma_H^2\epsilon^2\nu^{-4}k^{-7}\exp(-2ck/k_d) \quad (k \gg (\gamma_H^2\epsilon/\nu^3)^{1/4}). \tag{4.8}$$

Therefore, the k^{-7} law is an artifact of the neglect of any destructive interference between molecular and eddy viscosity, qualitatively accounted for by (4.7).

Even aside from the prediction of the unphysical dissipation-range spectrum (4.6), Heisenberg's transfer expression is not entirely satisfactory. The dominant contribution to $\nu_e(K)$ with the spectrum (4.5) comes from $k = 0(K)$, but the notion of eddy viscosity is not strictly appropriate to describe energy transfer between wavenumbers of the same order of magnitude. For the transfer process to be describable in terms of a viscosity coefficient, it is necessary that the energy-receiving eddies reach dynamic equilibrium in a time scale much shorter than that of the eddies whose dynamics is under consideration.

In a sense, Leith's (1967) transfer approximation is complementary to Heisenberg's. Leith bases his theory on the assumption that energy transfer is a diffusion process in k-space. For isotropic turbulence, Leith writes

$$T(k,t) = -\partial\epsilon(k,t)/\partial k, \tag{4.8}$$

which defines $\epsilon(k, t)$ as the energy flux across a sphere of radius k. Leith

makes the diffusion approximation

$$\epsilon(k, t) = -\mathrm{d}(k)\, \partial e(k, t)/\partial k, \tag{4.9}$$

where $\mathrm{d}(k) = \gamma_L k^{13/2}$, $e(k, t) = [E(k, t)]^{3/2} k^3$, and $\gamma_L > 0$. These forms for $d(k)$, $e(k, t)$ are obtained by consideration of dimensional correctness and the possibility of inviscid equipartition solutions. The latter solutions, discussed in § 5.2, require that $T(k, t) = 0$ if $E(k) \propto k^2$.

In the equilibrium range, Leith's approximation gives a Kolmogorov inertial-range spectrum

$$E(k) = (2\epsilon/11\gamma_L)^{2/3} k^{-5/3}. \tag{4.10}$$

Consistency with a Kolmogorov constant $C_K \approx 1{\cdot}5$ requires $\gamma_L \approx 0{\cdot}10$. In the dissipation range, there is a cutoff wavenumber k_c'' such that $E(k) = 0$ for $k \gg k_c''$. It is found that $k_c'' \approx 6{\cdot}32\ (\gamma^2\epsilon/\nu^3)^{1/4}$. This cutoff is evidently a consequence of the improper local nature of (4.9).

All these energy transfer approximations suffer from the difficulty that they approximate $T(k, t)$ characterized by interactions among *triads* of Fourier modes [cf. (2.34)] by transfer expression involving interactions between *pairs* of wavenumbers [as in (4.3)]. A consequence of this artificiality is that interactions among distant wavenumbers are misrepresented. This is the origin of the difficulties that these theories have in portraying plausible dissipation-range dynamics. Effectively, the third wavenumber of a triad modulates the interaction between the other two. When $k \ll p$ then also $k \ll q$ since $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$, so that the interaction between $\mathbf{u}(\mathbf{k})$ and $\mathbf{u}(\mathbf{p})$ is modulated by the factor $\mathbf{u}(\mathbf{q})$ which is very small for q in the dissipation range. The absence of this destructive interference in Heisenberg's theory, where all pair interactions are given essentially the same weight, leads to the unphysical power-law dissipation-range spectrum. Another difficulty of theories of the type discussed here is generalizing them to inhomogenous turbulence problems.

Orszag and Raila (1973) compared the results of Heisenberg's and Leith's theories with numerical simulations of the decay of moderate Reynolds number homogenous turbulence. Leith's theory gave very unsatisfactory results, while Heisenberg's gave satisfactory energy spectra but unsatisfactory transfer spectra.

4.3 Expansion in Powers of the Reynolds Number

Formal solutions to the Navier–Stokes equations are obtained by expansion in powers of the Reynolds number. The formal solutions obtained in this way are used to give formal series expansions of various statistically averaged quantities. The latter expansions provide a convenient framework within which various analytical theories can be compared and the nature of

diverse approximations to high-Reynolds-number turbulence understood. In addition, Reynolds number expansions form the basis of the theory of low-Reynolds-number turbulence.

It is convenient to introduce dimensionless variables by measuring distances, velocities, and times in units of the initial values of L_p, v_{rms}, and L_p^2/ν, respectively. In these units, (2.31) takes the form

$$\left[\frac{\partial}{\partial t}+k^2\right] u_\alpha(\mathbf{k},t) = -\frac{1}{2}iRP_{\alpha\beta\gamma}(\mathbf{k})\int u_\beta(\mathbf{p},t)u_\gamma(\mathbf{k}-\mathbf{p},t)\mathrm{d}\mathbf{p}, \tag{4.11}$$

where $R = v_{\mathrm{rms}} L_p/\nu$. If $\mathbf{u}(\mathbf{k}, t)$ is formally expanded in the series

$$u_\alpha(\mathbf{k},t) = \sum_{n=0}^{\infty} R^n u_\alpha^{(n)}(\mathbf{k},t),\ u_\alpha^{(0)}(\mathbf{k},0) = u_\alpha^{(0)}(\mathbf{k},0) \tag{4.12}$$

and substituted in (4.11), equations for $u_\alpha^{(n)}(\mathbf{k}, t)$ are obtained by equating the coefficients of like powers of R.

By this procedure, it follows that

$$\left[\frac{\partial}{\partial t}+k^2\right] u_\alpha^{(0)}(\mathbf{k},t) = 0,$$

whence

$$u_\alpha^{(0)}(\mathbf{k},t) = \mathrm{e}^{-k^2t}u_\alpha(\mathbf{k},0). \tag{4.13}$$

Similarly,

$$u_\alpha^{(1)}(\mathbf{k},t) = -\frac{i}{2}P_{\alpha\beta\gamma}(\mathbf{k})\int_0^t \mathrm{d}s\ \mathrm{e}^{-k^2(t-s)}\int u_\beta^{(0)}(\mathbf{p},s)u_\gamma^{(0)}(\mathbf{k}-\mathbf{p},s)\,\mathrm{d}\mathbf{p}. \tag{4.14}$$

Proceeding in this way from order to order, it is possible to obtain $\mathbf{u}^{(n)}(\mathbf{k}, t)$ as a homogeneous functional of degree $n + 1$ in $\mathbf{u}^{(0)}$.

The formal solution (4.12) of the Navier–Stokes equations may be used to express any moment in terms of the initial velocity field. For example,

$$\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{p},t')\rangle = \sum_{n=0}^{\infty} R^n \sum_{m=0}^{n} \langle u_\alpha^{(m)}(\mathbf{k},t)u_\beta^{(n-m)}(\mathbf{p},t')\rangle. \tag{4.15}$$

The coefficient of R^n is a homogeneous functional of degree $n + 2$ in the initial velocity field. Therefore, $\langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t')\rangle$ for t or $t' > 0$ depends on moments of all orders of the initial distribution (Kraichnan, 1966a).

If the initial velocity distribution is multivariate Gaussian, then all moments may be expressed in terms of $\langle u_\alpha(\mathbf{k}, 0)u_\beta(\mathbf{p}, 0)\rangle$ so that the only nonvanishing cumulant of the initial distribution is $S(\mathbf{k}, 0)$, which is arbitrary. In particular, all odd-number cumulants (and moments) vanish initially. The resulting series for $\langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t')\rangle$ contains only even powers of R. The coefficient of R^{2n} consists of at least $9^n(2n)!/(2^{n+1}n!)$ terms each

of which is an n-fold integral over wavevectors of a quantity of functional power $n + 1$ in $S_{\alpha\beta}(\mathbf{k}, 0)$. This proliferation of terms is due to: (a) the algorithm for computing $2n$th order moments of a Gaussian distribution [factor $(2n)!/(2^n n!)$]; and (b) the number of possible branchings in (4.15) when $\mathbf{u}^{(m)}$ is expressed in terms of $\mathbf{u}^{(0)}$ [factor $(1/2)3^{2n}$].

This accounting of terms in the series for $S_{\alpha\beta}(\mathbf{k}; t, t')$ suggests that truncations of expansions in powers of R are not likely to be convenient approximations unless R is very small. Even aside from questions of convergence or divergence, the number and complexity of terms rises so rapidly with increasing order that calculation is not feasible. Kraichnan (1966a) points out that to calculate $S_{\alpha\beta}(\mathbf{k})$ through order R^{40} requires evaluation of considerably more than $40!/(2^{20}20!) \approx 3 \cdot 2 \times 10^{23}$ terms.

The convergence properties of (4.12), and by implication (4.15), remain in doubt. Consideration of various model dynamical systems suggests that, for a typical realization, the expansion (4.12) has at most a finite radius of convergence for $t > 0$. For example, the model equation

$$\frac{dx}{dt} = -R \operatorname{sgn}(x) x^2, \quad x(0) = x_0,$$

where $\operatorname{sgn}(x)$ is the sign of x, has the solution

$$x(t) = x_0/(1 + R|x_0|t).$$

Expansion of this $x(t)$ in powers of R converges only for $R < (|x_0|t)^{-1}$. This example also suggests that the radius of convergence of (4.12) approaches ∞ as $t \to 0$ and 0 as $t \to \infty$. These results for model dynamics do not really prove anything about (4.12), but they do show that infinite radius of convergence of (4.12) should not be assumed.

If, for typical realizations, (4.12) has only a finite radius of convergence, it may be concluded that (4.15) and similar series expansions of higher-order moments probably have zero radius of convergence. This conclusion is reached as follows. For each realization, define a Reynolds number $R' = v'_{\text{rms}} L'_p / \nu$, where v'_{rms}, L'_p are obtained as spatial averages for the particular realization. The Reynolds number $R = v_{\text{rms}} L_p / \nu$ entering (4.15) is a kind of average of R' over the ensemble. Now consider a realization with Reynolds number R' at $t = 0$ and for which the series (4.12) has radius of convergence $R_0(t)$ with $R_0(t) < \infty$ for $t > 0$. Note that $R_0(t)$ may depend on k. Since the initial ensemble is assumed Gaussian, the initial flow obtained by increasing all velocities by an arbitrary constant factor A is also a member of the ensemble with positive probability. However, the initial Reynolds number of this accelerated flow is AR', which for $A > R_0(t)/R'$ is beyond the radius of convergence of (4.12). It follows that for any $t > 0$ and any $R > 0$, a Gaussian ensemble has realizations for which (4.12) diverges. Therefore, (4.15) probably diverges for any $R > 0$ if t or $t' > 0$. This argument also shows that

the Reynolds number expansions of averages over any *unbounded* ensemble probably diverge for all R after the initial instant.

However, even if (4.15) is divergent, it is still likely that it is an asymptotic expansion as $R \to 0$. The limit $R \ll 1$ occurs in one situation of interest in turbulence theory, viz., the final period of decay of unforced turbulence (Reissner, 1938; Batchelor, 1953, §5.4). If $t = 0$ is chosen as an instant within this final period, $u_\alpha(\mathbf{k}, t)$ is approximated for $t \geq 0$ by $u_\alpha^{(0)}(\mathbf{k}, t)$. It follows that

$$S_{\alpha\beta}(\mathbf{k}, t) \approx e^{-2k^2 t} S_{\alpha\beta}(\mathbf{k}, 0) \tag{4.16}$$

for the principally excited wavevector range during the final period. The result (4.16) can only be valid when $k^2 t = 0(1)$. When $k^2 t \gg 1$, the series (4.15) is not uniformly valid. For it was argued in §3.1 that spectra in the far dissipation range steeper than e^{-k} could not be dynamically consistent. The same argument shows that nonlinear transfer maintains the excitation of wavenumbers with $k^2 t \gg 1$ above the level (4.16). Although the problem of the spectrum for $k^2 t \gg 1$ in the final period remains unsolved at this time, it has been shown by Benney and Lang (1970) that the series (4.15) remains uniformly valid as an expansion in R for all t provided $k^2 t = 0(1)$.

4.4 Quasi-Normal Theory: Single-Time Moments

An obvious way to close the hierarchy of Sec. 2.4, and so obtain deterministic equations for a finite number of cumulants, is to neglect the effect of all cumulants above a certain order. For example, neglect of triple moments, $\mathbf{T}(\mathbf{k}, \mathbf{p})$, in equation (2.32) for the energy-spectrum tensor gives

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] S_{\alpha\beta}(\mathbf{k}, t) = 0,$$

whose solution (4.16) includes only the effect of viscous dissipation of energy. Since triple moments are neglected, there is no nonlinear energy transfer in this approximation. However, it may be hoped that, by making similar closures at higher order, consistent theories are obtained which include nonlinear effects and whose solutions approximate the exact dynamics with increasing accuracy. Closures obtained by the procedure just outlined are called cumulant-discard approximations.

The second closure within this scheme involves neglecting fourth-order cumulants, $\mathbf{U}(\mathbf{k}, \mathbf{p}, \mathbf{q})$, in equation (2.33) for third-order cumulants. In this way, there results the closed set of equations

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] S_{\alpha\beta}(\mathbf{k}, t) = -\frac{i}{2} P_{\alpha\rho\sigma}(\mathbf{k}) \int T_{\beta\rho\sigma}(-\mathbf{k}, -\mathbf{p}, t)\, d\mathbf{p}$$

$$-\frac{i}{2}P_{\beta\rho\sigma}(-\mathbf{k})\int T_{\alpha\rho\sigma}(\mathbf{k},\mathbf{p},t)\,d\mathbf{p} \tag{4.17}$$

$$\left[\frac{\partial}{\partial t}+\nu(k^2+p^2+q^2)\right]T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},t) = -iP_{\alpha\rho\sigma}(\mathbf{k})S_{\beta\rho}(\mathbf{p},t)S_{\gamma\sigma}(\mathbf{q},t)$$

$$-iP_{\beta\rho\sigma}(\mathbf{p})S_{\alpha\rho}(\mathbf{k},t)S_{\gamma\sigma}(\mathbf{q},t)-iP_{\gamma\rho\sigma}(\mathbf{q})S_{\alpha\rho}(\mathbf{k},t)S_{\beta\sigma}(\mathbf{p},t), \tag{4.18}$$

where $\mathbf{q} = -\mathbf{k} - \mathbf{p}$ (Millionshtchikov, 1941; Proudman and Reid 1954; Tatsumi, 1957).

Equations (4.17) and (4.18) may be derived if it is assumed that the statistical distribution of Fourier modes is approximately Gaussian with zero mean. More specifically, it is only necessary to assume that fourth-order moments are related to second-order moments in the same way as for a Gaussian distribution. In this case, (4.17) and (4.18) follow directly by taking second and third moments of (2.31). In this "derivation", the assumption of a Gaussian distribution of Fourier amplitudes is applied inconsistently, since $T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},t)$ does not have the value appropriate to a Gaussian zero-mean distribution, which is zero. To derive (4.17), (4.18) consistently, it is necessary to justify an asymptotic ordering in which the neglected fourth-order cumulants remain small compared to the terms retained in (4.18). This has not been done, except at very low Reynolds number. The present application of the Gaussian hypothesis to fourth-order moments, but not third, is called the quasi-Gaussian or quasi-normal approximation. It is analogous to the random phase approximation of many-body physics (Bohm and Pines, 1953).

If the probability distribution at $t = 0$ is exactly Gaussian, then $T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},0) \equiv 0$. At later times, evolution according to (4.18) implies that T is no longer necessarily zero, so that the probability distribution typically does not remain Gaussian. In fact, the solution to (4.18) with $T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},0) = 0$ is

$$T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},t) = -i\int_0^t ds\, e^{-\nu(k^2+p^2+q^2)(t-s)}\left[P_{\alpha\rho\sigma}(\mathbf{k})S_{\beta\rho}(\mathbf{p},s)S_{\gamma\sigma}(\mathbf{q},s)\right.$$

$$\left.+P_{\beta\rho\sigma}(\mathbf{p})S_{\alpha\rho}(\mathbf{k},s)S_{\gamma\sigma}(\mathbf{q},s)+P_{\gamma\rho\sigma}(\mathbf{q})S_{\alpha\rho}(\mathbf{k},s)S_{\beta\sigma}(\mathbf{p},s)\right], \tag{4.19}$$

which may be substituted in (4.17) to give an integro-differential equation for $S_{\alpha\beta}(\mathbf{k},t)$.

If the turbulence is isotropic, the equations of the theory simplify when $\mathsf{S}(\mathbf{k},t)$ and $\mathsf{T}(\mathbf{k},\mathbf{p},t)$ are expressed in terms of the scalar functions $E(k,t)$, $\Phi(k,p,q,t)$, $\Psi(k,p,q,t)$ according to (2.27) and (2.30). In terms of these scalar functions, (4.17) becomes

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]E(k,t) = 4\pi k^4\,\mathrm{Im}\int d\mathbf{p}\left[-2a_1(k,p,q)\Phi(k,p,q,t)\right.$$

$$\left.+b_1(k,p,q)\Phi(p,q,k,t)+k^2c_1(k,p,q)\Psi(k,p,q,t)\right], \tag{4.20}$$

where

$$a_1(k, p, q) = -p_\alpha P_{\beta\gamma\delta}(\mathbf{k}) P_{\beta\gamma}(\mathbf{p}) P_{\alpha\delta}(\mathbf{q})/(2k^2) \tag{4.21}$$

$$b_1(k, p, q) = -p_\alpha P_{\alpha\beta\gamma}(\mathbf{k}) P_{\beta\delta}(\mathbf{p}) P_{\gamma\delta}(\mathbf{q})/(2k^2) \tag{4.22}$$

$$c_1(k, p, q) = -p_\alpha k_\beta k_\gamma P_{\alpha\rho\sigma}(\mathbf{k}) P_{\beta\rho}(\mathbf{p}) P_{\gamma\sigma}(\mathbf{q})/(2k^4) \tag{4.23}$$

and $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$ so that k, p, q are the legs of a triangle. The geometrical coefficients are discussed further in the next Section. Their complication is due directly to the pressure in (1.21) and the incompressibility constraint (1.22). It is important to note that (4.20) follows directly from (2.32) and isotropy, so that (4.20) is generally valid for isotropic turbulence independently of the quasi-normal approximation.

Similarly, (4.19) reduces to the two scalar equations

$$\Phi(k, p, q, t) = -i(4\pi kpq)^{-2} \int_0^t \mathrm{d}s\, \mathrm{e}^{-\nu(k^2+p^2+q^2)(t-s)} \cdot E(q, s)\left[k^2 E(p, s) - p^2 E(k, s)\right] \tag{4.24}$$

$$\Psi(k, p, q, t) = 0. \tag{4.25}$$

Finally, a single integro-differential equation for $E(k, t)$ is obtained by substitution of these results in (4.20):

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E(k, t) = \int \mathrm{d}\mathbf{p} \int_0^t \mathrm{d}s\, \mathrm{e}^{-\nu(k^2+p^2+q^2)(t-s)} \frac{k^2}{4\pi p^2 q^2} \cdot \{2k^2 a(k, p, q) E(p, s) E(q, s) - E(k, s)\left[p^2 b(k, p, q) E(q, s) + q^2 b(k, q, p) E(p, s)\right]\}, \tag{4.26}$$

where

$$a(k, p, q) = P_{\alpha\beta\gamma}(\mathbf{k}) P_{\beta\delta}(\mathbf{p}) P_{\gamma\rho}(\mathbf{q}) P_{\alpha\delta\rho}(\mathbf{k})/(4k^2) \tag{4.27}$$

$$b(k, p, q) = -P_{\alpha\beta\gamma}(\mathbf{k}) P_{\gamma\alpha\delta}(\mathbf{p}) P_{\beta\delta}(\mathbf{q})/(2k^2) \tag{4.28}$$

and $\mathbf{q} = -\mathbf{k} - \mathbf{p}$.

4.5 Properties of Wavevector Integrations and Kinematical Coefficients

The coefficients a_1, b_1, c_1, a, b defined in the last section are not peculiar to the quasi-normal approximation. In fact, these kinematical coefficients appear naturally in most low-order closure approximations. In this section, some properties of the kinematical coefficients are explained. First, it is convenient to investigate some properties of the wavevector integrations appearing, e.g., in (4.26).

As mentioned several times in the foregoing, the condition $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$ implies that $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ form a triangle with sides of length k, p, q. It follows that k, p, q necessarily satisfy the triangle inequality

$$|k - p| \ll q \ll k + p$$

with equality only if the triangle degenerates into a line segment. Consequently, for fixed k, the only possible triad interactions involve wavevectors lying in the "wavenumber slot" shown in Fig. 4.1. It is possible to convert wavevector integrations of the type appearing in the hierarchy equations into integrations over the wavenumber slot. If $F(k, p, q)$ is an (isotropic) function of triangle shape and size but not of triangle orientation, then for any fixed k it follows that

$$\int_{\mathbf{p}+\mathbf{q}=-\mathbf{k}} F(k, p, q)\, d\mathbf{p} = 2\pi \iint_{\Delta} \frac{pq}{k} F(k, p, q)\, dp\, dq, \tag{4.29}$$

where $\iint_{\Delta} dp\, dq$ indicates integration over the slot shown in Fig. 4.1. This result follows from $d\mathbf{p} = 2\pi p^2\, dp\, dz$ and $q^2 = k^2 + p^2 - 2kpz$ (law of cosines), where z is the cosine of the angle included between $\mathbf{k}$ and $\mathbf{p}$, since $\partial(p, q)/\partial(p, z) = -kp/q$. Using (4.29), the quasi-normal closure (4.26) assumes the

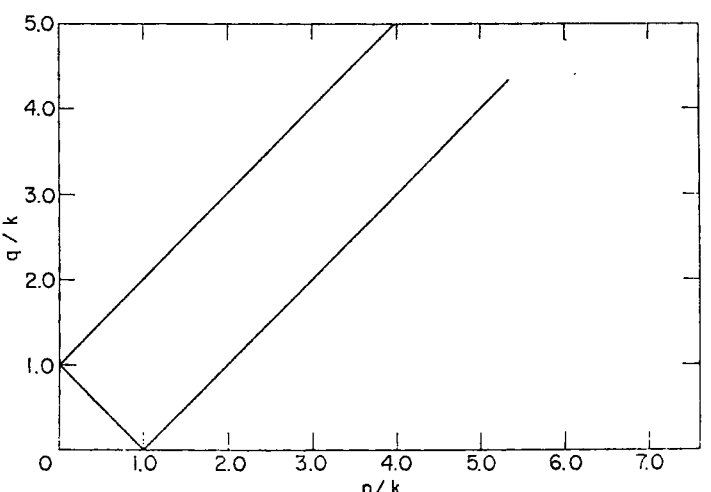

Figure 4.1 Region of integration in the "triangle integrals" (4.29).

symmetrical form

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right]E(k, t) = \frac{1}{2}\iint_{\Delta} \mathrm{d}p\,\mathrm{d}q \int_0^t \mathrm{d}s\, \mathrm{e}^{-\nu(k^2+p^2+q^2)(t-s)}\, kp^{-1}q^{-1}$$

$$\cdot \{2k^2a(k, p, q)E(p, s)E(q, s) - E(k, s)[p^2b(k, p, q)E(q, s) +$$

$$+ q^2b(k, q, p)E(p, s)]\}. \tag{4.30}$$

The kinematical coefficients defined by (4.21)–(4.23), (4.27), (4.28) are most conveniently evaluated in terms of the angle cosines of the triangle formed from $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$. Let α, β, γ denote the interior angles opposite k, p, q, respectively. Also, let $x = \cos\alpha, y = \cos\beta, z = \cos\gamma$, so that $kpz = -\mathbf{k}\cdot\mathbf{p}$, etc. Then, it follows by straightforward calculation from the definitions that†

$$a(k, p, q) = \tfrac{1}{2}(1 - xyz - 2y^2z^2) \tag{4.31}$$

$$b(k, p, q) = \frac{p}{k}(xy + z^3) \tag{4.32}$$

$$a_1(k, p, q) = a(k, p, q) + \tfrac{1}{2}(z^2 - y^2) \tag{4.33}$$

$$b_1(k, p, q) = \frac{p}{2k}(1 - z^2)xy - \frac{q}{2k}(1 - y^2)xz \tag{4.34}$$

$$c_1(k, p, q) = \tfrac{1}{2}(1 - y^2)(1 - z^2)(1 - 2pz/k). \tag{4.35}$$

For example, (4.31) follows by expansion of (4.27), noting that

$$k_\alpha k_\beta P_{\alpha\beta}(\mathbf{p}) = k^2(1 - z^2), \quad P_{\alpha\beta}(\mathbf{k})P_{\alpha\beta}(\mathbf{p}) = 1 + z^2$$

etc. It is also convenient to note that since $\beta + \gamma = \pi - \alpha$

$$x = (1 - y^2)^{1/2}(1 - z^2)^{1/2} - yz, \quad x^2 + y^2 + z^2 = 1 - 2xyz.$$

Also, the law of sines implies that $p/k = (1 - y^2)^{1/2}/(1 - x^2)^{1/2}$, etc.

Various manipulations, some outlined below, establish the following results:

$$0 \leqslant a(k, p, q) = a(k, q, p) \leqslant \tfrac{9}{16} \tag{4.36}$$

$$k^2b(k, p, q) = p^2b(p, k, q) \tag{4.37}$$

$$2a(k, p, q) = b(k, p, q) + b(k, q, p) \tag{4.38}$$

$$b_1(k, p, q) = -b_1(k, q, p) \tag{4.39}$$

†Leith (1971) gives the two-dimensional analogs of the geometrical coefficients $b(k, p, q)$, etc. A particularly elegant form of the two-dimensional $b(k, p, q)$, due to Lilly, is

$$b(k, p, q) = \frac{4}{\pi}k^{-3}p^{-1}(k^2 - q^2)(p^2 - q^2)(1 - x^2)^{1/2}.$$

$$b(k, p, q) = a_1(k, p, q) + b_1(k, p, q) \tag{4.40}$$

$$c_1(k, p, q) = -c_1(k, q, p). \tag{4.41}$$

For example, nonnegativity of $a(k, p, q)$ follows from (4.31) since $(1 - y^2)^{1/2}(1 - z^2)^{1/2} + yz \ll 1$ for $|y| \ll 1, |z| \ll$. Alternatively, nonnegativity of $a(k, p, q)$ may be demonstrated directly from (4.27) by introducing unit vectors $\mathbf{n}_1(\mathbf{p}), \mathbf{n}_2(\mathbf{p})$ $[\mathbf{n}_1(\mathbf{q}), \mathbf{n}_2(\mathbf{q})]$ defined so that $\mathbf{p}, \mathbf{n}_1(\mathbf{p}), \mathbf{n}_2(\mathbf{p})$ $[\mathbf{q}, \mathbf{n}_1(\mathbf{q}), \mathbf{n}_2(\mathbf{q})]$ form an orthogonal basis of three-dimensional space. Then since

$$P_{\alpha\beta}(\mathbf{p}) = n_{1\alpha}(\mathbf{p})n_{1\beta}(\mathbf{p}) + n_{2\alpha}(\mathbf{p})n_{2\beta}(\mathbf{p}),$$

it follows that

$$4k^2a(k, p, q) = \sum_{i,j=1}^{2} [P_{\alpha\beta\gamma}(\mathbf{k})n_{i\beta}(\mathbf{p})n_{j\gamma}(\mathbf{q})][P_{\alpha\rho\sigma}(\mathbf{k})n_{i\rho}(\mathbf{p})n_{j\sigma}(\mathbf{q})] \geqslant 0,$$

where $n_{1\alpha}(\mathbf{p})$ are the components of $\mathbf{n}_1(\mathbf{p})$, and so on.

Some insight into the nature of the kinematical coefficients $a(k, p, q)$, $b(k, p, q)$ is gotten by examining the contour plots shown in Figs. 4.2 and 4.3, respectively. Only the values of $a(k, p, q)$ and $b(k, p, q)$ within the wavenumber slot are relevant, so that only these values are plotted. It is apparent that $b(k, p, q)$ shows much structure, while $a(k, p, q) \approx 0{\cdot}5$ over most of the slot.

At this point, it is appropriate to make some comments on numerical

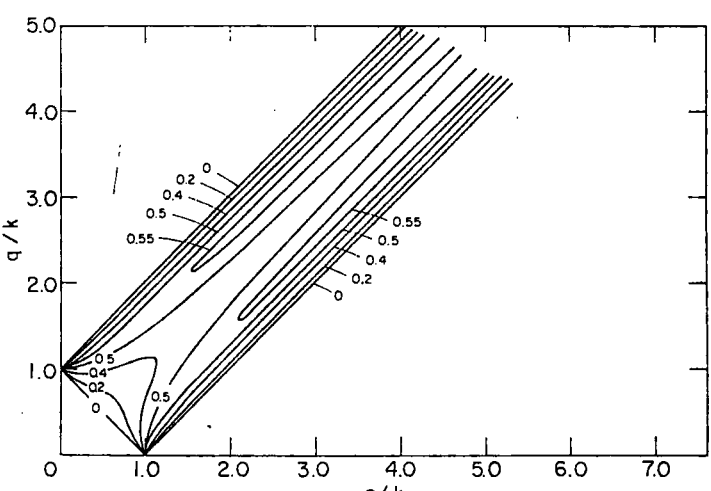

Figure 4.2 Contour plot of $a(k, p, q)$ within the wavenumber slot shown in Fig. 4.1. The contours are labelled by the values of $a(k, p, q)$.

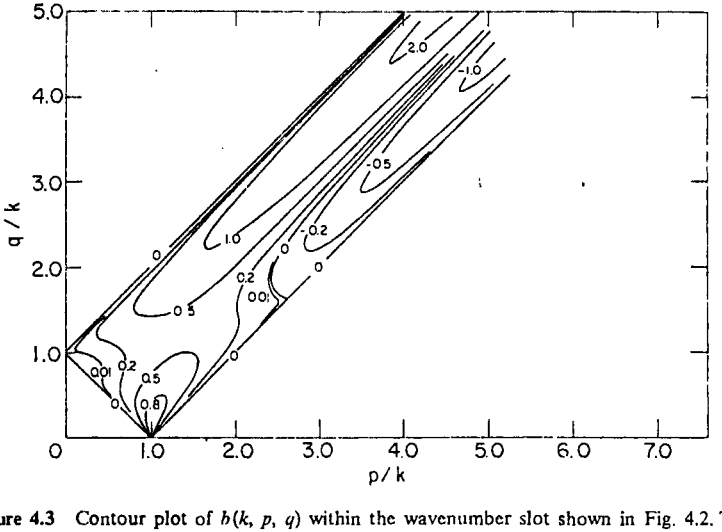

Figure 4.3 Contour plot of $b(k, p, q)$ within the wavenumber slot shown in Fig. 4.2. The contours are labelled by the values of $b(k, p, q)$.

methods to evaluate triangle integrals

$$I = \iint_{\Delta} F(k, p, q)\, \mathrm{d}p\, \mathrm{d}q$$

of the sort that appear on the right-hand side of (4.30), for example. Suppose that the region of integration is truncated by requiring k, p, q to lie in the finite range $(k_{\mathrm{bot}}, k_{\mathrm{top}})$, where k_{bot} may or may not be zero. If I represents the triangle integral in (4.30), then $(k_{\mathrm{bot}}, k_{\mathrm{top}})$ must include all wavenumbers with appreciable excitation, requiring that $k_{\mathrm{bot}} \ll L_p^{-1}$, $k_{\mathrm{top}} \gg k_d$. In practice, the choices $k_{\mathrm{bot}} = 0$, $k_{\mathrm{top}} \gtrsim k_d \propto R_\lambda^{3/2}/L_p$ suffice. The resulting truncated triangle integral is evaluated as a Riemann sum by dividing the range $(k_{\mathrm{bot}}, k_{\mathrm{top}})$ with $N + 1$ discrete wavenumbers $k_0 = k_{\mathrm{bot}}, k_1, \ldots, k_N = k_{\mathrm{top}}$. If the N intervals $\Delta k_n = k_n - k_{n-1}$ $(n = 1, \ldots, N)$ are chosen equal, then the requirement that the mesh be sufficiently fine to resolve structure at the integral scale L_p requires that $\Delta k_n \lesssim L_p^{-1}$, so that $N \gtrsim k_{\mathrm{top}} L_p \gtrsim R_\lambda^{3/2}$. In this case, evaluation of I involves order R_λ^3 arithmetic operations for each value of k. With equally spaced discrete wavenumbers, the requirement $N \gtrsim R_\lambda^{3/2}$ is so severe that, at high Reynolds numbers, solution of (4.30) (or any structurally-similar turbulence theory) involves nearly as much (or more) computation as numerical solution of the Navier–Stokes equations. If the use of equally spaced discrete wavenumbers were the only way to

evaluate triangle integrals, there would seem little point to turbulence theory, except for aesthetics.

Fortunately, another choice of $k_0, \ldots, k_N$ considerably reduces the amount of arithmetic computation necessary to get acceptable accuracy in the evaluation of I. The point is that the integrand of the triangle integral in (4.30) should be a smooth function of k, p, q with typically power-law behavior, if the interpretation of $E(k)$ as an ensemble average makes sense. Therefore, an adequate approximation representation of the integrand is gotten by choosing $\{k_n\}$ so that $\{\log_2 k_n\}$ are eq ally spaced. With constant logarithmic steps, $k_n = r^n k_{\text{bot}}$ where $r = \log_2 k_n - \log_2 k_{n-1}$, so that there are a fixed number of steps $m = 1/\log_2 r$ per octave of wavenumber. Then, $N = m \log_2(k_{\text{top}}/k_{\text{bot}}) = \frac{1}{2}m \log_2 R_\lambda + 0(m)$ so that evaluation of I requires order $m^2(\log_2 R_\lambda)^2$ arithmetic operations for each value of k. In numerical calculations of the equations of turbulence theories, it is found that $m = 8$ (eighth-octave bands) is usually sufficient for accurate wavevector integrations. At high R_λ, the computational savings afforded by constant logarithmic steps is evident. Some of the details of the numerical methods used to evaluate triangle integrals with constant logarithmic wavenumber intervals are given by Kraichnan (1964a) and Leith and Kraichnan (1972). Numerical methods are discussed further in Section VI.

4.6 Properties of the Single-Time Qnasi-Normal Theory

The single-time quasi-normal approximation (4.17), (4.18) will be shown unsatisfactory, in the sense that its predictions are not a valid approximation to the properties of turbulence. Before showing this, a number of plausible features of the quasi-normal theory will be indicated.

The quasi-normal theory conserves energy by nonlinear interaction, in the sense that (2.35) remains valid. Conservation follows from (4.30) using (4.36) – (4.38) and $\int_0^\infty dk \iint_\Delta dp\, dq = \iiint_\Delta dk\, dp\, dq$.

Further insight into the nonlinear dynamics of the quasi-normal theory is gotten by considering the evolution of enstrophy $\Omega(t) = \int_0^\infty k^2 E(k, t)\, dk$, when the Reynolds number is infinite. An equation for $\Omega(t)$ is obtained using the following relations, whose derivation is straightforward but tedious,

$$\int_{|p-q|}^{p+q} k^5 a(k, p, q)\, dk = \frac{2}{3} pq(p^4 + q^4)$$

$$\int_{|k-q|}^{k+q} pb(k, p, q)\, dp = \frac{q}{12k^3}(13k^4 - 8k^2q^2 + 3q^4)$$

$$+ \frac{(k^2 - q^2)^3}{8k^4} \ln\left|\frac{k+q}{k-q}\right|.$$

powers of R or $t - t_0$ indicates that the theories are asymptotically exact for small R or small $t - t_0$. The fact that cumulant-discard approximations retain terms of all orders in $R[t - t_0]$ suggests that solutions to the cumulant-discard equations may be acceptable at finite or large $R[t - t_0]$ where simple truncations of the power series fail. Unfortunately, this is not the case. The domains of validity of the cumulant-discard approximations and the simpler power-series truncations are apparently the same, namely, small R or $t - t_0$. Outside this domain of validity, both sets of approximations give physically unacceptable results.

In numerical integrations of (4.30) from various initial spectra, Ogura (1963) found that regions of negative $E(k, t)$ developed if R was sufficiently large. Some spectra obtained by numerical integration of (4.30) are shown in Figs. 4.4–4.6 (Orszag, 1970c). The only novel feature of the numerical scheme used to obtain these results is the use of constant logarithmic wavenumber steps, for reasons described at the end of §4.5. The initial spectrum in all three cases is

$$E(k, 0) = 16(2/\pi)^{1/2} v_{\text{rms}}^2 k_{\text{max}}^{-5} k^4 \exp\left[-2(k/k_{\text{max}})^2\right], \tag{4.47}$$

where k_{max} is the wavenumber of maximum initial excitation. In the runs plotted in Figs. 4.4–4.6, $k_{\text{max}} = 4 \times 2^{1/2}$ and $v_{\text{rms}} = 1$ (initially). The values of other relevant parameters are stated in the figure captions. The behavior shown in these figures is typical and not materially affected by the choice of initial spectrum (4.47).

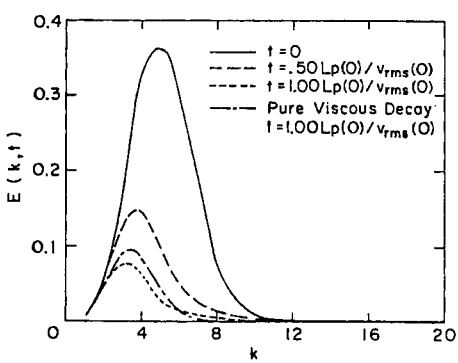

Figure 4.4 Decay calculation using the quasi-normal equation (4.30) with $\nu = 0{\cdot}08$ and (4.47) for the initial energy spectrum with $v_{\text{rms}}(t = 0) = 1$, $k_{\text{max}} = 4 \times 2^{1/4} = 4{\cdot}75683$. Wavevector space was truncated into quarter-octave bands between $k_{\text{bot}} = 2^{1/8} = 1{\cdot}09051$ and $k_{\text{top}} = 2^{43/8} = 41{\cdot}4990$ and the time step was $\Delta t = 0{\cdot}0025$. These initial conditions give the initial Reynolds number $R_\lambda(0) = 5{\cdot}25$, and Taylor microscale $\lambda(0) = 0{\cdot}4202$. $L_p(0) = 0{\cdot}5248$ is the initial longitudinal integral scale. The curve marked pure viscous decay is computed using (4.16).

At the lowest Reynolds number ($R_\lambda \sim 5$), the results plotted in Fig. 4.4 show that viscous decay is dominant and evolution according to (4.30) differs little from evolution according to (4.16). This low Reynolds number behavior is quite plausible. However, for $R_\lambda \gtrsim 15$, the behavior predicated by (4.30) is distinctly unphysical. For isotropic turbulence, (2.23) requires

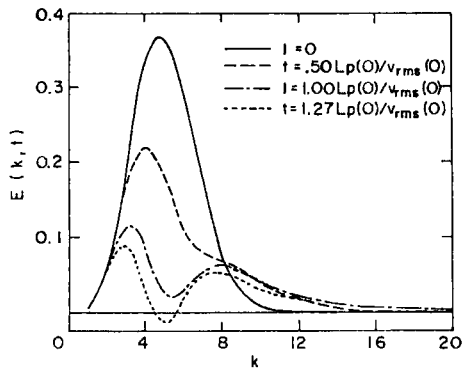

Figure 4.5 Decay calculation using the quasi-normal equation with $\nu = 0.02$ and all other parameter values, except $R_\lambda(0)$, identical to those given in the caption to Fig. 4.4. The initial Reynolds number for this case is $R_\lambda(0) = 21$.

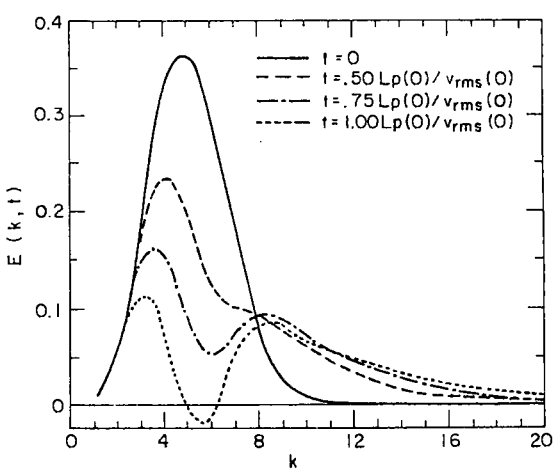

Figure 4.6 Decay calculation using the quasi-normal equation with $\nu = 0.01$ and all other parameter values, except $R_\lambda(0)$, identical to those given in the caption to Fig. 4.4. The initial Reynolds number for this case is $R_\lambda(0) = 42$.

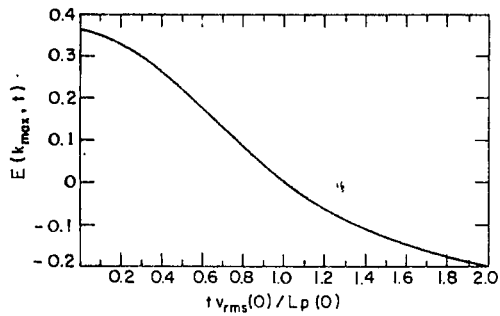

Figure 4.7 Evolution of $E(k_{\max}, t)$ for the calculation shown in Fig. 4.6. Here $k_{\max} = 4{\cdot}75683$ and $R_\lambda(0) = 42$.

that the energy spectrum $E(k, t)$ be nonnegative, which is grossly violated by the solutions plotted in Figs. 4.5 and 4.6. In these latter figures, it is seen that the energy density in modes that are most strongly excited initially becomes negative after about one circulation of the energy-containing eddies.† In Fig. 4.7, the evolution at $R_\lambda \sim 42$ of the energy density in the mode with the highest initial excitation is plotted as a function of time to illustrate that $E(k_{\max}, t)$ goes smoothly through zero.

It is important to emphasize that violation of the realizability inequality (2.23) is not in itself disqualifying. After all, the quasi-normal theory is at best an approximate theory, so that if the negative energy densities only appeared in weakly excited, uninteresting regions of Fourier space, the theory might still give a useful description of other features of the turbulence. Violation of realizability by the quasi-normal approximation is disqualifying because the violation occurs strongly in dynamically important regions of Fourier space.

If the initial flow is not restricted to be Gaussian, it is easy to concoct initial energy and transfer spectra which are realizable‡ but which evolve according to (4.17), (4.18) to violate realizability. The idea is to choose $T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}, 0)$ large in magnitude and of sign such that $\partial S_{\alpha\beta}(\mathbf{k}, t)/\partial t|_{t=0}$ is very negative, for some particular $\mathbf{k}$. Then for fixed finite ν, $S_{\alpha\beta}(\mathbf{k}, t)$ evolves according to (4.17) to become negative in a time scale short compared to

†The results of numerical simulation of the Navier–Stokes equation with initial spectra (4.47) at a Reynolds number near that used for Fig. 4.6 are reported in § 6.6. The spectra evolve smoothly.

‡The energy and transfer spectra are said to be realizable if they can be expressed as averages over a suitable ensemble.

that required for $T_{\alpha\beta\gamma}$ (**k**, **p**, t) to evolve significantly according to (4.18). The initial conditions are realizable since the unspecified fourth-order cumulants may be chosen large enough and of such sign to satisfy (2.24). Of course, it is the neglect of the large fourth-order cumulants in deriving (4.18) that leads to the violation of (2.23).

If only Gaussian initial conditions are allowed, the above argument explaining the origin of negative energy spectra in the quasi-normal theory is not applicable. Therefore, we return to (4.30) to see whether there is any tendency to preserve the positivity of energy spectra. For simplicity, only the extreme case $\nu = 0$ is considered. Suppose that $E(k, t) > 0$ for all k and all $t < t_1$, but that (4.30) evolves the energy spectrum so that $E(k_1, t) = 0$. Does (4.30) prevent $E(k_1, t) < 0$ for $t > t_1$?

With $\nu = 0$ and $E(k_1, t_1) = 0$, it follows from (4.30) and (4.36) that

$$\left.\frac{\partial^2 E(k_1, t)}{\partial t^2}\right|_{t=t_1} = \frac{1}{2} \iint_{\Delta} \frac{k_1}{pq} \left[2a(k_1, p, q)E(p, t_1)E(q, t_1)\right] \mathrm{d}p\, \mathrm{d}q > 0. \tag{4.48}$$

However, because the right-hand side of (4.30) involves the complete history of $E(k, t)$ for $t < t_1$, $\partial E(k_1, t)/\partial t|_{t=t_1}$ is not restricted positive by (4.30). Since there is no restriction on the sign of the first time-derivative at $t = t_1$, the positivity of the second derivative is not sufficient to prevent $E(k_1, t)$ from becoming negative. In the next section, we shall (drastically) modify (4.30) so that the left-hand side of (4.48) is replaced by $\partial E(k_1, t)/\partial t|_{t=t_1}$. In this modified form, (4.48) implies that $E(k_1, t)$ remains positive for $t > t_1$.

It is still necessary to explain how the conditions assumed at $t = t_1$ can be realized in normal evolution according to (4.30). Again, suppose that ν is negligible in the wavenumber range of interest. If $E(k, 0)$ is chosen arbitrarily, we should expect, on the basis of the statistical mechanical principles argued in §I, initial approach to an asymptotic state. This latter behavior which is expected of the exact dynamics should persist in the quasi-normal approximation, because the quasi-normal theory has plausible small-time behavior (since it reproduces the first four terms of the formally exact expansion in powers of t). Suppose for the sake of argument that (4.30) evolves $E(k, t)$ so that $E(k, t_2)$ is very close to the asymptotic state. Now, at $t = t_2$, the memory integral on the right-hand side of (4.30) involves quantities like

$$I = \int_0^{t_2} E(p, s)\, E(q, s)\, \mathrm{d}s$$

since ν is negligible. In the expression I, equal weight is given instants near $s = 0$ when $E(k, s)$ is far from the asymptotic state as is given instants near $s = t_2$ when $E(k, s)$ is close to the asymptotic state. It follows that, at large

Reynolds number, the right-hand side of (4.30) gives finite weight to instants when the energy spectrum differs greatly from that of the asymptotio state, even though the current spectrum may be closely approximated by that of the asymptotic state. We infer that energy spectra evolving according to (4.30) overshoot the asymptotic state and do not immediately relax. In fact, this argument suggests that the spectral regions requiring the most internal reorganization to reach the asymptotic state are those which overshoot most. These regions, in particular wavenumbers containing much of the initial excitation in the examples shown in Figs. 4.5, 4.6 are those which first evolve to the condition $E(k, t) = 0$ and thence negative.

The basic trouble is that the right-hand side of (4.30) includes too much memory of past dynamical evolution. The only limitation on the memory integrals in (4.30) is provided by viscosity. This viscous cutoff occurs at a time of order $(\nu k^2)^{-1}$, which is very large for ν small and k not too large. Actually, nonlinear effects (summarily called nonlinear scrambling) should cut off interactions more efficiently than viscosity.† In the inertial range, the relevant time scale for nonlinear scrambling is the local time (3.4), which is independent of ν. In this case, the memory integrals would cut off after a much shorter time than $(\nu k^2)^{-1}$. Using this idea of nonlinear scrambling to assist the loss of memory, we shall later be led to theories which do not significantly overshoot an asymptotic state. A simple theory of this type is described in the next section.

Violation of realizability by the quasi-normal theory may also be demonstrated using Betchov's inequality (2.47). The assumption of vanishing fourth-order cumulants requires $F_0 = 3$, so that Betchov's inequality implies $|S_0| \ll 0{\cdot}756$. This bound on S_0 is violated by the quasi-normal theory, as shown by (4.47). Therefore, the quasi-normal approximation is incompatible with a nonnegative probability distribution of velocity.

4.7 Quasi-Normal Theory: Markovian Modification

It is possible to make a simple, though crude, modification of (4.30) that eliminates many of the deficiencies of the quasi-normal theory discussed in §§4.4, 4.6. The modification is

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] E(k, t) = \frac{1}{2} \iint_{\Delta} \mathrm{d}p\, \mathrm{d}q\, \theta(k, p, q; t) k p^{-1} q^{-1} \cdot$$

$$\cdot \{2k^2 a(k, p, q) E(p, t) E(q, t) - E(k, t)[p^2 b(k, p, q) E(q, t)$$

†This effect of nonlinearity was apparently first realized by Lord Kelvin (1887). Kelvin termed the effect "vitiating rearrangement", but we prefer "nonlinear scrambling", the latter term apparently due to S. C. Crow.

$$+ q^2 b(k, q, p)E(p, t)]\} \tag{4.49a}$$

$$\theta(k, p, q; t) = \int_0^t \exp\left[-\int_s^t \{\eta(k, r) + \eta(p, r) + \eta(q, r)\}\, dr\right] ds, \tag{4.49b}$$

where $\eta(k)$ is an "eddy viscosity" coefficient included to account for the effects of nonlinear scrambling. The form of $\eta(k)$ is discussed below. In addition to the inclusion of $\eta(k)$, there is another important modification of (4.30) included in (4.49). The energy spectra evaluated at the intermediate time s in the memory integral on the right-hand side of (4.30) are evaluated at the current time t in (4.49).† Both these modifications of (4.30) are made *a posteriori* and lack fundamental justification. Our purpose in discussing (4.49) is not to propose it as a basic theory of turbulence but rather to illustrate the sort of effects that must be included in a satisfactory theory.

It is not difficult to show that (4.49) preserves the realizability of $E(k, t)$, whatever be $\eta(k)$. In particular, $E(k, t)$ remains nonnegative for $t > 0$ if $E(k, 0)$ is nonnegative. For assume that $E(k_1, t_1) = 0$ while $E(k, t) > 0$ for all k and all $t < t_1$. It follows from (4.49) and the nonnegativity of $a(k, p, q)$ that $\partial E(k_1, t)/\partial t|_{t=t_1} > 0$, so that $E(k_1, t) > 0$ for $t > t_1$. Similarly, for the theory of anisotropic turbulence obtained from (4.17) and (4.18) in the same way that (4.49) is obtained from (4.30), it may be shown that the realizability inequality (2.23) is maintained during time evolution.

The feature of (4.49) that preserves realizability of the solutions is the dependence of $\partial E(k, t)/\partial t$ only on current values of the energy spectrum. When $E(k, t)$ approaches an asymptotic state, (4.49) predicts that $\partial E(k, t)/\partial t$ becomes small, even at large Reynolds number. In contrast to the behavior of solutions to (4.30) discussed in the last section, there is no significant overshoot of an asymptotic state by solutions of (4.49). It is the property that the rate of change of the energy spectrum depends only on the current values of the energy spectrum that suggests calling (4.49) the Markovian quasi-normal theory.

Another feature of (4.49) is that its solutions approach an asymptotic state. This behavior may qualitatively be understood by considering the nature of the transfer term on the right-hand side of (4.49). Since $a(k, p, q)$ is nonnegative, the term involving a represents a *positive* flow of energy to mode k. Also, examination of Fig. 4.3 shows that $b(k, p, q)$ is *typically* positive, so that the terms in (4.49) involving b represent a typically *negative* flow of energy to k. The net flow is the sum of these *absorption* and *emission*

†It follows from (4.49b) that

$$\frac{d\theta}{dt} = 1 - [\eta(k, t) + \eta(p, t) + \eta(q, t)]\theta$$

so that (4.49) does not in fact involve any memory integrals in t.

terms. The quantity in curly brackets in (4.49) may be written

$$k^2p^2q^2\left\{b(k,p,q)\frac{E(q,t)}{q^2}\left[\frac{E(p,t)}{p^2}-\frac{E(k,t)}{k^2}\right]+\right.$$

$$\left.+\,b(k,q,p)\frac{E(p,t)}{p^2}\left[\frac{E(q,t)}{p^2}-\frac{E(k,t)}{k^2}\right]\right\}.$$

The form of this term and the fact that $b(k,p,q)$ is typically positive suggests that, aside from effects of molecular viscosity, the energy spectrum tends toward an equipartition state in which $E(k) \propto k^2$. Furthermore, the structure of the emission and absorption terms shows that equipartition is maintained stably against small perturbations, again neglecting viscous effects. The approach toward equipartition has been verified by numerical integrations (not reported here) of the inviscid form of (4.49), and is in accord with the detailed description of inviscid equipartition states given in §5.2.

With nonzero viscosity, no matter how small, excitation in high wavenumbers is depleted by viscous action and equipartition is not attainable. It follows from the structure of the transfer term that there is energy flow from strongly excited low-wavenumber modes to weakly excited high-wavenumber modes. The balance between viscous dissipation at large wavenumbers and the tendency of the transfer terms to establish equipartition determines the form of a quasi-steady asymptotic state. Further, since the emission terms in (4.49) are proportional to $E(k,t)$ but the absorption term is not, the asymptotic state is stable. For if $E(k)$ is increased locally above the value appropriate to wavenumber k in the asymptotic state, the emission terms will be increased appreciably while the absorption term will be little affected, showing that the asymptotic state tends to maintain itself.

The properties of realizability and approach to an asymptotic state are demonstrated in an elementary way by showing that (4.49) is the *exact* equation for the ensemble-average energy spectrum of a model dynamical system. The present model is closely related to one that will be discussed in §6.2 in connection with the direct-interaction theory. Models of the kind discussed here have been given by Phythian (1969), Kraichnan (1970a), Leith (1971), and Herring and Kraichnan (1972). In the case of (4.49), the model is the Langevin equation

$$\frac{\partial \hat{u}_\alpha(\mathbf{k},t)}{\partial t} = -\gamma(k,t)\hat{u}_\alpha(\mathbf{k},t) + q_\alpha(\mathbf{k},t), \tag{4.50}$$

where $\hat{\mathbf{u}}(\mathbf{k},t)$ is a hypothetical stochastic variable satisfying

$$\langle \hat{u}_\alpha(\mathbf{k},t)\hat{u}_\beta(\mathbf{p},t)\rangle = P_{\alpha\beta}(\mathbf{k})\frac{E(k,t)}{4\pi k^2}\delta(\mathbf{k}+\mathbf{p}), \tag{4.51}$$

$\gamma(k,t)$ is a nonstochastic damping to be specified below, and $\mathbf{q}(\mathbf{k},t)$ is a white-noise stochastic variable also chosen below. The variable $\hat{\mathbf{u}}(\mathbf{k},t)$ is a

mathematical artifice that should not be confused with the Fourier coefficients $\mathbf{u}(\mathbf{k}, t)$. Since $\gamma(k, t)$ is nonrandom, it follows from (4.50), (4.51) that

$$\frac{\partial E(k, t)}{\partial t} = -2\gamma(k, t)E(k, t) + Q(k; t, t) \tag{4.52a}$$

$$\langle q_\alpha(\mathbf{k}, t)\hat{u}_\alpha(\mathbf{p}, t')\rangle = \frac{Q(k; t, t')}{4\pi k^2}\delta(\mathbf{k} + \mathbf{p}). \tag{4.52b}$$

Further, since the solution to (4.50) is

$$\hat{u}_\alpha(\mathbf{k}, t) = G(k; t, 0)\,\hat{u}_\alpha(\mathbf{k}, 0) + \int_0^t G(k; t, s)q_\alpha(\mathbf{k}, s)\,ds,$$

where

$$G(k; t, s) = \exp\left[-\int_s^t \gamma(k, r)\,dr\right] \tag{4.53a}$$

it follows that

$$Q(k; t, t) = G(k; t, 0)Q(k; t, 0) + \int_0^t G(k; t, s)F(k; t, s)\,ds \tag{4.53b}$$

$$\langle q_\alpha(\mathbf{k}, t)q_\alpha(\mathbf{p}, s)\rangle = \frac{F(k; t, s)}{4\pi k^2}\delta(\mathbf{k} + \mathbf{p}). \tag{4.53c}$$

In order to recover (4.49) from (4.52), (4.53), the following choices of $\gamma(k, t)$, $\mathbf{q}(\mathbf{k}, t)$ and $\hat{\mathbf{u}}(\mathbf{k}, 0)$ are made. The choice

$$\gamma(k, t) = \nu k^2 + \tfrac{1}{2}\int_\Delta\int dp\,dq\,kp^{-1}q^{-1}\theta(k, p, q; t)\,b(k, p, q)E(q, t) \tag{4.54}$$

ensures that (4.52) reproduces the viscous term and the last two terms on the right-hand side of (4.49a). The dependence of $\gamma(k, t)$ on $E(q, t)$ may seem unusual—for clarity, this dependence is explained further below. The damping factor $\gamma(k, t)$ is a kind of eddy viscosity coefficient. In fact, a self-consistent theory without arbitrary constants or functions is obtained if $\gamma(k, t) = \eta(k, t)$ so that (4.49b) is replaced by

$$\theta(k, p, q; t) = \int_0^t \exp\left[-\int_s^t \{\gamma(k, r) + \gamma(p, r) + \gamma(q, r)\}\,dr\right]ds. \tag{4.55}$$

The resulting integral equation (4.54) together with (4.49a) determines the functions $\gamma(k, t)$ and $E(k, t)$—this type of self-consistent theory is closely related to those of Edwards (1964) and Herring (1965, 1966). We explore this self-consistent theory further in §4.9. However, the only truly self-consistent theory developed to date is the direct-interaction approximation to be discussed in Section VI. In §4.8, we regress by investigation of semi-empirical formulas for $\eta(k, t)$ in (4.49b) to specify $\theta(k, p, q; t)$.

The random force $\mathbf{q}(\mathbf{k}, t)$ is chosen to be proportional to the white noise process $a(t)$, defined as a Gaussian variable with $\langle a(t)\rangle = 0$, $\langle a(t)a(s)\rangle = \delta(t-s)$. It is also necessary to introduce two *statistically independent* fields $\mathbf{v}(\mathbf{k}, t)$ and $\hat{\mathbf{w}}(\mathbf{k}, t)$ that both satisfy (4.51) with the same $E(k, t)$ as $\hat{\mathbf{u}}(\mathbf{k}, t)$. The required choice of $\mathbf{q}(\mathbf{k}, t)$ is

$$q_\alpha(\mathbf{k}, t) = -iP_{\alpha\beta\gamma}(\mathbf{k}) \int a(t) \left[\theta(k, p, q; t)\right]^{1/2} \hat{v}_\beta(\mathbf{p}, t)\, \hat{w}_\gamma(\mathbf{q}, t)\, d\mathbf{p}, \quad (4.56)$$

where $\mathbf{q} = -\mathbf{k}-\mathbf{p}$. It follows from (4.27), (4.29), (4.51) and the statistical independence of $\mathbf{v}$ and $\hat{\mathbf{w}}$ that

$$F(k; t, s) = 2 \iint_\Delta dp\, dq\, k^3 p^{-1} q^{-1}\, \theta(k, p, q; t) a(k, p, q) E(p, t) E(q, t) \cdot \delta(t-s).$$

Finally, $\hat{\mathbf{u}}(\mathbf{k}, 0)$ is chosen statistically independent of $\mathbf{q}(\mathbf{k}, t)$ so that $Q(k; t, 0) = 0$. With these choices of $\gamma(k, t)$, $\mathbf{q}(\mathbf{k}, t)$, and $\hat{\mathbf{u}}(\mathbf{k}, 0)$, the energy equation (4.49a) follows from (4.52), (4.53).† This model gives a direct demonstration that $E(k, t)$ is realizable; in fact, (4.51) shows that $E(k, t)$ is realizable as the ensemble-average energy-spectrum of the random process $\hat{\mathbf{u}}(\mathbf{k}, t)$.

The dependence of the terms of the Langevin equation (4.50) on the energy spectrum $E(k, t)$ requires some explanation. Perhaps the most straightforward way of viewing the model is as a scheme for stepping $E(k, t)$ forward in time from t to $t + \Delta t$. At time t, $E(k, t)$ is assumed known so that $\gamma(k, t)$ is known and the random force $\mathbf{q}(\mathbf{k}, t)$ may be constructed for each realization of the ensemble. Then (4.50) may be used to step $\hat{\mathbf{u}}(\mathbf{k}, t)$ forward in time to give $\hat{\mathbf{u}}(\mathbf{k}, t + \Delta t)$ and thus $E(k, t + \Delta t)$ after ensemble averaging.

The model (4.50) involves two competing terms. The term proportional to $\hat{\mathbf{u}}(\mathbf{k}, t)$ involves eddy-viscous and molecular-viscous dissipation of energy in mode $\mathbf{k}$, while the random force $\mathbf{q}(\mathbf{k}, t)$ provides a diffusive input of energy into mode $\mathbf{k}$ from all triad interactions. The latter interpretation of the effect of $\mathbf{q}(\mathbf{k}, t)$ is justified because $Q(k; t, t)$ is nonnegative so that it represents an input of energy into mode $\mathbf{k}$. The nature of the stochastic balance expressed in (4.50) between emission (viscous) and absorption (diffusive) terms suggests strongly that the Markovian model exhibits plausible relaxation to an asymptotic state for typical non-negative $\eta(k)$.

4.8 Semi-Local Inertial-Range Structure

In this section, we consider the inertial-range structure of turbulence theories whose energy equation is of the form (4.49a). The results of this

†It is assumed that $\int_0^\infty \delta(t)dt = 1/2$.

investigation are widely applicable and are not restricted to the Markovian model discussed in the §4.7. Since an inertial range is (nearly) statistically stationary and (nearly) inviscid, it follows that the inertial-range form of (4.49a) is

$$\frac{1}{2}\iint_{\Delta} \mathrm{d}p\,\mathrm{d}q\;\theta(k,p,q)kp^{-1}q^{-1}\{2k^2a(k,p,q)E(p)E(pq)$$

$$-E(k)[p^2b(k,p,q)E(q)+q^2b(k,q,p)E(p)]\}=0. \tag{4.57}$$

We assume that $E(k)$ and $\theta(k,p,q)$ satisfy power laws in the inertial range so that

$$E(k)\propto k^{3n},\quad \theta(ak,ap,aq)=a^{-m}\theta(k,p,q) \tag{4.58}$$

and that $\theta(k,p,q)=\theta(p,q,k)=\theta(q,p,k)=\ldots$

The principal result is that if (4.58) holds with $|m|<2$ then (4.57) is satisfied in the inertial range when

$$n=2-\tfrac{1}{2}m. \tag{4.59}$$

Under these conditions, we say that there is a semi-local inertial range (Orszag and Kruskal, 1968). Orszag and Kruskal derived (4.59) by consideration of the energy transfer through mode k. Here we verify directly that (4.58) with (4.59) satisfies (4.57). The symmetry of (4.57) in p and q together with (4.38) implies that (4.57) is equivalent to

$$N(k)\equiv\iint_{\Delta}\mathrm{d}p\,\mathrm{d}q\theta(k,p,q)kp^{-1}q^{-1}b(k,p,q)E(q)[k^2E(p)-p^2E(k)]$$

$$=0 \tag{4.60}$$

The proof of (4.60) is accomplished by the change of variables $z=k^2/p$, $w=kq/p$. Since $\mathrm{d}p\,\mathrm{d}q=k^3z^{-3}\mathrm{d}z\,\mathrm{d}w$, it follows from (4.37), (4.58), and (4.60) that

$$\begin{aligned}N(k)&=\iint_{\Delta}\mathrm{d}z\,\mathrm{d}w\,k^3z^{-3}\theta(k,k^2/z,kw/z)k^{-2}w^{-1}z^2b(k,k^2/z,kw/z)\\&\qquad\cdot E(kw/z)[k^2E(k^2/z)-k^4z^{-2}E(k)]\\&=\iint_{\Delta}\mathrm{d}z\,\mathrm{d}w\,(k/z)^{3-m-n}w^{-1}\,\theta(z,k,w)b(k^2/z,k,kw/z)E(w)\\&\qquad\cdot[k^{2-n}z^nE(k)-k^{4-n}z^{-2+n}E(z)]\\&=\iint_{\Delta}\mathrm{d}z\,\mathrm{d}w\,kz^{-1}w^{-1}\theta(k,z,w)b(k,z,w)E(w)\\&\qquad\cdot(z/k)^{2n-4+m}[z^2E(k)-k^2E(z)]\\&=-N(k)\quad\text{if}\quad n=2-\tfrac{1}{2}m.\end{aligned}$$

The requirement $|m| < 2$ is necessary to ensure that the integral (4.57) converges with the inertial range form (4.58). It is assumed that $\theta(k, p, q) = O(1)$ as $k \to 0$ and $\theta(k, p, q) = O(k^{-m})$ as $k \to \infty$, which follow from the interpretation of θ as a triple moment correlation time (Orszag and Kruskal, 1968). The convergence of the integrals (4.57) in the inertial range implies that energy transfer is accomplished locally in wave-space, as required by the description of the inertial range given in §3.1. If the integrals (4.57) diverged with (4.58), it would imply a significant effect of other regions of the spectrum on the inertial-range dynamics. In this case, the actual finite value of the integral (4.57) would have to be computed using the exact energy spectrum, not (4.58), including the energy containing range where (4.58) does not hold. Convergence of (4.57) in the inertial range implies that inertial range energy transfer depends only on inertial-range spectra so that energy cascade is local.

In the Kolmogorov inertial range of §3.1, the triple-moment correlation time $\theta(k, p, q)$ must, by dimensional analysis, satisfy (4.58) with $m = 2/3$ [cf. (3.4)], so that (4.59) gives $E(k) \propto k^{-5/3}$, as it should. This inertial range is an example of a completely local cascade, in the sense that all single-time cumulant information is transferred locally in wave-space (Orszag and Kruskal, 1968). However, the result (4.59) is more general, in the sense that it requires only local energy transfer. It can be used to determine the inertial-range spectra for turbulence theories of the form (4.49a) which do not yield the inertial-range time (3.4).

The simplest choice of $\eta(k)$ in the Markovian model of §4.7 is $\eta(k) = \nu k^2$, in which case nonlinear scrambling is neglected except in so far as it suggests the replacement of $E(k, s)$ on the right-hand side of (4.30) by $E(k, t)$ in (4.49a). With this choice of $\eta(k)$, there are two possible inertial-range spectral laws according to whether $t \ll (\nu k^2)^{-1}$ or $t \gg (\nu k^2)^{-1}$ for k in the inertial range (Tatsumi, 1960). If $t \gg (\nu k^2)^{-1}$, then $\theta(k, p, q; t) \approx t$ for $k \sim p \sim q$. It follows that $m = 0$ in (4.58) so $E(k) \propto k^{-2}$. In this limit, the theory reduces to the Markovian random-coupling model recently investigated by Frisch *et al.* (1973).

In the limit $t \gg (\nu k^2)^{-1}$, $\theta(k, p, q; t) \approx [\nu(k^2 + p^2 + q^2)]^{-1}$ for $k \sim p \sim q$, so that $m = 2$ in (4.58). It follows that $E(k) \propto k^{-1}$, though logarithmic corrections should be expected because m lies at the boundary of the region of convergence of (4.57).

As discussed in §3.2, inertial range spectra proportional to k^{-1} or k^{-2} are not consistent with experiment, which should not be unexpected considering the severity of the assumption on $\eta(k)$. The trouble is that the inertial range time (3.4) is not obtained so that although the cascade is local in wave-space the wrong power law is obtained. The choice $\eta(k) = \nu k^2$ is more appropriate to Burgers' model turbulence where there is no inviscid relaxation effect as pressure provides on three-dimensional

turbulence (cf. §4.9). In fact, $E(k) \propto k^{-2}$ is believed correct for the inertial range of Burgers' turbulence and (4.49) with $\eta(k) = \nu k^2$ should give qualitatively correct results (Kraichnan, 1968a).

For three-dimensional turbulence, the Kolmogorov $k^{-5/3}$ law is recovered by the choice

$$\eta(k, t) = \nu k^2 + b\left[k^3 E(k, t)\right]^{1/2} \tag{4.61}$$

in (4.49b). With this dimensionally correct choice of $\eta(k)$, it follows that (3.3) is recovered with (Orszag, 1967)

$$bC_K^{-3/2} = 0 \cdot 19. \tag{4.62}$$

In order to reproduce the observed value $C_K \approx 1.5$, it follows that $b \approx 0{\cdot}35$. Using this value of b, numerical calculations of (4.49), (4.61) have been made that are in very good agreement with available experimental results. This form of the theory has been investigated in detail by Leith (1971) who terms it the eddy-damped Markovian approximation.

4.9 The Test-Field Model

In §4.7, it was shown that the self-consistent Markovian model (4.49a) with (4.54) and (4.55) gives a realizable theory of turbulence similar to the theories of Edwards (1964) and Herring (1965, 1966).

In the inertial range, (4.49a) and (4.54) may be approximated noting that

$$E(k, t) \approx E(k), \quad \gamma(k, t) \approx \gamma(k), \quad \nu k^2 \ll \gamma(k),$$

$$\theta(k, p, q; t) \approx \left[\gamma(k) + \gamma(p) + \gamma(q)\right]^{-1}$$

for inertial range wavenumbers. In this case, (4.54) becomes

$$\gamma(k) \approx \frac{1}{2} \int\int_{\Delta} \mathrm{d}p\, \mathrm{d}q\, kp\, q^{-1}\left[\gamma(k) + \gamma(p) + \gamma(q)\right]^{-1} b(k, p, q) E(q). \tag{4.63}$$

The dominant contribution to the right-hand side of (4.63) comes from $q \sim L_p^{-1}$, i.e. the energy-containing eddies. In fact, if $q \ll k, p$ then

$$b(k, p, q) \approx 1 - y^2 \tag{4.64}$$

by (4.32) with $x \approx -y$, $z \approx 1$. If the assumption is made (to be checked later) that $\gamma(q) \ll \gamma(k)$ when $q \ll k$, then the dominant form of (4.63) in the inertial range is

$$\gamma(k) \approx \frac{1}{2}\, v_{\mathrm{rms}}^2\, \frac{1}{\gamma(k)}\, k^2. \tag{4.65}$$

In fact, the contribution to the right-hand side of (4.63) from wavenumbers

$q \sim k$ is clearly at most of order

$$\frac{1}{\gamma(k)} k^3 E(k) \ll \frac{1}{\gamma(k)} v_{\text{rms}}^2 k^2$$

the latter being the contribution of $q \sim L_p^{-1}$. The above inequality is equivalent to $kE(k) \ll v_{\text{rms}}^2$, which holds because the inertial range contains negligible energy of the flow (cf. §3.1). It follows from (4.65) that

$$\gamma(k) \approx \frac{1}{\sqrt{2}} v_{\text{rms}} k \tag{4.66}$$

so that $\gamma(q) \ll \gamma(k)$ for $q \ll k$ and that $m = 1$ in (4.58) for $\theta(k, p, q)$. Consequently, (4.59) implies that the inertial range spectrum is

$$E(k) = C_{sc} \epsilon^{1/2} v_{\text{rms}}^{1/2} k^{-3/2} \tag{4.67}$$

where the appropriate dimensional factors have been included.

As discussed in §3.2, the inertial range spectrum (4.66) is inconsistent with both experiment and the Kolmogorov spectrum (3.3). The origin of this discrepancy is the result

$$\theta(k, p, q) \approx \sqrt{2}\, v_{\text{rms}}^{-1} (k + p + q)^{-1}$$

for inertial range wavenumbers. Thus $\theta(k, p, q)$ scales like the sweeping time $\tau_s(k) = (k v_{\text{rms}})^{-1}$ rather than the inertial range time (3.4). The trouble is that θ determined by (4.54), (4.55) is not a measure of the time scale for internal distortion of an inertial range eddy. In §4.7, an interpretation of (4.49a) was given in terms of absorption (diffusion) and emission (dissipation) in mode k by interaction with modes p, q. In (4.54), the eddy-damping rate $\gamma(k, t)E(k, t)$ is set equal to the emission terms in (4.49a). However, although these emission terms are dominated by energy-containing modes $q \ll k$, as in (4.65), there is a near cancellation with the absorption terms in (4.49a) with $q \ll k$. The use of (4.54) as an intrinsic rate of distortion of an inertial range eddy greatly overestimates the distortion rate and thereby leads to inconsistency with the Kolmogorov theory, the latter requiring little internal distortion of inertial range eddies by large scale shear.

One crude way to restore the Kolmogorov spectrum within the self-consistent model is to perform the wavevector integration in (4.63) only over wavenumbers $p, q > Ck$. This artificial restriction ensures that dimensional analysis may be used to estimate $\theta(k, p, q)$ in the inertial range, with the result $m = 2/3$ in (4.58) and the spectrum (3.3). Unfortunately, the results of this trunction procedure depend rather sensitively on the cutoff C (Kraichnan, 1964b). [In effect, this procedure artificially imposes statistical Galilean invariance on the model (cf. §6.5).]

In order to derive systematically a self-consistent theory without serious

overestimation of internal distortion rates, Kraichnan (1971a) proposed the following scheme, called the test field model. Kraichnan observed that in a Lagrangian coordinate frame (moving with the local fluid velocity), internal distortion of an inertial range eddy is due to the pressure force, since viscous forces and external stirring forces are not significant at inertial range scales (cf. §6.5).† However, there are significant complications in working through the theory in a Lagrangian frame. Thus Kraichnan proposed to work the theory in an Eulerian frame. He noted that one way to gauge the effect of pressure in an Eulerian frame is simply to turn it off, so that compressive parts of the velocity field are generated from the solenoidal (incompressible) part of the field. Kraichnan suggested that a measure of the inherent rate of distortion due to the pressure is given by the rate of transfer between solenoidal and compressive parts of the velocity field in the absence of pressure. Kraichnan proposed a scheme for computing this rate by introduction of a "test field" governed by pressure-less dynamics.

The analysis leading to the test-field model equations will not be reproduced here (see Kraichnan, 1971a, and Herring and Kraichnan, 1972). The resulting equations for isotropic turbulence are (4.49a) with

$$\theta(k, p, q; t) = \int_0^t \exp\Big[-\int_s^t (\gamma^s(k, r) + \gamma^s(p, r) + \gamma^s(q, r))\, dr\Big]\, ds \tag{4.68}$$

$$\theta^G(k, p, q; t) = \int_0^t \exp\Big[-\int_s^t (\gamma^c(k, r) + \gamma^s(p, r) + \gamma^s(q, r))\, dr\Big]\, ds, \tag{4.69}$$

where

$$\gamma^s(k, t) = \frac{1}{2} g^2 \iint_\Delta dp\, dq\, kpq^{-1} b^G(k, p, q)\theta^G(p, q, k; t)E(q, t) \tag{4.70}$$

$$\gamma^G(k, t) = g^2 \iint_\Delta dp\, dq\, kpq^{-1} b^G(k, p, q)\theta^G(k, p, q; t)E(q, t). \tag{4.71}$$

Here

$$b^G(k, p, q) = \frac{1}{2} k_l k_j p_m P_{ij}(\mathbf{q}) P_{mn}(\mathbf{k})/k^2 p^2, = \frac{1}{2}(1 - y^2)(1 - z^2), \tag{4.72}$$

†Here lies the essential difference between Burgers' model turbulence (Burgers, 1948) and Navier–Stokes turbulence. In Burgers' model, there is no pressure relaxation so that the rate of internal distortion of an inertial range eddy is small, being controlled by the large-scale distortion of the flow. The result is that $m = 0$ in (4.58) leading to $E(k) \propto k^{-2}$ in the inertial range. In Navier–Stokes turbulence, high-frequency inertial-range pressure fluctuations limit the buildup of correlations, giving a Kolmogorov inertial range.

where y, z are the cosines of the angles opposite p, q, respectively, in the triangle k, p, q. Also, the arbitrary constant factor g is not determined by the theory and must be fit by comparison with experiment.

Since the test-field model falls within the general class of Langevin models (4.50), realizability is guaranteed. Also, it may be shown that the model is consistent with the Kolmogorov theory of §3.1. The Kolmogorov spectrum follows because

$$b^G(k, p, q) \sim \tfrac{1}{2}q^2k^{-2}(1 - y^2) \tag{4.73}$$

when $q \ll k$, so that the contribution of small q to (4.70), (4.71) is proportional to the enstrophy in large scales [$q^2E(q)$, which is small in the energy-containing range] rather than $E(q)$ which dominates the contribution (4.65) when (4.54) is applied. In the inertial range, viscous damping is negligible and

$$\theta(k, p, q) \approx \left[\gamma^s(k) + \gamma^s(p) + \gamma^s(q)\right]^{-1},$$

$$\theta^G(k, p, q) \approx \left[\gamma^c(k) + \gamma^s(p) + \gamma^s(q)\right]^{-1}$$

so that dimensional analysis gives

$$E(k) = C_{TFM}g^{2/3}\,\varepsilon^{2/3}\,k^{-5/3},\ \gamma^s(k) = A_{TFM}g^{4/3}\,\varepsilon^{1/3}\,k^{2/3},\ \gamma^c(k) = A'_{TFM}g^{4/3}\,\varepsilon^{1/3}\,k^{2/3}. \tag{4.74}$$

Dimensional analysis is applicable because inertial range modes dominate all contributions in (4.68)–(4.71). The constants C_{TFM}, A_{TFM}, A'_{TFM} are evaluated by numerical integration of (4.68)–(4.71) with the result (Kraichnan, 1971b)

$$C_{TFM} = 1{\cdot}342, \quad A_{TFM} = 0{\cdot}343 \quad A'_{TFM} = 0{\cdot}742. \tag{4.75}$$

It follows that the inertial-range spectrum (3.3) is obtained with

$$C_K = 1{\cdot}342g^{2/3}. \tag{4.76}$$

Thus, the test-field model with $g = 1{\cdot}064$ (recommended by Kraichnan) gives $C_K = 1{\cdot}40$, which is in good agreement with experiment.

Numerical solutions of the test-field model for decaying two- and three-dimensional turbulence are discussed in §6.6 [the test-field equations for two-dimensions are given by Kraichnan (1971a)]. It is found that the test field results with $g \sim 1$–$1{\cdot}5$ in three dimensions and $g \sim 0{\cdot}6$–$1{\cdot}0$ in two dimensions are in very good agreement with numerical experiments (Orszag and Patterson, 1972, Herring *et al.*, 1974).

Kraichnan (1972) has given the generalization of the test-field model to inhomogeneous turbulence. The generalization is complicated, but Kraichnan proposes some simplifications that should make numerical solutions of the equations economical. It is a moot point whether the resulting equations

are significantly better than obtained by a semi-empirical choice of damping factor $\eta(k, t)$ as in §4.8. However, theories of the general type of those discussed in the last three Sections under the general heading of eddy-damped Markovian approximations seem to be the simplest analytical theories of turbulence that give plausible results. They are sufficiently simple that they can be applied directly to a wide variety of inhomogeneous turbulence, so their practical application should be expected to increase in the future.

V The Statistical Mechanics of Turbulence

5.1 Introduction

There are two essential features of the statistical behavior of solutions of the Navier–Stokes equations. First, there exist an infinite number of degrees of freedom (per unit volume of fluid). Second, individual degrees of freedom (Fourier components) can exhibit complicated behavior and "phase mix", as illustrated in §1.4. The latter effect is the heart of both turbulence and equilibrium statistical mechanics and will be examined here. The large number of degrees of freedom is also important in a sense to be explained below.

It is convenient to deal first with a finite mode analog of the Navier-Stokes equations (2.31). If the fluid is imagined confined to a cube of side L with periodic boundary conditions applied on all sides, the velocity field is expansible as

$$\mathbf{v}(\mathbf{x}, t) = \sum \mathbf{u}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{5.1}$$

where the sum is over all $\mathbf{k}$ of the form $\mathbf{k} = (2\pi/L)\mathbf{n}$ where $\mathbf{n}$ has integer components. The Navier–Stokes equations become

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k}, t) = -\frac{i}{2} P_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{p}} u_\beta(\mathbf{p}, t)\, u_\gamma(\mathbf{k} - \mathbf{p}, t). \tag{5.2}$$

A finite-mode model is obtained if the sums in (5.1) and (5.2) are truncated to $|\mathbf{k}| \ll K$ by the constraint

$$\mathbf{u}(\mathbf{k}, t) = 0 \quad (|\mathbf{k}| > K).$$

The result is the "cutoff" Navier–Stokes equation

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k}, t) = -\frac{1}{2} P_{\alpha\beta\gamma}(\mathbf{k}) \sum_K{}' u_\beta(\mathbf{p}, t) u_\gamma(\mathbf{k} - \mathbf{p}, t), \tag{5.3}$$

where Σ_K' indicates sum over all wavevectors $\mathbf{p}$ satisfying $|\mathbf{p}| \ll K$ and $|\mathbf{k} - \mathbf{p}| \ll K$.

It may easily be verified that, when $\nu = 0$, (5.3) conserves kinetic energy,

$$\frac{\partial}{\partial t}\frac{1}{2}\sum_{|\mathbf{k}|<K}|\mathbf{u}(\mathbf{k})|^2 = 0. \tag{5.4}$$

Even when $\nu > 0$, the rate of energy dissipation $\epsilon_K(t)$ is severly limited by the spectral cutoff K. In fact, it is easy to show that

$$\frac{\partial}{\partial t}\frac{1}{2}\sum_{|\mathbf{k}|<K}|\mathbf{u}(\mathbf{k},t)|^2 = -\epsilon_K(t)$$

$$0 \ll \epsilon_K(t) = \nu\sum_{|\mathbf{k}|<K}k^2|\mathbf{u}(\mathbf{k},t)|^2 \ll \nu K^2\sum_{|\mathbf{k}|<K}|\mathbf{u}(\mathbf{k},t)|^2 \tag{5.5}$$

so that (5.5) implies, since $\epsilon_K(t) \gg 0$,

$$0 \ll \epsilon_K(t) \ll \nu K^2\sum_{|\mathbf{k}|<K}|\mathbf{u}(\mathbf{k},0)|^2. \tag{5.6}$$

Consequently, $\epsilon_K = O(\nu)$ as $\nu \to 0$, in contrast to the result (2.49) that $\epsilon = O(1)$ as $\nu \to 0$ in turbulence. In other words, the limits $\nu \to 0$ and $K \to \infty$ do not commute.

This result has important consequences for numerical simulations of turbulence, which must use some kind of finite discretized approximation to the Navier–Stokes equations. In order to simulate properly the dynamics of the energy-containing range, it is necessary that the simulation give accurate values of $\epsilon(t)$. This requires inclusion of all scales that contribute appreciably to ϵ or, applying the theory of §3.1, $K \gg k_d$. Since $k_d L_p = O(R^{3/4})$ where $R = v_{rms}L_p/\nu$, it follows that the range of scales that must be included scales as $R^{3/4}$. Since space is three-dimensional, the number of degrees of freedom that must be retained scales as $R^{9/4}(\propto R_\lambda^{9/2})$. This growth with R limits direct numerical simulation to rather moderate Reynolds numbers. On the other hand, (5.6) implies that if $k_d \gtrsim K$, the damping is weak and the behavior of the system differs drastically from turbulence. In the limit, $K \ll k_d$, the behavior of the system resembles weakly-damped equilibrium statistical mechanics. A faithful simulation of turbulence is obtained only when $K \gtrsim k_d$.

5.2 Inviscid Equipartition Ensembles

The formal dynamical property of existence of inviscid equipartition ensembles follows by application of Gibbs' statistical mechanics to the cutoff Navier–Stokes equation (5.3) with $\nu = 0$ [Hopf, 1952; Lee, 1952; Kraichnan, 1958]. The essential fact that must be established here is that Liouville's

theorem is satisfied by (5.3). Using this and the property of conservation of energy (5.4), the existence of equipartition ensembles follows easily.†

Liouville's theorem may be verified by first resolving $u(k, t)$ into its real and imaginary parts,

$$u_\alpha(\mathbf{k}, t) = \hat{a}_\alpha(\mathbf{k}, t) + i\hat{b}_\alpha(\mathbf{k}, t) \tag{5.7}$$

so that incompressibility and reality imply, respectively,

$$k_\alpha \hat{a}_\alpha(\mathbf{k}, t) = k_\alpha \hat{b}_\alpha(\mathbf{k}, t) = 0 \tag{5.8}$$

$$\hat{a}_\alpha(\mathbf{k}, t) = \hat{a}_\alpha(-\mathbf{k}, t), \quad \hat{b}_\alpha(\mathbf{k}, t) = -\hat{b}_\alpha(-\mathbf{k}, t). \tag{5.9}$$

On account of (5.8) and (5.9), not all Fourier components are independent. In order to find a suitable set of independent components, we introduce two orthogonal *unit* vectors, $\mathbf{n}_1(\mathbf{k})$ and $\mathbf{n}_2(\mathbf{k})$, in the plane whose normal is $\mathbf{k}$, so that $\mathbf{k}$, $\mathbf{n}_1(\mathbf{k})$, $\mathbf{n}_2(\mathbf{k})$ form an orthogonal triad. Then, for each $\mathbf{k}$, independent components of $\hat{\mathbf{a}}(\mathbf{k}, t)$, $\hat{\mathbf{b}}(\mathbf{k}, t)$ are defined by

$$\begin{aligned} a_r(\mathbf{k}, t) &= \mathbf{n}_r(\mathbf{k}) \cdot \hat{\mathbf{a}}(\mathbf{k}, t) \qquad (\mathbf{r} = 1,2) \\ b_r(\mathbf{k}, t) &= \mathbf{n}_r(\mathbf{k}) \cdot \hat{\mathbf{b}}(\mathbf{k}, t) \end{aligned} \tag{5.10}$$

The components (5.10) are not all independent functions of $\mathbf{k}$ because (5.9) must hold. A complete set of independent components $a_r, b_r (r = 1, 2)$ corresponds to the set of wavevectors $I_K = \{\mathbf{k} \| \mathbf{k}| < K$ and $k_1 > 0$ or $k_1 = 0$, $k_2 > 0$ or $k_1 = k_2 = 0$, $k_3 > 0\}$. It is sometimes more convenient to use as independent variables $u_\alpha(\mathbf{k}, t)$ and $u_\alpha(-\mathbf{k}, t)$ regarded as *independent*, rather than the a_r, b_r amplitudes of the set I_K.

The cutoff Navier–Stokes equation (5.3) may be rewritten in terms of $\hat{a}_r(\mathbf{k}, t)$, $\hat{b}_r(\mathbf{k}, t)$ $[\mathbf{k} \in I_K, r = 1, 2]$ to give equations for $\dot{a}_r(\mathbf{k}, t) = \partial a_r(\mathbf{k}, t)/\partial t$ and $\dot{b}_r(\mathbf{k}, t)$. It may be verified that when $\nu = 0$

$$\sum_{\mathbf{k} \in I_K} \sum_{r=1}^{2} \left[\frac{\partial \dot{a}_r(\mathbf{k}, t)}{\partial a_r(\mathbf{k}, t)} + \frac{\partial \dot{b}_r(\mathbf{k}, t)}{\partial b_r(\mathbf{k}, t)} \right] = 0. \tag{5.11}$$

Equation (5.11) is the essential content of Liouville's theorem and possesses a simple physical interpretation developed below.

We introduce a phase space Γ defined as an N-dimensional Euclidean space, where N is four times the number of elements of I_K, with the N coordinates of a point $P \in \Gamma$ being the excitations in the N independent $a_r(\mathbf{k})$, $b_r(\mathbf{k})$. At each instant of time, the state of a solution to the cutoff Navier–Stokes equations, i.e., the amplitudes $a_r(\mathbf{k})$, $b_r(\mathbf{k})$ $[\mathbf{k} \in I_K, r = 1, 2]$, is specified by a point $P \in \Gamma$. The complete time evolution of a solution to (5.3) is

†Unless stated otherwise, the discussion in §§5.2–5.9 applies only to inviscid $\nu = 0$ dynamics.

depicted by a continuous sequence of points in Γ, i.e., a curve in phase space. An ensemble of realizations, each of which satisfies (5.3), is represented at each instant of time by a cluster of points in phase space. The evolution of the ensemble in time is represented by evolution of the cluster in phase space. The important fact the follows from (5.11) is that, during evolution of an ensemble of solutions to (5.3), the volume of the phase-space cluster representing the ensemble remains invariant. This result follows because an infinitesimal phase-space volume element is expanded from time 0 to time t by the magnitude of the Jacobian $J(t\,|\{a_r(\mathbf{k},0), b_r(\mathbf{k},0)\})$ of the transformation from $a_r(\mathbf{k},0)$, $b_r(\mathbf{k},0)$ $[\mathbf{k} \in I_K, r = 1, 2]$ to $a_r(\mathbf{k},t)$, $b_r(\mathbf{k},t)$ $[\mathbf{k} \in I_K, r = 1, 2]$. The Jacobian J is identically unity for all t and all starting values since $J(t + \Delta t\,|\{a_r(\mathbf{k},0) b_r(\mathbf{k},0)\}) = J(t\,|\{a_r(\mathbf{k},0), b_r(\mathbf{k},0)\})\, J(\Delta t\,|\{a_r(\mathbf{k},t), b_r(\mathbf{k},t)\})$ and $J(\Delta t\,|\{a_r, b_r\}) = 1 + 0(\Delta t^2)$, the latter result following from $J(0\,|\{a_r, b_r\}) = 1$ and $\mathrm{d}J(0\,|\{a_r, b_r\})/\mathrm{d}t = 0$ by (5.11).

While (5.11) is the essential content of Liouville's theorem, it is more usual to formulate the theorem using the complete probability distribution function of the velocity field. We introduce the single-time probability distribution $F(\{a_r(\mathbf{k},t), b_r(\mathbf{k},t)\,|\,\mathbf{k} \in I_K, r = 1, 2\}; t)$ defined in the usual way as the probability distribution for the cluster points of an ensemble in the phase space Γ. It is sometimes convenient to consider F as a function of independent $u_\alpha(\mathbf{k},t)$, according to the convention given previously. It is assumed that F is normalized to unity so that $\int F\,\mathrm{d}\mu_K = 1$, where the volume element in phase space is given by

$$\mathrm{d}\mu_K = \prod_{\mathbf{k}\in I_K} \prod_{r=1}^{2} \mathrm{d}a_r(\mathbf{k})\,\mathrm{d}b_r(\mathbf{k}). \tag{5.12}$$

Any single-time (simultaneous) moment of the velocity field is expressible as an integral of F. Realizability of the ensemble requires $F \geqslant 0$. In the limits $L \to \infty$ (L is the periodicity length) or $K \to \infty$, integrals of F involve functional integration over an infinite number of degrees of freedom and their general evaluation is troublesome.

Conservation of probability in phase space requires that F satisfy

$$\frac{\partial F}{\partial t} + \sum_{\mathbf{k}\in I_K} \sum_{r=1}^{2} \left[\frac{\partial}{\partial a_r(\mathbf{k})} (\dot{a}_r(\mathbf{k}) F) + \frac{\partial}{\partial b_r(\mathbf{k})} (\dot{b}_r(\mathbf{k}) F) \right] = 0.$$

With (5.11), there obtains Liouville's theorem

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \sum_{\mathbf{k}\in I_K} \sum_{r=1}^{2} \left[\dot{a}_r(\mathbf{k}) \frac{\partial F}{\partial a_r(\mathbf{k})} + \dot{b}_r(\mathbf{k}) \frac{\partial F}{\partial b_r(\mathbf{k})} \right] = 0. \tag{5.13}$$

Here $\mathrm{d}/\mathrm{d}t$ is the convective derivative along an orbit in phase space, an "orbit" being a complete flow evolving according to (5.3). The interpretation of (5.13) is that the density F remains constant along an orbit in phase space.

The general time-independent solution of (5.13) is a function of the con-

stants of motion of (5.3). If (5.3) involves N independent components, then there are $N-1$ independent integrals $C_n(\{a_r, b_r\})$, $n = 1, \ldots, N-1$, that do not involve time explicitly and satisfy $dC_n/dt = 0$ whenever $\{a_r, b_r\}$ satisfy (5.3). The Nth integral of motion corresponds to the irrelevant origin of time in (5.3). Since F is constant along orbits in phase space, while $\{C_n\}$ is the largest set of independent quantities that are constant along orbits and do not involve time explicitly, if follows that if F is stationary then F must be a function of $\{C_n\}$. The result that time-independent solutions of (5.13) depend only on $\{C_n\}$ is sometimes called Jeans' theorem.

One integral of motion is particularly important, viz., the energy

$$E = \sum_{\mathbf{k}\in I_K} \sum_{r=1}^{2} [a_r(\mathbf{k})^2 + b_r(\mathbf{k})^2] \tag{5.14}$$

which is conserved by (5.3). Therefore, $F = f(E)$, with arbitrary f, is a time-independent solution of (5.13) that describes an equilibrium ensemble of solutions to (5.3) invariant under time evolution. A special, but important, choice of f is an exponential giving the Gaussian equipartition ensemble

$$F = \mathcal{N} \exp[-E/C] \tag{5.15}$$

where $\mathcal{N}$ is a normalizing constant and C is an arbitrary constant. The distribution (5.15) has the property

$$\langle u_\alpha(\mathbf{k}, t) u_\beta(-\mathbf{k}, t)\rangle = CP_{\alpha\beta}(\mathbf{k}) \quad (|\mathbf{k}| \ll K). \tag{5.16}$$

The factor $P_{\alpha\beta}(\mathbf{k})$ preserves incompressibility. Equation (5.16) justifies referring to (5.15) as an equipartition ensemble, since each mode has equal energy associated with it. Similarly, it may be verified from (5.15) that all higher-order cumulants are identically zero. A little algebra shows that the appropriately cutoff hierarchy of cumulant equations given in §2.4 are satisfied by $S_{\alpha\beta}(k) = CP_{\alpha\beta}(k)$ for $|\mathbf{k}| \ll K$ with all cumulants of order three and higher identically zero.

In the limit $K \to \infty$, the equipartition ensemble has quite singular properties. The total energy density diverges at large wavenumber as $K \to \infty$ (an "ultraviolet catastrophe"). Also, as $K \to \infty$, the velocity-correlation tensor $R_{\alpha\beta}(\mathbf{r})$ is asymptotically proportional to $[4\pi\delta_{\alpha\beta}\delta(\mathbf{r}) + (\partial^2/\partial r_\alpha \partial r_\beta)(1/r)]$. These singularities are further evidence that $\nu \to 0$ and $K \to \infty$ do not commute. The proper formulation of equipartition ensembles is in terms of the cutoff Navier–Stokes equations.

The Gaussian equipartition ensemble (5.15) is not as special as it may at first seem. Two complementary arguments support the importance of (5.15). The first argument, which is really a hypothesis, is that the phase-space flow is ergodic on surfaces of constant energy. If F varies smoothly as a function of its phase-space coordinates and is time-independent, then it follows that

F must be a function of only the time-independent isolating integrals. Thus, if E is the only isolating integral, then $F = f(E)$.

The second point concerns the fact that, with a large number N of degrees of freedom ($N \propto K^3L^3$), a large class of plausible phase-space distributions have low-order moments that are indistinguishable from the low-order moments of the Gaussian distribution (5.15). This result is just a special case of the central limit theorem (Khinchin, 1949). It should be noted that the first argument, i.e. ergodicity, requires only that N be large enough that (5.3) with $\nu = 0$ is ergodic on constant energy surfaces, while the second result, i.e. the central limit theorem, requires that $N \to \infty$.

The ergodicity properties of the inviscid cutoff Navier–Stokes equations have been studied by numerical solution of (5.3). Some results of a three-dimensional computer experiment (Orszag and Patterson, 1972) with $K = 16$ involving 4,096 Fourier modes to represent each velocity component are plotted in Fig. 5.1. In the figure, $\overline{U}(k_0)$ is the average modal energy $|\mathbf{u}(\mathbf{k})|^2$ for $\mathbf{k}$ in the band $k_0 - 1 \leqslant |\mathbf{k}| < k_0 + 1$. Arbitrary initial energy distributions should evolve towards the equipartition spectrum (5.16) as $t \to \infty$ if the system mixes. The numerical results plotted in Fig. 5.1 are consistent with this behavior, but they are by no means conclusive.

The situation in two-dimensions is somewhat more complicated. The system (5.3) then has at least two isolating integrals of motion, energy (5.4) and enstrophy $\Omega = \Sigma k^2 E(k, t)$. As noted in §3.4, the two-dimensional Euler equations have an infinity of other simple integrals of motion. However, most (if not all) of these are lost by the spectral truncation imposed on (5.3). If energy and enstrophy are the only remaining isolating integrals, then the limiting distribution, analogous to (5.15), is

$$F = \mathcal{N} \exp(-E/C - \Omega/D),$$

where C and D are arbitrary constants and $\mathcal{N}$ is a normalizing constant. It follows that (Kraichnan, 1967)

$$\langle u_\alpha(\mathbf{k}, t) u_\beta(-\mathbf{k}, t) \rangle = \frac{CD}{D + Ck^2} P_{\alpha\beta}(\mathbf{k}). \tag{5.17}$$

Arbitrary initial energy distributions should evolve towards equilibrium spectra of this form as $t \to \infty$ if the two-dimensional system mixes. Fox and Orszag (1973) report numerical simulations with $(128)^2$ modes to test this behavior. Their results are consistent with approach to the equilibrium spectrum (5.17) as $t \to \infty$.

Thompson (1972, 1973) has considered the equilibria of forced two-dimensional turbulence. He finds a k^{-3} energy spectrum at high wavenumber and a k^{-1} spectrum at low wavenumbers.

Kraichnan (1974b) has studied the self-consistency of the two dimensional equilibria (5.17) and their relation to inviscid discrete-vortex models (Joyce and Montgomery, 1973; Edwards and Taylor, 1974).

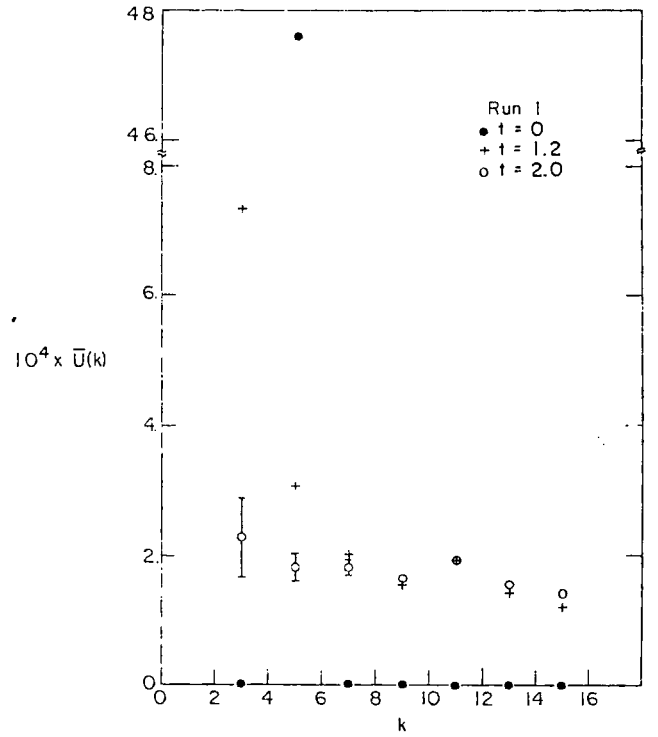

Figure 5.1 Evolution of modal energy spectra $U(k) = E(k)/2\pi k^2$ vs k. Here the cutoff $K = 16$ and $\nu = 0$. The initial energy spectrum is zero, except in the wavenumber band $4 \ll k < 6$. Gaussian random initial conditions, constructed as indicated at the end of §2.3, are used, with $v_{rms}(0) = 0{\cdot}9428$, $L_p(0) = 0{\cdot}4956$. (For further details, see Orszag and Patterson, 1972.)

Finally, it should be emphasized that the equilibrium spectra (5.16) and (5.17) have little direct connection with turbulence. They relate only to a dissipation-free system to which classical equilibrium statistical mechanics applies, while turbulence is a critically damped system to which equilibrium statistics does not apply. Ruelle and Takens (1971) have recently made progress in developing a theory of ergodic behavior appropriate to dissipative systems like turbulence, based on their notion of "strange attractors".

5.3 The Fluctuation-Dissipation Theorem

The fluctuation-dissipation theorem (Callen and Greene, 1952; Kubo, 1959; Kraichnan, 1958) relates the fluctuations of a conservative dynamic system in equilibrium to the generalized susceptibility of the system to an external force. The remarkable feature of the theorem is that it equates a property of

equilibrium, viz. fluctuations, with an irreversible process, viz. the relaxation (dissipation) of a perturbation. We prove the theorem for a general dynamical system with quadratic integral of motion, following Kraichnan (1958). The proof is given in detail here not because turbulence is in statistical equilibrium which it is not, but because the result points out the fundamental importance of Green's functions in nonequilibrium statistical mechanics.

Consider two unrelated conservative systems satisfying

$$\frac{dx_\alpha}{dt} = X_\alpha(\mathbf{x}, t), \quad \frac{dy_n}{dt} = Y_n(\mathbf{y}, t) \tag{5.18}$$

with $\alpha = 1, \ldots, N$ and $n = 1, \ldots, M$. It is supposed that these systems satisfy Liouville's theorem

$$\sum_{\alpha=1}^{N} \frac{\partial X_\alpha}{\partial x_\alpha} = 0, \quad \sum_{n=1}^{M} \frac{\partial Y_n}{\partial y_n} = 0 \tag{5.19}$$

and conserve energy,

$$E_1 = \frac{1}{2}\sum x_\alpha^2 \qquad \frac{dE_1}{dt} = \sum x_\alpha X_\alpha = 0$$

$$E_2 = \frac{1}{2}\sum y_n^2 \qquad \frac{dE_2}{dt} = \sum y_n Y_n = 0.$$

Let $\mathbf{x}(t)$ be a solution of (5.18). Now consider a system that differs from (5.18) by an infinitesimal external force $\delta\mathbf{f}(t)$ that vanishes rapidly as $t \to -\infty$, so there exists a solution of the modified system

$$\dot{x}'_\alpha = X_\alpha(\mathbf{x}', t) + \delta f_\alpha(t)$$

with $\lim_{t\to-\infty} x'_\alpha(t)/x_\alpha(t) = 1$. If $\delta\mathbf{f}$ is infinitesimal, the perturbation $\delta\mathbf{x}(t) = \mathbf{x}'(t) - \mathbf{x}(t)$ is *linearly* related to $\delta\mathbf{f}$, at least for finite t. Therefore, it is possible to introduce the generalized susceptibility $g_{\alpha\beta}(t, t')$ defined by

$$\delta x_\alpha(t) = \sum_{\beta=1}^{N} \int_{-\infty}^{\infty} g_{\alpha\beta}(t, t')\delta f_\beta(t')\,dt' + o(|\delta\mathbf{f}|). \tag{5.20}$$

Causality requires that $\delta x_\alpha(t)$ depend only on $\delta\mathbf{f}(t')$ for $t' \ll t$, so that $g_{\alpha\beta}(t, t') = 0$ for $t < t'$. Also, the special choice $\delta f_\alpha(t) = \epsilon\delta_{\alpha\beta}\delta(t - t')$ with ϵ infinitesimal gives $\delta x_\alpha(t) = \epsilon g_{\alpha\beta}(t, t')$. Therefore, $\epsilon g_{\alpha\beta}(t, t')$ is the response of x_α at time t to a small perturbation of strength ϵ introduced in x_β at time t'. In particular, it follows that $g_{\alpha\beta}(t' + 0, t') = \delta_{\alpha\beta}$. In view of this interpretation, it is natural to call $g_{\alpha\beta}(t, t')$ the infinitesimal-impulse-response tensor (Kraichnan, 1958). It is also called the Green's function. Similarly, we introduce the generalized susceptibility $h_{nm}(t, t')$ as the response in $y_n(t)$ to an infinitesimal perturbation in $y_m(t')$.

It is important to note that $g_{\alpha\beta}(t, t')$ $[h_{nm}(t, t')]$ depends implicitly on the

particular solution $x(t)[y(t)]$ of (5.18) that is perturbed, since $X(x,t)[Y(y,t)]$ is not necessarily linear in $x[y]$.

The fluctuation-dissipation theorem states that, in equilibrium,

$$\langle x_\alpha(t)x_\beta(t')\rangle = C\langle g_{\alpha\beta}(t, t')\rangle, t \gg t' \tag{5.21}$$

with C independent of α, β, t, t'. Before proving (5.21), it is helpful to consider the following heuristic argument. Define $\langle x'_\beta, t'|x_\alpha, t\rangle$ as the average value of x_α at time t given that $x_\beta(t') = x'_\beta$ precisely. The expectation $\langle x'_\alpha, t'|x_\alpha, t\rangle$ is called the regression of x_α (Callen and Greene, 1952). Now, for $t > t'$, it follows from the foregoing definition that

$$\langle x_\alpha(t)x_\beta(t')\rangle = \int x'_\beta\langle x'_\beta, t'|x_\alpha, t\rangle\, p_\beta(x'_\beta)\,dx'_\beta,$$

where $p_\beta(x_\beta)$ is the probability distribution of x_β allowing x_γ, $\gamma \neq \beta$, to be arbitrary. The distribution $p_\beta(x_\beta)$ is independent of time in equilibrium. But $\langle x'_\beta, t'|x_\alpha, t\rangle$ may be computed by imposing some kind of force on the system (5.18) to ensure that $x_\beta(s) = x'_\beta$ for $s < t'$. If the force is lifted at t', the average value of x_α observed at time t is $\langle x'_\beta, t'|x_\alpha, t\rangle$. However, the effect of removing a force at time t' is clearly related to the Green's function, so that there is basis to expect a relation such as (5.21).

We begin the proof of (5.21) by introducing an ensemble of realizations of the systems (5.18). Suppose that the time-independent phase-space distributions of systems x, y are, respectively,

$$f_1(\mathbf{x}) = N_1 \exp(-E_1/C), \quad f_2(\mathbf{y}) = N_2 \exp(-E_2/C), \tag{5.22}$$

where N_1 and N_2 are normalization factors. The distributions (5.22) are realized, for example, if x, y are thermodynamic systems in equilibrium at the same temperature T (with the choice $C = kT$, where k is Boltzmann's constant). Since the x and y systems are independent of each other, the time-independent phase-space distribution in the product phase-space $\mathbf{x} \otimes \mathbf{y}$ of x and y together is

$$F(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x})f_2(\mathbf{y}), = N_1N_2 \exp[-(E_1 + E_2)/C]. \tag{5.23}$$

The density $F(\mathbf{x}, \mathbf{y})$ is the Gaussian equipartition ensemble for the composite system with total energy $E = E_1 + E_2$, as is appropriate for two noninteracting systems in isolation. In (5.21), and in the following, ensemble averages are intended over these ensembles.

With these formal preliminaries attended to, we introduce a conservative coupling between the x and y systems so that (5.18) is altered to

$$\dot{x}_\alpha = X_\alpha(\mathbf{x}, t) + \epsilon \sum_{b=1}^{M} a_{\alpha n}(t)y_n(t) \qquad (\alpha = 1, \ldots, N)$$

$$\dot{y}_n = Y_n(y, t) - \epsilon \sum_{\alpha=1}^{N} a_{\alpha n}(t)x_\alpha(t) \qquad (n = 1, \ldots, M), \tag{5.24}$$

the proof of (5.21). It is only for these ensembles that the composition law (5.23) yields an ensemble invariant under the perturbation (5.24).

The fluctuation-dissipation theorem is a generalization of a classical theorem due to Nyquist (1928) concerning the thermal noise in an electrical resistance. Nyquist showed that, at temperature T,

$$\langle V^2 \rangle = \frac{2}{\pi} kTR\Delta\omega,$$

where V is the open-circuit voltage fluctuation across the resistance R and $\Delta\omega$ is the frequency bandwidth across which the noise is measured. Takahasi (1952) examines closely the relation between the fluctuation-dissipation theorem and Nyquist's theorem.

5.4 Equilibrium Properties of Cumulant-Discard Closures

The inviscid cutoff Navier–Stokes equations lead to a hierarchy of equations for statistical cumulants that is obtained from the heirarchy of §2.4 by the replacement of wavevector integrals by wavevector sums and the restriction of all wavevectors to the region $|\mathbf{k}| \ll K$. Cumulant-discard closures of this modified hierarchy are obtained as in §4.4 by neglecting the effect of all cumulants above a certain order. For example, the quasinormal closure is

$$\frac{\partial}{\partial t}\langle u_\alpha(\mathbf{k}, t)u_\beta(-\mathbf{k}, t)\rangle = -\frac{i}{2}P_{\alpha\rho\sigma}(\mathbf{k}) \sum_K' \langle u_\rho(\mathbf{p})u_\sigma(\mathbf{k} - \mathbf{p})u_\beta(-\mathbf{k})\rangle + \frac{i}{2}P_{\beta\rho\sigma}(\mathbf{k}) \sum_K' \langle u_\alpha(\mathbf{k})u_\rho(\mathbf{p})u_\sigma(-\mathbf{k} - \mathbf{p})\rangle \quad (5.30)$$

$$\begin{aligned}\frac{\partial}{\partial t}&\langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t)u_\gamma(\mathbf{q}, t)\rangle = \\ &-iP_{\alpha\rho\sigma}(\mathbf{k})\langle u_\beta(\mathbf{p})u_\rho(-\mathbf{p})\rangle\langle u_\gamma(\mathbf{q})u_\sigma(-\mathbf{q})\rangle \\ &-iP_{\beta\rho\sigma}(\mathbf{p})\langle u_\alpha(\mathbf{k})u_\rho(-\mathbf{k})\rangle\langle u_\gamma(\mathbf{q})u_\sigma(-\mathbf{q})\rangle - \\ &- iP_{\gamma\rho\sigma}(\mathbf{q})\langle u_\alpha(\mathbf{k})u_\rho(-\mathbf{k})\rangle\langle u_\beta(\mathbf{p})u_\sigma(-\mathbf{p})\end{aligned} \quad (5.31)$$

where $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$.

All these cumulant-discard closures possess time-independent Gaussian equipartition solutions satisfying

$$\begin{aligned}\langle u_\alpha(\mathbf{k}, t)u_\beta(-\mathbf{k}, t)\rangle &= CP_{\alpha\beta}(\mathbf{k}) \\ \langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t)\ldots u_\gamma(\mathbf{q}, t)\rangle &= 0,\end{aligned} \quad (5.32)$$

where C is a constant independent of α, β, $\mathbf{k}$, and t, and $\mathbf{k} + \mathbf{p} + \cdots + \mathbf{q} = 0$ but no nontrivial subset of $\{\mathbf{k}, \mathbf{p}, \ldots, \mathbf{q}\}$ sums to zero.

Consider an arbitrary *integrable* initial ensemble. If (5.3) is mixing, and we assume it is for KL sufficiently large, then it follows that averages over the ensemble approach equilibrium. If the number of degrees of freedom retained in (5.3) is very large, then, according to the argument of §5.2, low-order moments should approach Gaussian equilibrium values. This latter property is not needed for the argument to follow. Instead, the important fact is that the equilibrium state, with an arbitrary number of degrees of freedom, is such that all odd-order moments (and cumulants) are zero. The latter fact follows since the distribution function in equilibrium is a function of energy E alone, but E is a symmetric function of $\{u_\alpha(\mathbf{k})\}$ while odd-order moments are odd functions.

It will be shown that cumulant-discard closures, in particular, the quasi-normal theory, are not consistent with approach to the equilibrium state (5.32) and the notion of ensemble average. The crucial fact is that cumulant-discard closures for the hierarchy derived from the inviscid cutoff Navier–Stokes equations are *time reversible*. Time reversibility means that the transformation

$$
\begin{aligned}
&t \to -t \\
&\langle u_\alpha(\mathbf{k})u_\beta(-\mathbf{k})\rangle \to \langle u_\alpha(\mathbf{k})u_\beta(-\mathbf{k})\rangle \\
&\langle u_\alpha(\mathbf{k})u_\beta(\mathbf{p})u_\gamma(\mathbf{q})\rangle \to -\langle u_\alpha(\mathbf{k})u_\beta(\mathbf{p})u_\gamma(\mathbf{q})\rangle
\end{aligned}
\tag{5.33}
$$

etc., where all odd-order moments change sign while even-order moments remain unchanged, leaves the closure equations invariant. We carry through the argument for the quasi-normal closure, but the same argument applies to closures of every finite order. Suppose that (5.30), (5.31) is solved with nonequilibrium initial values for second- and third-order moments at $t = 0$. If the quasi-normal closure implies approach to equilibrium, there will exist, for arbitrarily small $\delta > 0$, a time t_0 when

$$
|\langle u_\alpha(\mathbf{k}, t_0)u_\beta(\mathbf{p}, t_0)u_\gamma(\mathbf{q}, t_0)\rangle| < \delta
$$

for all retained $\mathbf{k}, \mathbf{p}, \mathbf{q}$, α, β, γ. At $t = t_0$, we apply the *small* perturbation

$$
\begin{aligned}
&\Delta\langle u_\alpha(\mathbf{k}, t_0)u_\beta(-\mathbf{k}, t_0)\rangle = 0 \\
&\Delta\langle u_\alpha(\mathbf{k}, t_0)u_\beta(\mathbf{p}, t_0)u_\gamma(\mathbf{q}, t_0)\rangle = -2\langle u_\alpha(\mathbf{k}, t_0)u_\beta(\mathbf{p}, t_0)u_\gamma(\mathbf{q}, t_0)\rangle
\end{aligned}
$$

so that the perturbed values of second- and third-order moments are

$$
\begin{aligned}
&\langle u_\alpha(\mathbf{k}, t_0)u_\beta(-\mathbf{k}, t_0)\rangle' = \langle u_\alpha(\mathbf{k}, t_0)u_\beta(-\mathbf{k}, t_0)\rangle \\
&\langle u_\alpha(\mathbf{k}, t_0)u_\beta(\mathbf{p}, t_0)u_\gamma(\mathbf{q}, t_0)\rangle' = -\langle u_\alpha(\mathbf{k}, t_0)u_\beta(\mathbf{p}, t_0)u_\gamma(\mathbf{q}, t_0)\rangle,
\end{aligned}
$$

where the prime denotes perturbed value. The time reversibility of (5.30),

(5.31) implies that

$$\langle u_\alpha(\mathbf{k}, t)u_\beta(-\mathbf{k}, t)\rangle' = \langle u_\alpha(\mathbf{k}, 2t_0 - t)u_\beta(-\mathbf{k}, 2t_0 - t)\rangle$$

$$\langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t)u_\gamma(\mathbf{q}, t)\rangle' = -\langle u_\alpha(\mathbf{k}, 2t_0 - t)u_\beta(\mathbf{p}, 2t_0 - t)u_\gamma(\mathbf{q}, 2t_0 - t)\rangle \tag{5.34}$$

as may be verified by substitution. The perturbed system evolves from t_0 to $2t_0$ as the mirror image of the evolution of the unperturbed system from 0 to t_0. According to (5.34), the perturbed solution unwinds from t_0 to $2t_0$, so that at $t = 2t_0$ the perturbed second-order cumulant equals its initial unperturbed value, while the perturbed third-order moment equals the negative of its initial unperturbed value. The conclusion follows that, since the initial conditions at $t = 0$ are quite arbitrary, either (5.30), (5.31) do not give approach to equilibrium or they give approach to equilibrium but are unstable. The latter possibility is not satisfactory, since (5.30), (5.31) are equations for ensemble averages. Ensemble averages should not be extremely sensitive and should not exhibit the instabilities that plague individual realizations—otherwise, someone might get the idea to ensemble average the "turbulence" exhibited by the ensemble averages, etc., ad infinitum.

The argument given above is sufficiently basic to the subject that it deserves a name: the statistical reversibility argument. Some comments are in order concerning the present version of the argument. For one thing, the perturbation applied at $t = t_0$ is justifiably small because the number of perturbed quantities is finite when the number of modes and the order of closure are both finite. Under certain conditions explained in later sections, the restriction on the number of modes may be lifted. The finiteness of the number of perturbed third-order moments also guarantees the existence of a finite t_0 when they are all smaller than δ, if there is approach to equilibrium.

Numerical solution of (5.30), (5.31) shows that equipartition of energy is not approached as $t \to \infty$, the typical behavior being finite persistent oscillations about equilibrium (Orszag, 1970c). Other similar troubles of the quasi-normal theory are discussed by Orszag (1970c).

The cause of these troubles is qualitative and very basic: *Relaxation times for small departures from equilibrium should be determined by the random motions in the equilibrium state, not by the departure from equilibrium.* As discussed in Section IV, the memory integral in (4.30) should involve a dynamical cutoff when R is large, in order to represent properly the effect of the destruction of correlations by nonlinear scrambling in the random convection field. In actual turbulence, departures from equilibrium are large, but it should not be expected that the troubles incurred by cumulant-discard closure's misrepresentation of motions in the equilibrium state will disappear.

The critique of cumulant discard closures presented in this Section is important because it emphasizes the fundamental nature of the difficulties with the quasi-normal theory that were pointed out in Section IV. The difficulties persist to all orders of closure. On the other hand, it is not known whether higher-order cumulant-discard closures lead to negative energy-densities in the manner of the quasi-theory [but see the numerical study for Burgers' equation by Tanaka (1969)].

5.5 Stochastic Relaxation

In the next several sections, we attempt to explain the origin of irreversible dynamical effects in the hierarchy equations. The problem is made particularly acute by the following formal extension of the statistical reversibility argument presented in §5.4. It is easily verified that the hierarchy equations for the inviscid cutoff Navier–Stokes system (5.3) are formally time-reversible. In fact, transforming all odd-order moments (and cumulants) into the negatives of themselves, while keeping all even-order moments unchanged [cf. (5.3)], reverses the evolution of each and every equation of the inviscid hierarchy. It was mentioned in §5.4 that, at equilibrium, all odd-order moments are zero. Therefore, if the hierarchy predicts evolution to equilibrium, as it should, and if there exists a time t_0 when all odd-order moments are smaller than some arbitrarily small predetermined number δ, we can apply the "perturbation" that changes the sign of all odd-order moments while keeping even-order moments unchanged. The effect of this "perturbation" is to reverse the evolution of the complete set of hierarchy equations. The perturbed moments at $t = 2t_0$ equal, except-for a possible change of sign, the unperturbed moments at $t = 0$, which are arbitrary. It seems to follow that either the complete unclosed hierarchy equations cannot predict evolution to equilibrium or the hierarchy equations are unstable. This is a most embarrasing conclusion, since approach to equilibrium *is* expected on the basis of ergodicity arguments while the purpose of the ensemble is to obtain average quantities that are *not* extremely sensitive to perturbations.

Of course, the trouble is that the statistical reversibility argument is incorrect when applied to the unclosed hierarchy. There are two reasons: first, there does not necessarily exist an instant t_0 when *all* odd-order moments are arbitrarily small; and, second, even if such a t_0 did exist, changing the sign of the infinite number of odd-order moments is not necessarily a small perturbation, no matter how small δ. These two criticisms of the statistical reversibility argument applied to the complete hierarchy are illustrated by the following very simple example.

Consider the problem of determining the average behavior of a random

harmonic oscillator (Kraichnan, 1961)

$$\frac{dq_b}{dt} = -ibq_b(t), \tag{5.35}$$

where b is a zero-mean random variable with probability distribution $P(b)$. Since b is random, $q_b(t) = e^{-ibt}$ fluctuates. If $P(b)$ is Gaussian, i.e., if $P(b) = (2\pi < b^2 >)^{-1/2} \exp(-b^2/2 < b^2 >)$, then

$$G(t) \equiv \langle q_b(t) \rangle, \quad = \int q_b(t) P(b) db, \quad = \exp(-\tfrac{1}{2} \langle b^2 \rangle t^2). \tag{5.36}$$

The fact that $G(t) \to 0$ as $|t| \to \infty$ is a consequence of the Riemann–Lebesgue lemma and, therefore, holds true for any integrable distribution $P(b)$. Equation (5.35) is time reversible under the transformation $t \to -t$, $b \to -b$, but $G(t)$ exhibits irreversible decay towards 0 as $|t| \to \infty$. Clearly, each realization does not relax to 0 as $|t| \to \infty$; irreversible relaxation appears only after ensemble averaging.

The random harmonic oscillator problem may be treated in the usual way by constructing moment equations. Defining $G_n(t) = \langle b^n q_b(t) \rangle$, it follows that

$$\frac{dG_n(t)}{dt} = -iG_{n+1}(t) \tag{5.37}$$

for all nonnegative integers n. The system (5.37) is an unclosed hierarchy of equations for the moments G_n.† This hierarchy is formally time-reversible under the transformation $t \to -t$, $G_n \to (-1)^n G_n$. The only possible way that the solutions to be hierarchy equations can exhibit irreversible behavior is by a flow of information through the entire sequence of moments (or cumulants). Since $G_n(t)$ is a single-time average of the random field $q_b(t)$, there is no explicit memory of past dynamical behavior in terms of current values of the moments. The source of dynamical memory is a cooperative flow of information through the entire sequence of cumulants. This "cumulant-space" flow produces correlations that drive the system irreversibly to equilibrium.

If the flow of information from G_{n+1} to G_n through (5.37) is disturbed by a reversible closure, then irreversible relaxation is lost. For example, the closure obtained by arbitrarily setting $G_N = 0$ for some N and using (5.37) for $n = 1, \ldots, N-1$ gives the solution

$$G(t) (= G_0(t)) = \sum_{n=0}^{N-1} \frac{1}{n!} (-it)^n \langle b^n \rangle \tag{5.38}$$

†A closure problem arises even though (5.35) is linear in the dynamical variable q_b, because (5.35) is nonlinear in the stochastic variables b and q_b.

since $G_n(0) = \langle b^n \rangle$, $n = 1, \ldots, N-1$. The solution (5.38) includes the first N terms of the Taylor series expansion of the exact solution (5.36). All these closures give solutions that diverge as $|t| \to \infty$ and, of course, do not relax irreversibly to zero. Similarly, the cumulant-discard closures for the hierarchy (5.37) with Gaussian b are given successively by

$$G_2(t) = \langle b^2 \rangle G_0(t) \tag{5.39a}$$

$$G_3(t) = 3\langle b^2 \rangle G_1(t) \tag{5.39b}$$

$$G_4(t) = 6\langle b^2 \rangle G_2(t) - 3\langle b^2 \rangle^2 G_0(t) \tag{5.39c}$$

and so on. It is apparent from the statistical reversibility argument that none of the closures (5.39) can give relaxation to equilibrium. For example, the closure (5.39a) gives $G(t) = \cos(\langle b^2 \rangle^{1/2} t)$ which oscillates indefinitely. In order to get plausible relaxation from a closure, the closure must be irreversible and this means taking into account the effect of cumulants of all orders.

The fact that the statistical reversibility argument cannot be applied to the unclosed hierarchy is seen as follows. For Gaussian $P(b)$,

$$G_n(t) = i^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} e^{-1/2\langle b^2 \rangle t^2} \tag{5.40}$$

so that $G_n(t)$ does not tend to zero as $t \to \infty$ uniformly in n. In fact, it may be shown that

$$G_n(\langle b^2 \rangle^{-1/2}(4n+2)^{1/2}) \sim$$

$$(-i)^n (2\pi)^{1/2} 3^{2/3} \Gamma(\tfrac{2}{3})^{-1} \langle b^2 \rangle^{n/2} \mathrm{e}^{-1/2(3n+1)} n^{1/2n+1/6}$$

so that $\max_n |G_n(t)|$ becomes unbounded as $t \to \infty$. Therefore the "perturbation" considered in the extension of the statistical reversibility argument given at the beginning of this section is *not* small.

The process by which ensemble averages relax towards equilibrium through the existence of an infinite unclosed hierarchy may be called *stochastic relaxation*. For the random harmonic oscillator probllem, stochastic relaxation is the only possible mechanism available to drive $G(t)$ irreversibly to zero. Stochastic relaxation represents the principal new physics involved in the closure problem—it is the way the hierarchy represents nonlinear scrambling at large Reynolds number. Obviously, its proper treatment is fundamental to a satisfactory theory of turbulence. However, stochastic relaxation is rather unusual and has not received much attention in the literature in comparison with the other kinds of irreversible behavior discussed in the next section. The reason is simple: stochastic relaxation is not important in weakly-nonlinear dispersive systems that may be treated by multiple-scale perturbation theory and these latter

systems have been the subject of most contemporary research in the field of nonequilibrium statistical mechanics.

5.6 Mechanisms of Irreversible Relaxation

There are several ways that irreversible relaxation to equilibrium (or, more generally, relaxation to an asymptotic state of the kind argued for in §1.3) can occur. First, there is the possibility that *every* realization of the ensemble irreversibly relaxes to equilibrium (*without* averaging). This possibility will be termed *explicit irreversibility*. For example, consider the system obtained by neglecting the nonlinear terms of (5.2)

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k}, t) = 0. \tag{5.41}$$

The solution is

$$u_\alpha(\mathbf{k}, t) = u_\alpha(\mathbf{k}, 0)\, e^{-\nu k^2 t}$$

so that the Fourier amplitudes of all realizations decay irreversibly to zero as $t \to \infty$. Formally, (5.41) is explicitly irreversible since changing t into $-t$ changes the form of the equation [since it is not possible to change the sign of ν without violating $\nu > 0$]. Since each realization of an explicitly irreversible equation relaxes, averages over an arbitrary ensemble also show irreversible relaxation. In problems of this type, nothing is gained by averaging, except perhaps if it is necessary to average over fine-grained structure present at the initial instant. Significant new fine-grained structure does not develop. An example of explicitly irreversible dynamics is low-Reynolds-number turbulence which satisfies (5.41) to lowest order (§4.3).

In the cases of interest here, explicit irreversibility is not the principal cause of approach to an asymptotic statistical state. For example, (5.2) is explicitly irreversible due to the presence of viscosity. However, the viscous decay time is $(\nu k^2)^{-1}$, which is enormous if ν is small and k is not too large. On the other hand, nonlinear scrambling is the important relaxation mechanism at large Reynolds numbers.

Since (5.2) is manifestly time-reversible aside from the viscous term [$t \to -t$, $u_\alpha(\mathbf{k}) \to -u_\alpha(\mathbf{k})$ transforms the viscosity-independent terms into themselves] no other cause of explicit irreversibility is available to produce the required relaxation at large R. As discussed in Section 1, irreversible effects appear when the fine-grained structure developed by individual realizations is ensemble averaged. Thus, there is a second type of irreversibility in which individual realizations do not behave irreversibly, but in which statistically-averaged properties behave as if they do.

With explicitly reversible dynamics, the irreversible behavior of ensemble averages is a product of initial conditions. That is, the moment at which

the initial conditions are given is the "past" while the flow evolves into the "future". We define a symmetric ensemble to be such that, at the initial instant $t = t_0$, $\mathbf{v}(\mathbf{x})$ and $-\mathbf{v}(\mathbf{x})$ are given the same weight for all realizations $\mathbf{v}(\mathbf{x})$ of the ensemble. Any zero-mean Gaussian ensemble is symmetric. Assuming that the initial ensemble is symmetric, explicit reversibility of the underlying dynamics implies that averages evolve in the same way for $t - t_0 \to -\infty$ as for $t - t_0 \to \infty$. Relaxation implies that ensemble averages approach equilibrium values as $|t - t_0| \to +\infty$. In other words, relaxation in the "future" means that t_1 is "later" than t_2 if $t_1 > t_2 \geq t_0$ *or* $t_1 < t_2 \leq t_0$. The similar evolution for $t - t_0 \to \infty$ and $t - t_0 \to \infty$ indicates time symmetry. On the other hand, irreversible relaxation as $|t - t_0| \to \infty$ implies that for any $t_1 \neq t_0$ ensemble averages evolve in a different way for $t - t_1 > 0$ than for $t - t_1 < 0$. Evidently, an ensemble symmetric at $t = t_0$ is not symmetric for $t \neq t_0$.

Clearly, time reversibility of the underlying dynamics precludes monotonic approach to an asymptotic state for arbitrary ensembles. If some ensemble approaches an asymptotic state monotonically as $t - t_0 \to \infty$, as exhibited by, say, an "H-theorem", the time-reversed ensemble obtained by reversing the velocity of each realization of the given ensemble at time $t_1 > t_0$ does not exhibit monotonic relaxation. In fact, at $t = t_1 + (t_1 - t_0)$, the time-reversed ensemble is obtained from the given initial ensemble ($t = t_0$) by reversing all velocities. Thus, the time-reversed ensemble at $t = 2t_1 - t_0$ is almost surely not closer to an asymptotic state than at $t = t_1$. It follows from this argument that the strongest statement we can hope to make is that *typical* ensembles exhibit regular approach to an asymptotic state.

The purpose of ensemble averaging is to remove the fine-grained structure of the individual realizations. In order to be successful, the ensemble must not be too trivial. If we label realizations as in §1.3 by a parameter α and choose the distribution function to be

$$P(\alpha) = \frac{1}{N} \sum_{i=1}^{N} \delta(\alpha - \alpha_i) \tag{5.42}$$

i.e., an ensemble of N distinct realizations, ensemble averages of quantities with fine-grained structure will still have fine-grained structure. With (5.42), an ensemble average is just a finite arithmetic mean over the N realizations. In order for an ensemble to smooth fine-grained structure, $P(\alpha)$ cannot consist of a finite number of line spectra. Rather, $P(\alpha)$ must have some nontrivial structure, such as bands $\alpha_0 < \alpha < \alpha_1$ where $P(\alpha) > 0$ for all α in the band. For such "smoothing" ensembles, irreversible relaxation may be anticipated.

As stated above, the evolution of an ensemble-averaged quantity reverses its evolution if *every* realization of the ensemble is time reversed.

However, for a smoothing ensemble, this reversal requires than an infinite number of realizations be time reversed and this is not expected to be a typical result of evolution. This point is perhaps made more clearly by considering Poincaré's recurrence theorem. The Poincaré recurrence theorem (Chandrasekhar, 1943, Appendix V) states that if a finite-mode system (actually, any system with a constant-energy surface of finite measure) passes through the sequence of phase-space points $P_0, P_1, \ldots, P_n$ at the time $t_0, t_1, \ldots, t_n$, *respectively, then for any* $\delta > 0$ and almost every P_i there exists a time T such that $|P_i' - P_i| < \delta$ for $i = 0, \ldots, n$ where P_i' is the state of the system at $t = t_i + T$. Consequently, those properties of a finite-mode system that depend on the state of a realization at a finite number of times cannot relax to equilibrium and, in fact, have arbitrarily close recurrences to their initial values arbitrarily often. However, while nonsmoothing ensembles such as (5.42) must preserve the property of arbitrarily close recurrences for averages, smoothing ensembles may exhibit relaxation. The point is that a smoothing ensemble is analogous to a system with an infinite number of degrees of freedom, corresponding to the possible choices of α. While the Poincaré recurrence time T is finite for each realization of a smoothing ensemble for a finite mode system, there is no time when all the realizations need recur together. For example, each realization of the random harmonic oscillator problem (5.35) has an exact recurrence time $2\pi/|b|$, but there is no time when all the realizations corresponding to a smooth distribution of b's recur together. Consequently, $G(t)$ exhibits irreversible relaxation to 0 despite the recurrences of $q_b(t)$.

Since the formal structure of the cumulant-hierarchy equations does not depend on the ensemble, it follows from the existence of nonsmoothing ensembles that the hierarchy equations by themselves cannot give approach to equilibrium. Evidently, the initial conditions for cumulants of all orders provide the information necessary to distinguish a smoothing ensemble from an ensemble of the form (5.42).

Assuming that irreversible behavior results for the ensemble under consideration, the problem remains of explaining the source of the relaxation in the language of the hierarchy equations. The point is that irreversible behavior may result whenever information can get lost. For the hierarchy of §2.4 with ν so small that explicit irreversibility is not important, the possible sources of irreversible behavior are (i) a flow of information through the hierarchy from nth-order to $(n+1)$-order to "infinite-order" cumulants, (ii) the continuous range of the wavevector integrations in (2.32), (2.33), and (iii) the infinite range of wavevector integrations in (2.32), (2.33). Case (i), the flow of information through "cumulant space" by means of an unclosed hierarchy, is stochastic relaxation that

was discussed in §5.6. In case (ii), information is lost in the averaging process implied by integration, or more abstractly, in infinitesimal volume elements in k-space. This mechanism, which we call *dynamical incoherence*, is discussed briefly below. In case (iii), the possibility is that energy transfer to infinite wavenumber is accompanied by loss of information. This latter process will be called *cascade relaxation*. Notice that the three possible sources of irreversibility in the cumulant hierarchy are the "infinities" and "infinitesimals" associated with the hierarchy. It is only at an infinity (in wavenumber or "cumulant" space) or at an infinitesimal (wavevector-integration element) that information can get permanently and irrevocably lost, with the result irreversible relaxation.

Dynamical incoherence involves the loss of information by integration of a rapidly oscillating function, the prototype of which is the Riemann–Lebesque lemma

$$\lim_{|t|\to\infty} \int f(k) e^{ikt}\, dk = 0$$

for any integrable function f. Another example is Landau damping of plasma waves. Dynamical incoherence is the dominant mechanism of irreversibility in systems of weakly-nonlinear dispersive waves (Hasselmann, 1962; Benney and Saffman, 1966) and weakly interacting kinetic systems (Balescu, 1963). For these systems, closed kinetic equations for cumulants are derivable without closure assumptions in the asymptotic limit where a generalized Reynolds number (coupling constant) is small. These kinetic equations are derivable either by multiple-scale perturbation theory (Benney and Saffman, 1966) or partial summation of the "most divergent" terms of formal perturbation theory (Balescu, 1963).

In the case of approach to equipartition by solutions of the hierarchy equations for the inviscid, cutoff, Navier–Stokes equations, the only possible source of irreversible behavior is stochastic relaxation. With finite periodicity-box volume, the modes of the cutoff equations are discrete so that dynamical incoherence is impossible. Further, the cutoff prevents cascade relaxation, so that the only alternative is that relaxation is due to stochastic relaxation. In the limit of infinite periodicity-box volume $[L \to \infty]$, the set of permitted wavevectors becomes dense in the region $|k| \ll K$, so that dynamical incoherence cannot be ruled out *a priori*. However, since cumulants are averages which should remain *smooth* functions of their arguments, no information is irrevocably lost upon integration. Consequently, even in the limit $L \to \infty$, relaxation to equipartition can only be effected by stochastic relaxation.

In the case of approach to an asymptotic statistical state by solutions of the cumulant hierarchy for the Navier–Stokes equations, all three mechanisms

of irreversible behavior are possible *a priori*. However, dynamical incoherence can be ruled out because the cumulants for the Navier–Stokes equations, in common with the cumulants for the cutoff equations, must be smooth functions of wavevector. It is not so easy to rule out cascade relaxation as an important mechanism. In fact, energy cascade *is* important in determining the kind of asymptotic statistical state that develops, as should be amply clear from the discussion of Sections II, III.

The mechanism involved in cascade relaxation may be put roughly as follows. An arrow of time, i.e., a characteristic that distinguishes future from past, is provided by the rate of energy transfer which is typically outwards in Fourier space. At any given instant, the ensemble obtained by time-reversing all realizations of the given ensemble has energy transfer inwards. Since outwards transfer follows from the general argument given below, the reversed ensemble is not expected in typical evolution, and irreversible relaxation may be exhibited. Energy flow typically outwards is deduced from the eventual importance of viscosity at very large wavenumbers. Viscous action at large k means that high wavenumbers are typically excited less than is required for equipartition with low wavenumbers. However, since the ergodic (mixing) nature of nonlinear interaction tends toward equipartition, energy flow typically outwards in Fourier space results. Otherwise said, the mechanism of cascade relaxation is simply that fluctuations are transferred by nonlinear interaction to large k where they may easily be dissipated by viscosity.

At this time, it is not possible to state with certainty the relative effectiveness of cascade relaxation versus stochastic relaxation. However, it seems most likely that cascade relaxation is the secondary effect. If cascade relaxation were dominant over stochastic relaxation in the hierarchy, there would seem no need to introduce ensembles, because the possibility for cascade relaxation exists already with the infinite range of wavenumbers included in the Navier–Stokes equations. For example, if shutting off stochastic relaxation by imposing a formally reversible closure can give a plausible turbulence theory, then it would seem that cascade relaxation should also succeed in preventing individual realizations of the Navier–Stokes equations from becoming unstable and thence turbulent. However, the most telling argument against the dominance of cascade relaxation is that, in practice, imposing formally reversible closures on the hierarchy leads to results that are, in general, disastrous (Orszag, 1966).

In summary, it appears that a turbulence theory must properly account for stochastic relaxation, in the sense that a satisfactory, even though approximate, theory must account for effects of cumulants of *all* orders. It appears that without stochastic relaxation, a turbulence theory will itself exhibit "turbulence", i.e., the low-order moments of the theory will not relax to a (statistically) stable asymptotic state.

VI The Direct-Interaction Approximation

6.1 Direct-Interaction Approximation

The direct-interaction approximation (DIA), developed by Kraichnan (1958, 1959), is the only fully self-consistent analytical turbulence theory yet discovered. While its predictions do not accord with experiment at very high Reynolds numbers, the insights that it has given into the nature of turbulence are many and important. It is the only theory to account for nonlinear scrambling and stochastic relaxation in a fundamental way.

We first derive the approximation using the model dynamical equation

$$\frac{dy_i}{dt} + \nu_i y_i = \sum_{j,k} A_{ijk} y_j y_k \tag{6.1}$$

instead of the Navier–Stokes equations. Incompressibility, symmetry, and conservation of energy by nonlinear interaction require that the interaction coefficients satisfy, respectively,

$$A_{ijj} = 0 \tag{6.1a}$$

$$A_{ijk} = A_{ikj} \tag{6.2b}$$

$$A_{ijk} + A_{jki} + A_{kij} = 0. \tag{6.2c}$$

The Navier–Stokes equations can be reduced to a model of this form using the polarization vector representation of the velocity field developed in §5.2 to eliminate the longitudinal (compressive) degrees of freedom.

On the basis of the discussion of Sections IV and V, it is plausible to assert that the effect of the sea of excited Fourier modes (or eddies) on a given mode is to give both an effective eddy dissipation of the energy out of the mode and an effective eddy diffusion of energy into the mode. Therefore, it reasonable to try to account explicitly for these effects by separating out an explicit eddy-viscous term on both sides of (6.1):

$$\frac{dy_i}{dt} + \nu_i y_i + \int_0^t \eta_i(t, s) y_i(s)\, ds =$$

$$\lambda \left(\sum_{j,k} A_{ijk} y_j y_k + \int_0^t \eta_i(t, s) y_i(s)\, ds \right), \tag{6.3}$$

where λ is a formal perturbation parameter that will be set equal to 1 later. The eddy-damping factor $\eta_i(t, s)$, which is assumed non-random, will be determined below. It is also assumed that the initial values $y_i(0)$ are Gaussianly distributed with zero mean and that $y_i(t)$ may be expanded as

$$y_i(t) = y_i^{(0)}(t) + \lambda y_i^{(1)}(t) + \lambda^2 y^{(2)}(t) + \cdots, \tag{6.4}$$

where $y_i^{(0)}(0) = y_i(0)$ and $y_i^{(n)}(0) = 0$ for $n \geqslant 1$.

Substitution of the formal expansion (6.4) into (6.3) and equating coefficients of like powers of λ gives

$$\frac{dy_i^{(0)}}{dt} + \nu_i y_i^{(0)} + \int_0^t \eta_i(t, s) y_i^{(0)}(s)\, ds = 0 \tag{6.5}$$

$$\frac{dy_i^{(1)}}{dt} + \nu_i y_i^{(1)} + \int_0^t \eta_i(t, s) y_i^{(1)}(s)\, ds = q_i(t), \tag{6.6}$$

where

$$q_i(t) = \sum_{jk} A_{ijk} y_j^{(0)}(t) y_k^{(0)}(t) + \int_0^t \eta_i(t, s) y_i^{(0)}(s)\, ds. \tag{6.7}$$

If the (*nonrandom*) Green's function $G_i(t, t')$ is defined by

$$\frac{\partial G_i(t, t')}{\partial t} + \nu_i G_i(t, t') + \int_{t'}^t \eta_i(t, s) G_i(s, t')\, ds = 0 \qquad (t > t') \tag{6.8a}$$

$$G_i(t' + 0, t') = 1 \tag{6.8b}$$

$$G_i(t, t') = 0 \qquad (t < t') \tag{6.8c}$$

it follows that the solution to (6.5) is expressible as

$$y_i^{(0)}(t) = G_i(t, 0) y_i(0).$$

Consequently, $y_i^{(0)}(t)$ is Gaussian for all t. Also, the solution to (6.6) with $y_i^{(1)}(0) = 0$ becomes

$$y_i^{(1)}(t) = \int_0^t G_i(t, s)\, q_i(s)\, ds. \tag{6.9}$$

The two-time correlation $Y_i(t, t')$ is defined by

$$Y_i(t, t') = \langle y_i(t) y_i(t') \rangle. \tag{6.10}$$

If λ is small (which it is not!), it follows that

$$Y_i(t, t') \approx \langle y_i^{(0)}(t) y_i^{(0)}(t') \rangle \tag{6.11}$$

It follows from (6.3) and (6.10) that

$$\frac{\partial Y_i(t, t')}{\partial t} + \nu_i Y_i(t, t') + \int_0^t \eta_i(t, s)\, Y_i(s, t')\, ds$$

$$= \lambda \Big[\sum_{jk} A_{ijk} \langle y_j(t) y_k(t) y_i(t') \rangle + \int_0^t \eta_i(t, s)\, Y_i(s, t')\, ds \Big]. \tag{6.12}$$

The equations of the theory are completed by using lowest-order perturbation theory to evaluate the right-hand side of (6.12) and choosing

$\eta_i(t, s)$ to cancel exactly all the eddy viscous terms that appear. To first order in λ,

$$\langle y_j(t)\, y_k(t)\, y_i(t')\rangle = \langle y_j^{(0)}(t)\, y_k^{(0)}(t)\, y_i^{(0)}(t')\rangle + \lambda\langle y_j^{(1)}(t)\, y_k^{(0)}(t)\, y_i^{(0)}(t')\rangle$$
$$+ \lambda\langle y_j^{(0)}(t) y_k^{(1)}(t) y_i^{(0)}(t')\rangle + \lambda\langle y_j^{(0)}(t) y_k^{(0)}(t) y_i^{(1)}(t')\rangle + 0(\lambda^2). \tag{6.13}$$

The first term on the right is zero since $\mathbf{y}^{(0)}$ is zero-mean Guassian. Using (6.9) and (6.11), it follows from the Gaussian property of $\mathbf{y}^{(0)}$ that

$$\langle y_j^{(1)}(t) y_k^{(0)}(t) y_i^{(0)}(t')\rangle = \int_0^t G_j(t, s)\langle q_j(s) y_k^{(0)}(t) y_i^{(0)}(t')\rangle\, ds,$$
$$= \int_0^t G_j(t, s) \sum_{p,q} A_{jpq}\langle y_p^{(0)}(s) y_q^{(0)}(s) y_k^{(0)}(t) y_i^{(0)}(t')\rangle\, ds$$
$$+ \int_0^t ds \int_0^s dr\, G_j(t, s)\eta_j(s, r)\langle y_j^{(0)}(r) y_k^{(0)}(t) y_i^{(0)}(t')\rangle,$$
$$\approx 2A_{jkl}\int_0^t G_j(t, s) Y_k(s, t) Y_i(s, t')\, ds. \tag{6.14}$$

Similar evaluation of the other terms in (6.13) gives, to first order in λ,

$$\langle y_j(t) y_k(t) y_i(t')\rangle \approx 2\lambda A_{jkl}\int_0^t G_j(t, s) Y_k(s, t) Y_i(s, t')\, ds$$
$$+ 2\lambda A_{klj}\int_0^t G_k(t, s) Y_j(s, t) Y_i(s, t')\, ds$$
$$+ 2\lambda A_{ijk}\int_0^{t'} G_i(t', s) Y_j(s, t) Y_k(s, t)\, ds. \tag{6.15}$$

The first two terms on the right-hand side of (6.15) are of the form $\int_0^t \eta_i(t, s) Y_i(s, t)\, ds$ of the last term on the right-hand side of (6.12). Consequently, they give the eddy viscous effect we have sought and may be cancelled on the right-hand side of (6.12) by setting

$$\eta_i(t, s) = -4\sum_{j,k} A_{ijk} A_{jkl} G_j(t, s) Y_k(s, t). \tag{6.16}$$

where we have set $\lambda = 1$ and given up the pretense of perturbation expansion. With (6.16), (6.12) becomes

$$\frac{\partial Y_i(t, t')}{\partial t} + \nu_i Y_i(t, t') + \int_0^t \eta_i(t, s) Y_i(s, t')\, ds$$
$$= 2\sum_{j,k} (A_{ijk})^2 \int_0^{t'} G_i(t', s) Y_j(s, t) Y_k(s, t)\, ds. \tag{6.17}$$

The equations of the DIA are (6.8) (6.16), and (6.17).

The DIA equations for isotropic turbulence follow by identification of the terms of the model (6.1) with those of the Navier-Stokes equations (2.31)

[or (5.1)] using the two-time energy spectrum $E(k; t, t')$ defined by

$$\langle u_\alpha(\mathbf{k}, t) u_\beta(\mathbf{p}, t')\rangle = \frac{E(k; t, t')}{4\pi k^2} P_{\alpha\beta}(\mathbf{k}) \delta(\mathbf{k} + \mathbf{p})$$

and the isotropic Green's function $G(k; t, t')$, which is related to the Green's function $G_{\alpha\beta}(\mathbf{k}; t, t')$ (discussed below) by

$$G_{\alpha\beta}(\mathbf{k}; t, t') = P_{\alpha\beta}(\mathbf{k}) G(k; t, t').$$

The DIA equations for isotropic turbulence are, by analogy to (6.8), (6.16), (6.17), respectively,

$$\frac{\partial}{\partial t} G(k; t, t') + \nu k^2 G(k; t, t') + \int_{t'}^{t} \eta(k; t, s) G(k; s, t')\, ds = 0 \quad (t > t') \tag{6.18}$$

$$\eta(k; t, s) = \frac{1}{2} \iint_{\Delta} dp\, dq\, \frac{kp}{q} b(k, p, q) G(p; t, s) E(q; s, t) \tag{6.19}$$

$$\frac{\partial}{\partial t} E(k; t, t') + \nu k^2 E(k; t, t') + \int_{0}^{t} \eta(k; t, s) E(k; s, t')\, ds$$
$$= \frac{1}{2} \iint_{\Delta} dp\, dq \frac{k^3}{pq} a(k, p, q) \int_{0}^{t'} ds\, G(k; t', s) E(p; s, t) E(q; s, t), \tag{6.20}$$

where $G(k; t' + 0, t') = 1$ and the other notation follows that of §§ 4.4, 4.5.

The Green's functions associated with the eddy damping are an essential feature of the DIA. These functions may also be introduced in analogy with (5.20). Imagine that an infinitesimal force $\delta f_\alpha(\mathbf{k}, t)$ vanishing for $t < t_0$ for some t_0, is imposed on the right-hand side of (2.31).† Then the perturbation $\delta\mathbf{u}(\mathbf{k}, t) = \mathbf{u}'(\mathbf{k}, t) - \mathbf{u}(\mathbf{k}, t)$, where $\mathbf{u}'(\mathbf{k}, t) = \mathbf{u}(\mathbf{k}, t)$ for $t < t_0$, should be linearly related to δf:

$$\delta u_\alpha(\mathbf{k}, t) = \int_{t_0}^{t} ds \int d\mathbf{p}\, \hat{G}_{\alpha\beta}(\mathbf{k}, t; \mathbf{p}, s) \delta f_\beta(s) + o(|\delta \mathbf{f}|), \tag{6.21}$$

where $\hat{G}_{\alpha\beta}$ is independent of δf. The Fourier transform of (6.21) gives the corresponding expression in x-space:

$$\delta v_\alpha(\mathbf{x}, t) = \int_{t_0}^{t} ds \int d\mathbf{y}\, \hat{g}_{\alpha\beta}(\mathbf{x}, t; \mathbf{y}, s) \delta h_\beta(\mathbf{y}, s),$$

†Actually, the pressure is modified by the longitudinal part of δf so that $P_{\alpha\beta}(\mathbf{k})\delta f_\beta(\mathbf{k}, t)$ is the term that should be added to the right-hand side of (2.31).

where $\delta\mathbf{v}$ and $\delta\mathbf{h}$ are the Fourier transforms of $\delta\mathbf{u}$ and $\delta\mathbf{f}$, respectively, and

$$\hat{g}_{\alpha\beta}(\mathbf{x}, t; \mathbf{y}, s) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\mathbf{p}\, \hat{G}_{\alpha\beta}(\mathbf{k}, t; \mathbf{p}, s) e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{p}\cdot\mathbf{y}}. \tag{6.22}$$

Thus, $\hat{g}_{\alpha\beta}(\mathbf{x}, t; \mathbf{y}, s)$ is the response in v_α at $\mathbf{x}$, t to a unit impulse perturbation applied to v_β at $\mathbf{x}'$, t'. Homogeneity implies $\langle \hat{g}_{\alpha\beta}(\mathbf{x}, t; \mathbf{y}, s)\rangle$ is a function of $\mathbf{x}$-$\mathbf{y}$, so that (6.22) gives

$$\langle \hat{G}_{\alpha\beta}(\mathbf{k}, t; \mathbf{p}, s)\rangle = G_{ab}(\mathbf{k}; t, s)\delta(\mathbf{k} - \mathbf{p}). \tag{6.23}$$

Finally, the derivation of the DIA given in this section is little more than a heuristic plausibility argument in their favor. Therefore, it is important to investigate the properties of the DIA as well as other more justifiable derivations, as will be done in the next few Sections.

6.2 Consistency Properties of the Direct-Interaction Equations

The DIA equations (6.16)–(6.20) reduce to the quasi-normal equation (4.30) with the replacements

$$G(k; t_>, t_<) = e^{-\nu k^2 (t_> - t_<)}, \; E(k; t_>, t_<) = e^{-\nu k^2 (t_> - t_<)} E(k_>, t_<), \tag{6.24}$$

where $t_< \ll t_>$, upon noting that $\partial E(k, t)/\partial t = 2\partial E(k; t, t')/\partial t|_{t'=t}$. The difference between the quasi-normal and DIA theories lies in their treatments of past dynamical evolution. The effect of nonlinear scrambling is accounted for in the DIA by the Green's function, while it is not accounted for in the quasi-normal theory.

The spectrally cutoff versions of the DIA equations are consistent with the inviscid equipartition ensembles discussed in Section V and the irreversible relaxation of arbitrary initial states to it as $t \to \infty$. In fact, the fluctuation-dissipation property

$$Y_l(t_>, t_<) = CG_l(t_>, t_<) \; (t_< \ll t_>) \tag{6.25}$$

with arbitrary modal energy C satisfies (6.8), (6.16), (6.17) with $\nu_l \equiv 0$ provided the initial time 0 is moved to the infinite past (so equilibrium can be achieved) and $G_l(t, t') = G_l(t - t')$ satisfies

$$\frac{dG_l(t)}{dt} = 4C \sum_{jk} A_{ljk} A_{jkl} \int_0^t G_j(t - s) G_k(t - s) G_l(s)\, ds. \tag{6.26}$$

Similarly, the DIA for the inviscid cutoff Navier–Stokes equations yields equipartition ensembles with

$$U(k; t_>, t_<) = \frac{E(k; t_>, t_<)}{4\pi k^2} = CG(k; t_>, t_<) \; (t_< \ll t_>), \tag{6.27}$$

where $U(k; t_>, t_<)$ is the modal energy spectrum. The irreversible relaxation of arbitrary initial states to an equilibrium of the form (6.27) may be proved for the DIA as indicated in §6.3, but is illustrated here by the numerical results plotted in Fig. 6.1 (Kraichnan, 1964a). These results are consistent with relaxation to equilibrium as $t \to \infty$, i.e. $U(k; t, t) \to C$ as $t \to \infty$.

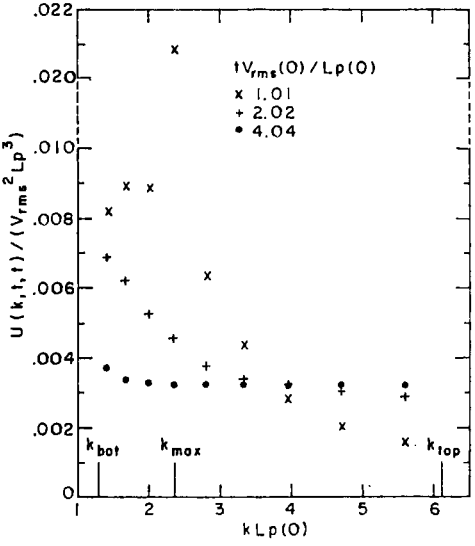

Figure 6.1 Approach to equipartition of inviscid truncated system in the DIA theory. The initial conditions are chosen so that the modal energy spectrum is zero except in the quarter-octave band centered on $k = 4.75683$. The initial values of v_{rms}, L_p are 1.0, 0.495329, respectively. (For further details, see Kraichnan 1964a.)

Other consistency properties of the DIA equations include (formal) conservation of energy by nonlinear interaction in the sense that (2.35) holds. Also, expansion of $E(k, t)$ given by the DIA into series of powers of time t or powers of Reynolds number R agree with the exact expansion of $E(k, t)$ developed from Gaussian initial conditions through terms of order t^3 and R^3, respectively (but not beyond).

The most important consistency property of the DIA is the guaranteed realizability of its solutions as averages of a real dynamical process. In particular, it follows that $E(k; t, t)$ is non-negative for all t if it is so initially and that

$$E(k; t, t')^2 \ll E(k; t, t)E(k; t', t').$$

Realizability is demonstrated by explicit construction of a Langevin statistical model whose exact solution satisfies the DIA equations (Phythian, 1969; Kraichnan, 1970a; Leith, 1971). The model, analogous to (4.50), is

$$\frac{\partial u(k,t)}{\partial t} + \nu k^2 u(k,t) + \int_0^t \eta(k;t,s)u(k,s)\,\mathrm{d}s = q(k,t), \tag{6.28}$$

where $\eta(k;t,s)$ is given by (6.19), $q(k,t)$ is a random process with covariance

$$\langle q(k,t)q(k,s)\rangle = \frac{1}{2}\iint_{\Delta} \mathrm{d}p\,\mathrm{d}q\,\frac{k^3}{pq}a(k,p,q)E(p;s,t)E(q;s,t), \tag{6.29}$$

and $u(k,t)$ is a random variable satisfying $E(k,t) = (1/2)\langle u(k,t)^2\rangle$ and such that $u(k,0)$ is statistically independent of $q(k,t)$ for all t. The equivalence of (6.28), (6.29) with (6.18)–(6.20) is easily demonstrated after noting that the solution of (6.28) is

$$u(k,t) = G(k;t,0)u(k,0) + \int_0^t G(k;t,s)q(k,s)\,\mathrm{d}s$$

so that

$$\frac{\partial}{\partial t}E(k;t,t') + \nu k^2 E(k;t,t') + \int_0^t \eta(k;t,s)E(k;s,t')\,\mathrm{d}s$$

$$= \int_0^{t'} G(k;t',s)\langle q(k,t)q(k,s)\rangle\,\mathrm{d}s.$$

It only remains to show that $q(k,t)$ satisfying (6.29) is realizable. This is done, recalling the "bootstrap" process explained in § 4.7 to march from t to $t + \Delta t$, by observing that

$$q(k,t) = \int \mathrm{d}^3p\,\frac{k^2}{pq}\sqrt{\frac{a(k,p,q)}{4\pi}}\,\xi(\mathbf{p},t)\xi'(\mathbf{k}-\mathbf{p},t)$$

gives (6.29). Here (4.29) and (4.36) are used, $k = |\mathbf{k}|$, and ξ and ξ' are chosen to be statistically independent of each other and the initial $u(k,0)$ distribution and to satisfy

$$\langle \xi(\mathbf{k},t)\xi(\mathbf{p},s)\rangle = \langle \xi'(\mathbf{k},t)\xi'(\mathbf{p},s)\rangle = E(k;t,s)\delta(\mathbf{k}-\mathbf{p}).$$

6.3 Random Coupling Model

Kraichnan (1961) devised a most remarkable model representation of the DIA equations. This "random coupling model" is, for (6.1), the system

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} + \nu_i y_i = \sum_{j,k} A'_{ijk}y_j y_k, \tag{6.30}$$

where $A'_{ijk} = \pm A_{ijk}$ with the choice of sign varying randomly from distinct triad i, j, k to triad. The choice of sign in A' is constant for all permutations of i, j, k in order that energy conservation (6.2c) be maintained. The choice of sign is also time independent. Kraichnan showed that the DIA equations (6.8), (6.16), and (6.17) are the *exact* equations for the evolution of $G_i(t, t') = \langle \delta y_i(t)/\delta f_i(t') \rangle$ [cf. (6.21)] and $Y_i(t, t') = \langle y_i(t) y_i(t') \rangle$, respectively.

Since the DIA equations are exact for (6.30), it follows that they are (a) realizable, (b) integrable, and (c) consistent with stochastic relaxation. Realizability was discussed above. Integrability requires, among other things,

$$\frac{\partial^2}{\partial t \partial t'} Y_i(t, t') = \frac{\partial^2}{\partial t' \partial t} Y_i(t, t'). \tag{6.31}$$

The importance of (6.31) is that it guarantees that the results for $Y_i(t, t')$ are independent of the path in the t, t' plane used to integrate the equations. The DIA is consistent with stochastic relaxation because the model (6.30) for which it is exact includes an infinity of interacting degrees of freedom and is such that an isolating integral of motion in addition to energy is even less likely than for (6.1).

It should be emphasized that the random coupling model (6.30) is an entirely different kind of approximation to the exact dynamics than is the more familiar random phase approximation (Bohm and Pines, 1953). The former requires only that the phase (sign) of the coupling constant A'_{ijk} be random and uncorrelated, and then treats the resulting dynamics exactly. On the other hand, the random phase approximation requires that the phases of the solution y_i be random and uncorrelated. For the turbulence problem, the random phase approximation yields the quasi-normal theory, which we have seen to be unsatisfactory.

The proof that (6.30) is a model for the DIA is based on the diagrammatic perturbation methods outlined in the next Section. However, the proof of equivalence of the random coupling model and the DIA will not be given here. The interested reader should consult the original paper by Kraichnan (1961).

6.4 Diagram Methods

The idea of diagram methods and their relation to the DIA is best explained for the dynamically-linear stochastic model

$$\frac{\partial q(t, t')}{\partial t} = -ib(t) q(t, t'), \quad q(t', t') = 1. \tag{6.32}$$

where $b(t)$ is a zero-mean Gaussian random vatiable with covariance

$$\langle b(t)b(s)\rangle = \Gamma(t - s).$$

The case $\Gamma(t) = \Gamma(0) = \langle b^2\rangle$ in which $b(t) \equiv b(0)$ was discussed in §5.5. The exact solution to (6.32) is

$$G(t - t') = G(t, t') = \langle q(t, t')\rangle = \exp\left[-\frac{1}{2}\int_{t'}^{t} ds \int_{t'}^{t} dr\Gamma(s - r)\right]. \tag{6.33}$$

This random oscillator problem is homologous, with reference to diagram structure, to linear wave propagation problems (Frisch, 1968).

A diagrammatic treatment of (6.32) may be based on the Neumann (perturbation) series

$$q(t, t') = H(t, t') - i\int_{t'}^{t} H(t, s)b(s)H(s, t')\,ds$$

$$+ (-i)^2\int_{t'}^{t} ds \int_{t'}^{s} drH(t, s)b(s)H(s, r)b(r)H(r, t'), \tag{6.34}$$

where $H(t, t') = 1$ if $t \gg t'$, 0 if $t < t'$. This series is represented by the diagram expansion shown in Fig. 6.2, where $\overline{t \quad t'}$ depicts the "bare propagator" $H(t, t')$, • depicts the "bare vertex" $-i$, and ¦ depicts $b(t)$. Intermediate times in the diagrams are assumed to be integrated between their limits.

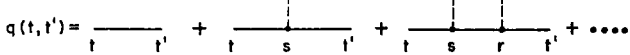

Figure 6.2 Diagrammatic representation of the Neumann series (6.34) for the solution of (6.32).

The only stochastic element on the right-hand side of (6.34) is $b(\cdot)$, so that the Gaussian property may be used to evaluate the average of each term of the series. The result of averaging over the Gaussian ensemble is shown in diagrammatic form in Fig. 6.3. A dashed line connecting time s to time r indicates the factor $\langle b(s)b(r)\rangle = \Gamma(s - r)$, while the heavy solid line $\overline{t \quad t'}$ represents the "renormalized" propagator $G(t, t')$. The series depicted in Fig. 6.3 has the analytic form, with term by term correspondence,

$$G(t, t') = H(t, t') + (-i)^2\int_{t'}^{t} ds\int_{t'}^{s} drH(t, s)H(s, r)H(r, t')\Gamma(s - r)$$

$$+ (-i)^4\int_{t'}^{t} ds\int_{t'}^{s} dr\int_{t'}^{r} du\int_{t'}^{u} dvH(t, s)H(s, r)H(r, u)H(u, v)H(v, t')$$

$$\cdot[\Gamma(s - r)\Gamma(u - v) + \Gamma(s - v)\Gamma(r - u) + \Gamma(s - u)\Gamma(r - v)] + \cdots \tag{6.35}$$

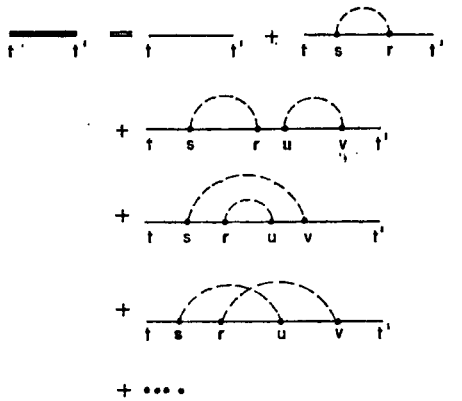

Figure 6.3 Bare diagram expansion for $G(t, t') = \langle q(t, t') \rangle$ with Gaussian $b(t)$. This diagram expansion is equivalent to the series (6.35).

The various analytical theories of turbulence have analogs for the problem at hand which correspond to partial resummations (or contractions or renormalizations) of the series (6.35). For example, the sum of the string of diagrams shown in Fig. 6.4a satisfies the integral equation depicted in Fig. 6.4b:

$$G_{QN}(t, t') = H(t, t') + (-i)^2 \int_{t'}^{t} ds \int_{t'}^{s} dr H(t, s) H(s, r) G_{QN}(r, t') \Gamma(s - r). \tag{6.36}$$

If a Neumann series is constructed by iteration of the right-hand side of (6.36), the resulting series expansion has the diagrammatic representation of Fig. 6.4a.

The partial summation $G_{QN}(t, t')$ is precisely the same as the quasi-normal

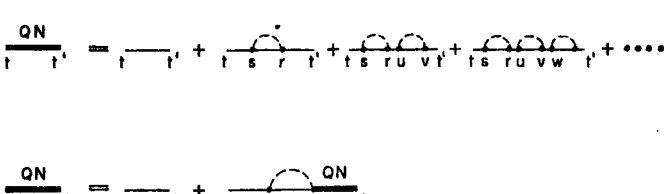

Figure 6.4 (a) Bare diagram representation of the quasi-normal approximation. (b) Resummation of the series of (a), resulting in the integral equation (6.36).

approximation. In fact, the closure (5.55) gives

$$\frac{\mathrm{d}G_{\mathrm{QN}}(t)}{\mathrm{d}t} = -\int_0^t \Gamma(t-s)G_{\mathrm{QN}}(s)\,\mathrm{d}s, \tag{6.37}$$

which is equivalent to (6.36), noting that $G_{\mathrm{QN}}(t-t') = G_{\mathrm{QN}}(t,t')$ is a function of $t - t'$ alone because of the statistical stationarity of the random process $b(t)$.

The solution to (6.36) is best effected by Laplace transformation; it follows that

$$G_{\mathrm{QN}}(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{p + \tilde{\Gamma}(p)}\,\mathrm{e}^{pt}\,\mathrm{d}p, \tag{6.38}$$

where $\tilde{\Gamma}(p)$ is the Laplace transform of $\Gamma(t)$ and the path of integration is to the right of the singularities of the integrand. If

$$\Gamma(t) = \langle b^2\rangle \mathrm{e}^{-t/\tau}$$

then, setting $\alpha^2 = \tau^{-2} - 4\langle b^2\rangle$, it follows that

$$G_{\mathrm{QN}}(t) = \frac{1}{2\alpha}\,\mathrm{e}^{-t/2\tau}\left[\left(\alpha + \frac{1}{\tau}\right)\mathrm{e}^{1/2\alpha t} + \left(\alpha - \frac{1}{\tau}\right)\mathrm{e}^{-1/2\alpha t}\right]. \tag{6.39}$$

On the other hand, the exact solution (6.33) is

$$G(t) = \exp\left[-\langle b^2\rangle(t\tau - \tau^2 + \tau^2\mathrm{e}^{-t/\tau})\right]. \tag{6.40}$$

If $R = \langle b^2\rangle^{1/2}\tau \ll 1$, then $\alpha = \tau^{-1} - 2\tau\langle b^2\rangle + 0(R^4\tau^{-1})$ so that

$$G_{\mathrm{QN}}(t) = \mathrm{e}^{-\langle b^2\rangle t\tau}\left[1 + 0(R^2 + R^4 t\tau^{-1})\right], \tag{6.41}$$

while

$$G(t) = \mathrm{e}^{-\langle b^2\rangle t\tau}\left[1 + 0(R^2)\right]. \tag{6.42}$$

Consequently, $G_{\mathrm{QN}}(t) \approx G(t)$ for all t if $R \ll 1$, and the relative error is small if $t \ll \tau/R^4$. R may be interpreted as a generalized Reynolds number. The conclusion is that the quasi-normal theory is asymptotically valid for all t if $R \ll 1$. This situation should be contrasted with the perturbation series solution which, to second order, consists of the first two terms on the right-hand side of (6.35), viz.

$$G_{\mathrm{pert}}(t) = 1 - \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\,\Gamma(s-r) = 1 - \langle b^2\rangle(t\tau - \tau^2 + \tau^2\mathrm{e}^{-t/\tau}). \tag{6.43}$$

When $R \ll 1$, $G_{\mathrm{pert}}(t) \approx G(t)$ only with the additional restriction $R^2 t/\tau \ll 1$. In contrast to the quasi-normal approximation, the perturbation series is not uniformly valid in t when R is small.

When $R \to \infty$, the quasi-normal approximation (6.39) gives, for $t \ll R^2\langle b^2\rangle^{-1/2}$,

$$G_{QN}(t) = e^{-1/2\langle b^2\rangle t/R}\cos(\langle b^2\rangle^{1/2}t)\left[1 + 0\left(\frac{1}{R}\right)\right] \tag{6.44}$$

while the exact solution (6.40) satisfies

$$G(t) \approx e^{-1/2\langle b^2\rangle t^2} \tag{6.45}$$

for $t \ll R\langle b^2\rangle^{-1/2}$. The quasi-normal solution (6.44) oscillates on the time scale $\langle b^2\rangle^{-1/2}$ and only approaches zero on the time scale $R\langle b^2\rangle^{-1/2}$, in contrast to the exact solution which approaches zero on the time scale $\langle b^2\rangle^{-1/2}$. The behavior of the quasi-normal approximation is qualitatively wrong when $R \to \infty$, as already found in Sec. 5.5 (for the case $R = \infty$). Frisch and Bourret (1970) find that, for a variety of $\Gamma(t)$, G_{QN} is accurate to within 5% for $R \ll 1/10$ and G_{QN} is qualitatively incorrect for $R \gg 1/4$.

The series (6.35) can be contracted to the "irreducible" diagram expansion shown in Fig. 6.5. Here a diagram is said to be reducible if it can be contracted by the replacement shown in Fig. 6.6 to a diagram with fewer vertices. For example, the third and fourth diagrams on the right-hand side of the equation depicted in Fig. 6.2 are reducible, while the fifth diagram on the right is irreducible. The second diagram on the right in Fig. 6.2 is defined to be irreducible even though it may be contracted. The explicit form of the first two terms of the diagram expansion of Fig. 6.5 is

$$G(t, t') = H(t, t') + (-i)^2 \int_{t'}^{t} ds \int_{t'}^{s} dr H(t, s)G(s, r)G(r, t')\Gamma(s - r) + \cdots \tag{6.46}$$

The equivalence of the diagram expansion shown in Fig. 6.5 to that of Fig. 6.2 is demonstrated by iterative expansion of the renormalized diagrams into a series of "bare" (uncontracted) diagrams.

The DIA corresponds to retaining only the first two diagrams on the right-hand side of Fig. 6.5, so that the equation for $G(t, t')$ is (6.46) neglecting all but the terms shown explicitly. The bare diagram expansion of the DIA

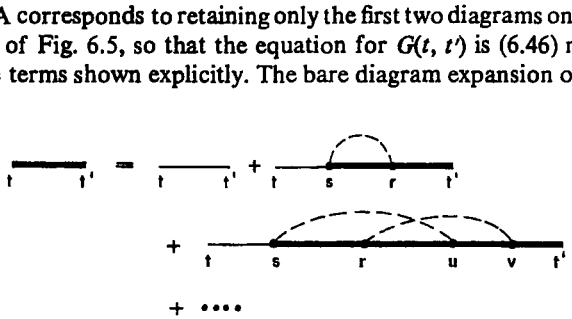

Figure 6.5 Irreducible diagram expansion of $G(t, t')$. The explicit form of this diagram series is given in (6.46).

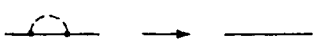

Figure 6.6 Replacement used to define a reducible diagram. If any part of a diagram can be removed by the contraction shown in this figure, the diagram is reducible.

is shown in Fig. 6.7; all diagrams consisting of series and "towers" of "bubble" diagrams are included, but all "crossing" diagrams, like the fifth on the right in Fig. 6.2, are excluded. It follows from (6.46) that

$$\frac{dG_{DIA}(t)}{dt} = -\int_0^t \Gamma(t-s)G_{DIA}(t-s)G_{DIA}(s)\,ds. \tag{6.47}$$

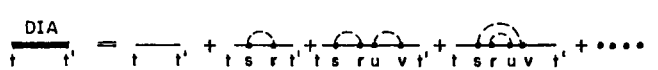

Figure 6.7 Bare diagram expansion of $G(t, t')$ in the direct-interaction approximation.

The closed form analytical solution of (6.47) is not known even for the special case $\Gamma(t) = \langle b^2\rangle e^{-t/\tau}$. However, if $R \ll 1$, it can be shown that $G_{DIA}(t) \approx G(t)$ for all t. Rigorous proofs can be given (Frisch and Bourret, 1970), but the simplest argument is that, when $R \ll 1$, the time integral in (6.47) can be approximated by $\Gamma(t-s)G_{DIA}(t-s) \approx \Gamma(t-s)$ since $G_{DIA}(t)$ deviates from 1 only on the time scale τ/R^2 while $\Gamma(t)$ approaches zero in the time scale τ. Thus,

$$\frac{dG_{DIA}(t)}{dt} \approx -\int_0^t \Gamma(t-s)G_{DIA}(s)\,ds \quad (R \ll 1)$$

which is precisely (6.37), so that $G_{DIA}(t) \approx G_{QN}(t) \approx G(t)$ when $R \ll 1$.

On the other hand, when $R \to \infty$, (6.47) reduces to

$$\frac{dG_{DIA}(t)}{dt} = -\langle b^2\rangle \int_0^t G_{DIA}(t-s)G_{DIA}(s)\,ds, \tag{6.48}$$

whose solution (found by Laplace transformation) is (Kraichnan, 1961)

$$G_{DIA}(t) = \frac{J_1(2\langle b^2\rangle^{1/2}t)}{\langle b^2\rangle^{1/2}t}, \tag{6.49}$$

where J_1 is the Bessel function of order 1. While (6.49) does differ quantitatively from the exact result (6.45), it shares the important qualitative feature with (6.45) that both approach zero on the time scale $\langle b^2\rangle^{-1/2}$. Frisch and Bourret (1970) solved (6.47) numerically for a variety of $\Gamma(t)$. They found that G_{DIA} is in error by less than 5% if $R \ll 1/2$, and that G_{DIA} and G behave in a qualitatively similar way for all R. It may be concluded that

the DIA is a significant qualitative, if not quantitative, improvement over cumulant-discard approximations.

The DIA equation (6.47) also follows from the random coupling model for the system (6.32). The details of this construction are given by Kraichnan (1961), who uses random couplings of M identical system satisfying (6.32) to generate a macrosystem for which the average Green's function $G_{\mathrm{DIA}}(t)$ satisfies (6.47) in the limit $M \to \infty$. Since the random coupling model involves couplings of a large number of systems, stochastic relaxation is guaranteed in the DIA. Similar random coupling models are possible for a very wide variety of stochastic systems. In terms of diagrams, all diagrams except the DIA diagrams shown in Fig. 6.7 vanish in the random coupling model.

The success of the DIA may give hope that successive truncations of the renormalized diagram expansion of Fig. 6.5 give approximations that converge rapidly to $G(t)$. Unfortunately, this is not the case. All higher-order truncations of the series in Fig. 6.5 give unsatisfactory approximations to $G(t)$, as shown by Kraichnan (1961). If all the terms shown explicitly in Fig. 6.5 are retained, the equation for $G(t)$ becomes, in the limit $R \to \infty$,

$$\frac{dG}{dt} = -\langle b^2\rangle\, G*G + \langle b^2\rangle^2 G*(G*(G*G)), \tag{6.50}$$

where $*$ denotes convolution, with $G(0) = 1$. The Laplace transform $g(p)$ of $G(t)$ satisfies

$$pg(p) - 1 = -\langle b^2\rangle\, g(p)^2 + \langle b^2\rangle^2 g(p)^4$$

so that

$$g(0)^2 = [1 \pm \sqrt{-3}]/2\langle b^2\rangle.$$

It follows that $\int_0^\infty G(t)\, dt$ cannot be real and finite. In fact, the solution to (6.50) grows faster than any power of t as $t \to \infty$.

The difficulty with higher-order ("indirect interaction") approximations obtained from the renormalized series shown in Fig. 6.5 is not that surprising since the Laplace transform of the renormalized series diverges (as does the Laplace transform of the bare series) when $R = \infty$. It may be shown that the number of diagrams in the renormalized series with $2n$ vertices is asymptotically $1/e$ times the number of diagrams in the bare-diagram series of Fig. 6.3. Since each diagram has a value independent of its topology (and dependent only on t, t' and the number of vertices) when $R = \infty$, it follows that the bare and renormalized series diverge together (Kraichnan, 1970b). The higher-order truncations of the renormalized series of Fig. 6.5 are evidently not realizable by model dynamics.

An improvement on the DIA is obtained by a vertex consolidation of the series of Fig. 6.5. An *exact* equation for $G(t, t')$ is given by the diagram

equation shown in Fig. 6.8; here the renormalized vertex ${}_t\otimes_{t'}$, denoted by $-iV(t, s, t')$, is defined by the diagram series of Fig. 6.9. The explicit form of the equation depicted in Fig. 6.8 is

$$G(t, t') = H(t, t') - \int_{t'}^{t} ds \int_{t'}^{s} dr \int_{t'}^{r} dv \int_{t'}^{v} du\, H(t, s)G(s, r)V(r, v, u)$$
$$G(u, t')\Gamma(s - v). \tag{6.51}$$

The problem of approximating $G(t)$ is then reduced to the problem of approximating the vertex $-iV(t, s, t')$, whose bare diagram expansion depicted in Fig. 6.9 is given explicitly through the first three terms on the right by

$$-iV(t, s, t') = -i\delta(t - s)\delta(s - t') + i\int_{s}^{t} dr \int_{t'}^{s} du H(t, r)\delta(r - s)$$
$$\delta(s - u)H(u, t')\Gamma(t - t') - i\int_{s}^{t} dr \int_{s}^{r} du \int_{t'}^{s} dv \int_{t'}^{v} dw\, H(t, r)H(r, u)$$
$$\delta(u - s)\delta(s - v)H(v, w)H(w, t')\Gamma(t - t')\Gamma(r - w) + \cdots, \tag{6.52}$$

where it is assumed that $\int_0^\epsilon \delta(x)dx = 1$ for all $\epsilon > 0$, $=0$ for all $\epsilon < 0$. Eq. (6.51) follows by classifying all bare diagrams serially from left to right within the diagram into bare propagator (H), propagator (G), vertex (V) "tied" to the leftmost H, and propagator (G) components. This decomposition of any diagram is unique, so no double counting of diagrams results. If the vertex is (6.51) is replaced by the bare vertex, the DIA results. Higher approximations are gotten as follows.

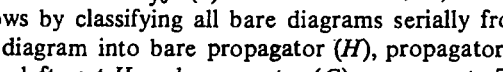

Figure 6.8 *Exact* integral equation for $G(t, t')$ in terms of the renormalized vertex $-iV(r, v, u)$. The explicit form of this equation is (6.51).

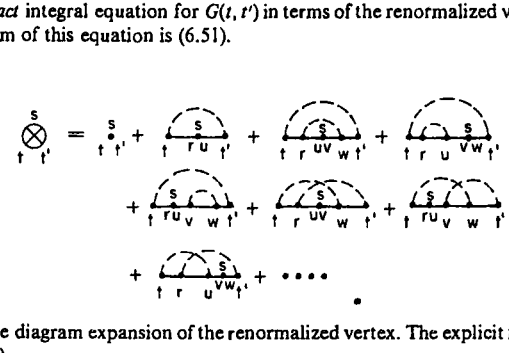

Figure 6.9 Bare diagram expansion of the renormalized vertex. The explicit form of this series is given in (6.52).

The vertex series in Fig. 6.9 may be resummed into the renormalized series shown in Fig. 6.10. Again, each diagram in Fig. 6.9 is uniquely classified to one of the "irreducible" vertex diagrams of Fig. 6.10 by consideration of its structure serially from left to right. Here "irreducible" means a vertex diagram that can not be contracted by the replacements shown in Figs. 6.6 or 6.11. The only irreducible diagrams on the right of Fig. 6.9 are the first, second (by assumption), and sixth.

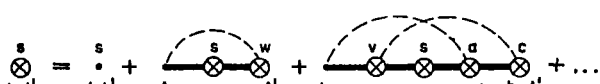

Figure 6.10 Irreducible diagram expansion of the renormalized vertex. Here a diagram is irreducible if it cannot be contracted by either of the replacements shown in Figs. 6.6 or 6.11.

Figure 6.11 Replacement used to define a reducible vertex diagram. If any part of a diagram can be removed by the contraction shown in this figure, the diagram is reducible.

Kraichnan's (1961) "second stochastic model" is obtained by retaining only the first two terms on the right side of Fig. 6.10. The resulting equation for the vertex is

$$V(t, s, t') = \delta(t - s)\delta(s - t') - \int_s^t \mathrm{d}r \int_{t'}^s \mathrm{d}u \int_{t'}^u \mathrm{d}v \int_{t'}^v \mathrm{d}w \; G(t, r)V(r, s, u)\, G(u, v)V(v, w, t')\Gamma(t - w). \tag{6.53}$$

The solution to the coupled equations (6.51), (6.53) is in excellent agreement with the exact solution (6.33) even at $R = \infty$ (Kraichnan, 1961). As such, it gives a substantial improvement over the DIA. However, the present vertex consolidation has a number of practical and philosophical drawbacks, including the apparent nonexistence of a model system yielding (6.51), (6.53) exactly, so that realizability can not be guaranteed. Also, no higher-order truncation of the series in Fig. 6.10 gives plausible approximations to $G(t)$ at large R. The latter trouble relates to the fact that the number of irreducible vertex diagrams with $2n + 1$ vertices is asymptotically $1/e^2$ of the total number of bare vertex diagrams with $2n + 1$ vertices as $n \to \infty$.

Extensions of the diagram methods surveyed here have been made to the nonlinear dynamics of the Navier–Stokes equations (Kraichnan, 1961; Wyld, 1961; Nakano, 1972; Martin, Siggia, and Rose, 1974; and others). Some care is required to avoid double counting problems, which Wyld's

expressions suffer from. Extensions may also be made to a variety of other dynamical systems (cf. Morton and Corrsin, 1970 for some applications to nonlinear oscillator problems). Finally, vertex modifications may be included. However, although the DIA has been amenable to numerical study (cf. Sec. 6.6), the equations of the vertex consolidation are so formidable that numerical solution has not yet been attempted even for the simplest cases of homogeneous turbulence. One additional reason for the lack of interest in vertex-consolidated models is that it is known (cf. §6.5) that they can not be consistent with the high-Reynolds-number inertial range dynamics of the $k^{-5/3}$ law.

6.5 Inertial-Range Dynamics of the DIA

The DIA equations (6.18)–(6.20) were analyzed at large Reynolds numbers by Kraichnan (1959). Consider a large wavenumber satisfying $kL_p \gg 1$. For such a wavenumber, the turbulence may be assumed in a statistically stationary state, so that

$$E(k; t, t') \approx E(k, |t - t'|), \; G(k; t, t') \approx G(k, |t - t'|). \tag{6.54}$$

The dominant contribution to the eddy damping (6.19) for k comes from $qL_p \sim 1$, i.e. the energy-containing eddies. In fact, it follows from (4.64) that the dominant contribution to (6.19) is

$$\eta(k; t, s) \approx k^2 v_{\text{rms}}^2 \, G(k, t - s). \tag{6.55}$$

It is assumed that $E(q; s, t) \approx E(q)$ for $q \sim L_p^{-1}$ and $|t - s|$ less than the decay time of $G(k, \tau)$. With (6.55), (6.18) becomes

$$\frac{\partial G(k, \tau)}{\partial \tau} + \nu k^2 \, G(k, \tau) + k^2 v_{\text{rms}}^2 \int_0^\tau G(k, \tau - s) G(k, s) \, ds = 0. \tag{6.56}$$

The solution satisfying $G(k, 0) = 1$ is [cf. (6.49)]

$$G(k, \tau) = e^{-\nu k^2 \tau} \frac{J_1(2k v_{\text{rms}} \tau)}{k v_{\text{rms}} \tau}. \tag{6.57}$$

In the inertial range, direct viscous effects are unimportant so that $G(k, \tau) \approx J_1(2k v_{\text{rms}} \tau)/(k v_{\text{rms}} \tau)$. The important result follows that the characteristic time for $G(k, \tau)$ to approach is $1/k v_{\text{rms}}$, viz. the sweeping time $\tau_s(k)$ defined in §3.1.

Similar approximations on (6.20) yield the result that, in the inertial range,

$$E(k; t, t') = E(k, |t - t'|) = E(k) r(k v_{\text{rms}} |t - t'|), \tag{6.58}$$

where $r(s)$, the inertial-range time-correlation function, can be shown to have vanishing Fourier transform outside the interval $[-2, 2]$; that is, $r(s) = \int_{-2}^{2} r(v) \exp(ivs) \, dv$. Hence, $E(k, \tau)$ approaches zero in the time scale $\tau_s(k)$, showing that Fourier modes become decorrelated with themselves after the sweeping time.

An equation for $E(k; t, t)$ follows from (6.20) by setting $t = t'$:

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right]E(k; t, t) + 2\int_0^t \eta(k; t, s)E(k; s, t)\,ds$$

$$= \iint_\Delta dp\,dq\,\frac{k^3}{pq}a(k, p, q)\int_0^t ds\,G(k; t, s)E(p; s, t)E(q; s, t). \qquad (6.59)$$

In the inertial range, this equation can be written in the form (4.57) with

$$\theta(k, p, q) = \int_0^\infty G(k, s)r(pv_{\text{rms}}s)r(qv_{\text{rms}}s)\,ds \qquad (6.60)$$

so that the theory of §4.8 applies. Since $\theta(k, p, q)$ is dominated by the sweeping time $\tau_s(k)$, it follows that $m = 1$ in (4.57), so that (4.58) gives $n = 3/2$. Introducing the proper dimensional scaling, it follows that the inertial range spectrum of the DIA is

$$E(k) = C_{\text{DIA}}\,\epsilon^{1/2}v_{\text{rms}}^{1/2}k^{-3/2}. \qquad (6.61)$$

As discussed in §3.2, the spectrum is not consistent with experimental results for high Reynolds number turbulence. This is the main failing of the DIA. The origin of the failure of the DIA in the inertial range is the fact that the energy equation (6.59) is governed by the sweeping time $\tau_s(k)$, so that the argument of §4.8 gives a $k^{-3/2}$ law. It is of some importance to understand how the sweeping time enters the direct-interaction approximation and thereby understand how the approximation breaks down. The following argument is directed at this goal.

Suppose that a very-low-wavenumber disturbance is superposed on a given state of turbulence. It is supposed for convenience that the very-low-wavenumber excitation is statistically isotropic so that it may be described by its energy spectrum $E_0(k)$, which is supposed confined to wavenumbers less than k_0. It is further supposed that the field described by $E_0(k)$ is, at $t = 0$, Gaussian and statistically independent of the undisturbed turbulence. The effect of the disturbance field on the given turbulence is then considered in the asymptotic limit $k_0 \to 0$ and

$$\frac{3}{2}v_0^2 \equiv \int_0^{k_0} E_0(k)\,dk, \quad = 0(1).$$

It is clear that, as $k_0 \to 0$, the very-low-wavenumber excitation approaches uniform convection over larger and larger scales, and longer and longer times. For finite time of evolution, the effect of the very weak shear associated with $E_0(k)$ should be well approximated by the effect of uniform convection. Since the superposed field is statistically independent of the given turbulence, and since uniform convection cannot distort an eddy and transfer energy, spectral properties should be undisturbed.

This latter result is investigated formally by approximating the effect of $E_0(k)$ by a random uniform convecting velocity with rms velocity $3v_0^2$. The effect of a uniform velocity $\mathbf{U}\delta(\mathbf{k})$ on (2.31) is to add $-i(\mathbf{k}\cdot\mathbf{U})u_\alpha(\mathbf{k}, t)$ to its right-hand side. The effect of this term on a solution of (2.31) is to replace $\mathbf{u}(\mathbf{k}, t)$ by

$$u'_\alpha(\mathbf{k}, t) = u_\alpha(\mathbf{k}, t)\exp(-i\mathbf{k}\cdot\mathbf{U}t). \tag{6.62}$$

Thus, the effect of uniform convection is merely a phase change, as is also apparent from (2.14). If averages with respect to $\mathbf{U}$ are denoted by $[\ ]$, reserving $\langle\ \rangle$ for averages with respect to the undisturbed turbulence, then

$$\left[\langle u_\alpha(\mathbf{k}, t)e^{-i\mathbf{k}\cdot\mathbf{U}t}u_\beta(\mathbf{p}, t)e^{-i\mathbf{p}\cdot\mathbf{U}t}\dots u_\gamma(\mathbf{q}, t)e^{-i\mathbf{q}\cdot\mathbf{U}t}\rangle\right] =$$

$$\langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t)\dots u_\gamma(\mathbf{q}, t)\rangle \tag{6.63}$$

since the moments are zero unless $\mathbf{k}+\mathbf{p}+\dots+\mathbf{q}=0$. Therefore, the effect of random uniform convection on single-time moments and cumulants vanishes. This *statistical Galilean invariance* is the basis for the neglect of the dynamical effect of large scales on small scales in the Kolmogorov theory of §3.1.

On the other hand, many-time moments are affected by large-scale convection. For example, the effect of *random* uniform convection on the two-time energy-spectrum tensor is

$$\left[\langle u'_\alpha(\mathbf{k}, t)u'_\beta(\mathbf{p}, t')\rangle\right] = \langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t')\rangle\left[\exp(-i\mathbf{k}\cdot U(t-t'))\right]$$

$$= \langle u_\alpha(\mathbf{k}, t)u_\beta(\mathbf{p}, t')\rangle\exp\left[-\tfrac{1}{2}v_0^2k^2(t-t')^2\right] \tag{6.64}$$

which is certainly not negligible. In fact, large scale convection affects the correlation (6.64) in the time scale $\tau_s^0(k) = (kv_0)^{-1}$. Similarly, it may be shown that random uniform convection should affect the Green's function $G(k; t, t')$ on the time scale $t - t' \sim \tau_s^0(k)$.

Consequently, the results (6.57), (6.58) for the Green's function and time-correlation function, respectively, in the DIA are qualitatively *correct*, especially with regard to their characteristic time scale. On the other hand, these sweeping times should not enter the equation for the single-time energy $E(k; t, t)$, which they do in (6.59) for the DIA. The trouble is that (6.30), for which DIA is exact, is not invariant to a random Galilean transformation in large scales, even if (6.1) is so invariant. In fact, the coupling coefficients A^t_{ijk} fluctuate randomly from triad to triad whereas Galilean invariance to large (but not infinite) scale requires that the coupling coefficients approach a constant times the wavevector as $k \to 0$.

The difficulty of DIA with time scales besets any self-consistent theory that deals with many-time moments in an essentially perturbative way in Eulerian coordinates. For example, the vertex renormalizations discussed in §6.4 are also inconsistent with the Kolmogorov theory of §3.1, for the same reason that the sweeping time scale enters the single-time energy dynamics.

One way to avoid this problem with DIA is to rework the entire theory in Lagrangian coordinates. In this way it is possible to account explicitly for the absence of effects of large-scale convection on small scales within low-order perturbation treatments. In a remarkable series of papers, Kraichnan (1965b, 1966b) developed the Lagrangian history-direct interaction (LHDI) approximation. The heart of this theory is a generalized Lagrangian velocity field $\mathbf{v}(\mathbf{x}, t|r)$, defined as the velocity *measured* at time r for the fluid particle that passed through $\mathbf{x}$ at time t. The time r is called the measuring time, while the time t is called the labelling time. The usual Lagrangian velocity $\mathbf{w}(\mathbf{a}, r)$, defined as the velocity measured at time r for the fluid particle that was at $\mathbf{a}$ at the initial instant $t = 0$, is merely $\mathbf{w}(\mathbf{a}, r) = \mathbf{v}(\mathbf{a}, 0|r)$. The advantage of the generalized Lagrangian velocity is that it satisfies simpler differential equations than $\mathbf{w}(\mathbf{a}, r)$. In fact, the equation satisfied by $\mathbf{v}(\mathbf{x}, t|r)$ as a function of its *labelling coordinates* is

$$\frac{\partial}{\partial t}\mathbf{v}(\mathbf{x}, t|r) + \mathbf{v}(\mathbf{x}, t)\cdot\nabla\mathbf{v}(x, t|r) = 0. \tag{6.65}$$

Here $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t|t)$ is the usual Eulerian velocity measured at $\mathbf{x}, t$. Eq. (6.65) follows because the velocity measured at time r is independent of the labelling coordinate along the path line of the fluid particle passing through $\mathbf{x}, t$; (6.65) states that the measured velocity does not change as the fluid particle executes its path, when the measuring time is constant.

Unfortunately, the resulting LHDI approximation is exceedingly complicated and of dubious fundamental validity. The problem is that in the course of reworking the theory in Lagrangian coordinates, Kraichnan was able to achieve statistical Galilean invariance but he was not able to maintain contact with the random coupling model. Thus, there is no fundamental guarantee that the results of the LHDI theory are self-consistent or realizable. Nevertheless, numerical integration of the LHDI equations gives results in excellent agreement with experiment for the velocity field (Kraichnan, 1966b). There is evidence that the predictions of LHDI are not so satisfactory in the case of passive scalar convection (Kraichnan, 1966c, 1968b), in the sense that LHDI overestimates scalar transport efficiency because, in the absence of molecular diffusivity, Lagrangian scalar correlations persist longer than Eulerian scalar transport correlations.

6.6 Comparison with Experiment

Herring and Kraichnan (1972) give extensive comparisons of the test-field model (§4.9) and the DIA with laboratory experiments and numerical simulations of moderate Reynolds number homogeneous turbulence, as performed by Orszag and Patterson (1972). The choice $g \approx 1{\cdot}2$ gives the best fit of the test-field model with the numerical experiments.

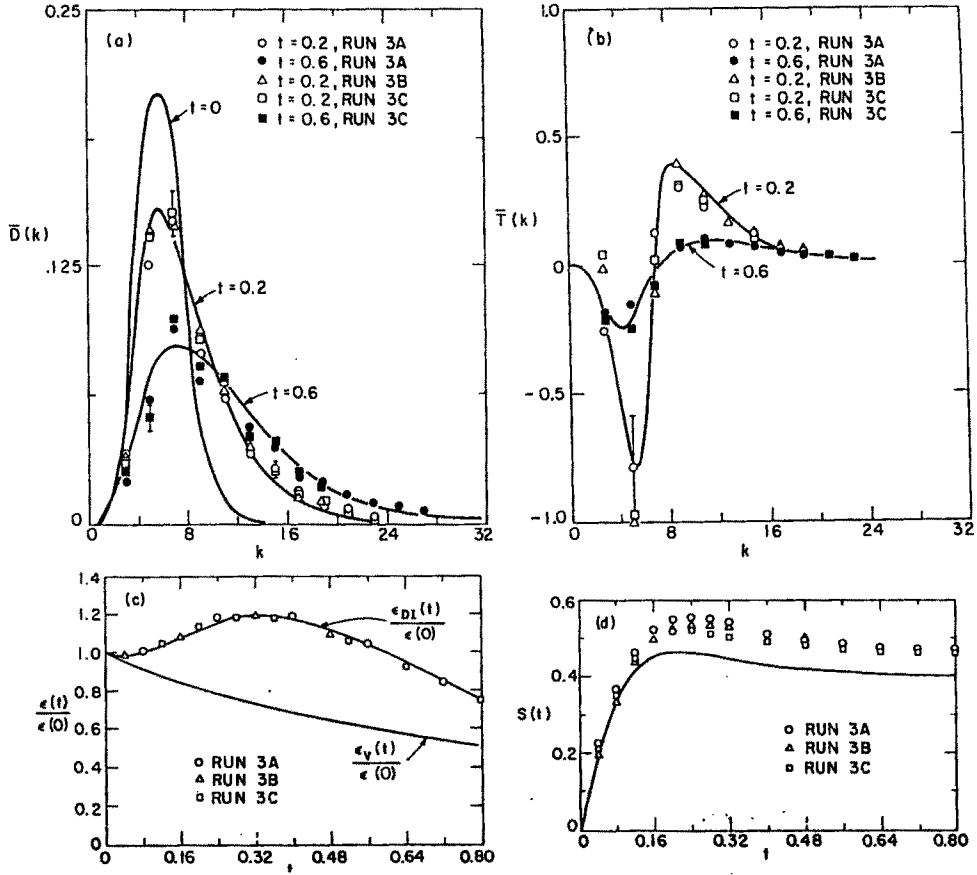

Figure 6.12 Comparisons between numerical simulations (data points) and direct-interaction (DI) theory (solid curves). Run 3 is numerical simulation performed using the initial energy spectrum (4.47) with $u_{rms}(0) = 1$, $k_{max} = 4{\cdot}75683$, $\nu = 0{\cdot}01189$, $L_p(0) = 0{\cdot}54$, $\lambda(0) = 0{\cdot}42$, $R_\lambda(0) = 35{\cdot}4$. Other details are given by Orszag and Patterson (1972). (a) Dissipation spectrum $D(k) = 2k^2E(k)$; (b) transfer spectrum $T(k)$ [cf. (2.34)]; (c) energy dissipation rate. The curve labelled ϵ_ν is for pnre viscous decay using (4.16). (d) Skewness of the longitudinal velocity derivative [cf. (2.45)].

In Fig. 6.12, we show the comparison of the DIA with the homogeneous turbulence numerical simulations of Orszag and Patterson. At these moderate Reynolds numbers, DIA does a good job of predicting the flow evolution, except for the skewness S, which is significantly underestimated by the DIA. The latter result is consistent with the general notions of the accuracy of the DIA, since the skewness is a nondimensional measure of the efficiency of energy transfer and energy transfer is inhibited in the DIA. This inhibition is due to the cutoff in the buildup of triple correlations after the sweeping time.

It is interesting to compare the computational effort to compute with the DIA, the test-field model, and direct numerical solution of the Navier–Stokes equations. For homogeneous turbulence, the test-field equations are the most efficient to integrate because (a) they are ordinary differential equations in time in contrast to the DIA equations that are integro-differential equations in time, and (b) discretization points may be spaced logarithmically in wave-space as discussed in §4.5, in contrast to direct simulation that requires inclusion of all modes (but see Lorenz, 1972, for an interesting attempt to include only a sparse distribution of modes in a direct simulation). At grid turbulence Reynolds numbers $R_\lambda \sim 50$, numerical solution of the DIA equations is perhaps only an order of magnitude more efficient than direct numerical solution of the Navier–Stokes equations.

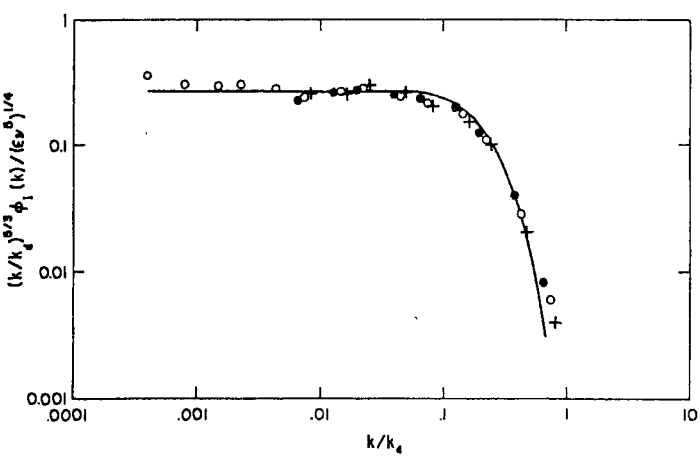

Figure 6.13 Nondimensionalized one-dimensional spectrum function $k^{5/3}\phi_1(k)$ in the inertial and dissipation ranges as computed by the LHDI approximation (solid curve) and compared with the October 1959 data of Grant, Stewart and Moilliet (1962). Here k_d is given by (3.2) and $\phi_1(k)$ is related to the isotropic energy spectrum $E(k)$ by $E(k) = k^3 \mathrm{d}/\mathrm{dk}\,[1/k\,\mathrm{d}/\mathrm{d}k\,\phi_1(k)]$. For further details, see Kraichnan (1966b).

However, at these Reynolds numbers, direct simulation is quite competitive if the turbulence is anisotropic and, especially, inhomogeneous. At higher Reynolds numbers, the work estimates for direct numerical simulation show that the number of computer operations required grows as R_λ^6. Consequently, since the work for DIA or the test-field model grows more slowly with Reynolds number (Kraichnan, 1972), numerical solution of the analytical theories is much more efficient than numerical solution of the Navier–Stokes equations (but see below).

In Fig. 6.13, we plot the equilibrium range (inertial and dissipation) range spectra determined by the LHDI approximation as compared with high Reynolds number geophysical data. The results of the test-field model and the eddy-damped Markovian approximation are quite similar.

Herring *et al.* (1974) compare the results of numerical simulations of two-dimensional turbulence with the predictions of the DIA and the test-field model. In Fig. 6.14 we plot the enstrophy dissipation spectrum $k^4E(k)$ vs k for the DIA and the test-field model with $g = 0{\cdot}65$, $1{\cdot}0$ as well as numerical simulation data. It is apparent that the DIA fails badly in this case. The reason for this failure can also be traced to the incorrect time scales of the DIA. As pointed out in §3.4, the inertial range time in two dimensions is constant, whereas the DIA enstrophy transfer is still affected by the sweeping time $(kv_{\text{rms}})^{-1}$. Since the analog of (4.59) for an enstrophy-cascade inertial range is $n = 3 - m/2$, where n is the power-law exponent of $E(k)$ and m is the power-law exponent of the characteristic time scale, it follows that the DIA gives an enstrophy-transfer inertial range in two dimensions with $E(k) \propto k^{-5/2}$. The inertial range spectrum in two dimensions is actually a log-corrected k^{-3} law, so that DIA is off by $k^{1/2}$; this should be contrasted with three dimensions where the DIA $k^{-3/2}$ inertial range law is only off by $k^{1/6}$ from $k^{-5/3}$. In other words, the DIA error is larger in two than in three space dimensions, so that the poor results plotted in Fig. 6.14 are plausible. On the other hand, the test-field model, which accounts properly for the dynamical time scales, does a much better job of predicting the evolution of these two-dimensional turbulent flows.

Finally, we mention an intriguing possibility pointed out by Orszag and Israeli (1974) and Herring *et al.* (1974), viz. that numerical simulation of turbulence may be workable at moderate Reynolds numbers for the simulation of very high Reynolds number flows. It has been observed in numerical simulations that there is a very strong degree of Reynolds number independence in the flows. It appears that if one is interested only in wavenumbers $k < K$, it may be sufficient to simulate a flow whose energy dissipation spectrum in three dimensions [enstrophy dissipation spectrum in two dimensions] peaks at a wavenumber somewhat larger than K, even though the Reynolds number of the latter flow may be several orders of magnitude

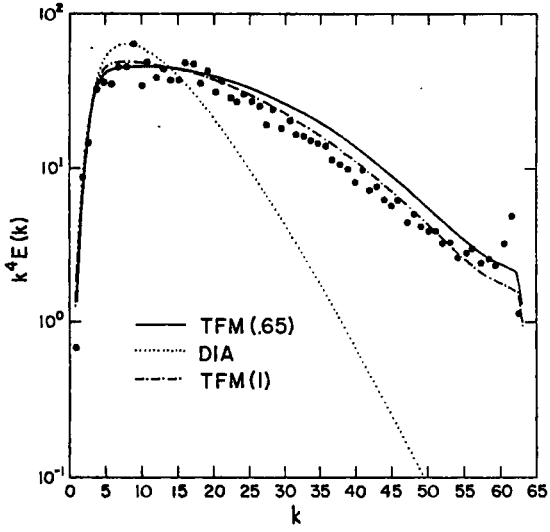

Figure 6.14 Enstrophy dissipation spectrum $k^4E(k)$ vs k for two-dimensional turbulence numerical experiments and theories. The data points are the results of numerical simulations, while the curves labelled TFM (g) are for the test-field model (§4.9) with the indicated value of g, and that labelled DIA is for the direct-interaction approximation. The initial energy spectrum is $E(k) = \frac{1}{2}k\exp(-\frac{1}{2}k)$, while $\nu = 0{\cdot}0025$. The length scale L_2 given by (3.32) is initially $L_2(0) = 0{\cdot}7388$, while $v_{\rm rms}(0) = (1{\cdot}5)^{1/2} = 1{\cdot}225$ and $R(0) = v_{\rm rms}L_2/\nu = 362$. The plots are at $t = 0{\cdot}8$, which is well into the evolution of this flow. Further details are given by Herring *et al.* (1974).

less than the flow of interest. The scales $k < K$ should be Reynolds number independent. It follows that the amount of computational work necessary to do a direct numerical simulation of a turbulent flow may not grow as fast as R_λ^6. This exciting possibility must be left to be judged in the future.

References

Bass, J. (1949) Sur les bases mathématiques de la théorie de la turbulence d'Heisenberg. *C.R. Acad. Sci., Paris*, **228**, 228–229.

Baslecu, R. (1963) *Statistical Mechanics of Charged Particles*. Interscience.

Batchelor, G. K. (1953) *The Theory of Homogeneous Turbulence*. Cambridge University Press.

Batchelor, G. K. (1969) Computation of the energy spectrum in homogeneous two-dimensional turbulence. *Phys. Fluids* (Suppl. 2) **12**, 233–239.

Batchelor, G. K. and Proudman, I. (1956) The large-scale structure of homogeneous turbulence. *Phil. Trans. Roy. Soc.* **A248**, 369–405.

Batchelor, G. K. and Townsend, A. A. (1947) Decay of vorticity in isotropic turbulence. *Proc. Roy. Soc.* **A191**, 534–550.

Batchelor, G. K. and Townsend, A. A. (1949) The nature of turbulent motion at large wave-numbers. *Proc. Roy. Soc.* **A199**, 238–255.

Batchelor, G. K. and Townsend, A. A. (1956) Turbulent diffusion. *Surveys in Mechanics*, 352–399. Cambridge University Press.

Benney, D. J. and Lange, C. G. (1970) The asymptotics of nonlinear diffusion. *Stud. in Appl. Math.* **49**, 1–19.

Benney, D. J. and Saffman, P. G. (1966) Nonlinear ineractions of random waves in a dispersive medium. *Proc. Roy. Soc.* **A289**, 301–320.

Betchov, R. (1956) An inequality concerning the production of vorticity in isotropic turbulence. *J. Fluid Mech.* **1**, 497–504.

Bohm, D. and Pines, D. (1953) A collective description of electron interactions: III. Coulomb interactions in a degenerate electron gas. *Phys. Rev.* **92**, 609–625.

Burgers, J. M. (1948) A mathematical model illustrating the theory of turbulence. *Advances in Mechanics*, **1**, 171–199.

Busse, F. H. (1970) Bounds for turbulent shear flow. *J. Fluid Mech.* **41**, 219–240.

Callen, H. B. and Greene, R. F. (1952) On a theorem of irreversible thermodynamics. *Phys. Rev.* **86**, 702–710.

Chandrasekhar, S. (1943) Stochastic problems in physics and astronomy. *Rev. Mod. Phys.* **15**, 1–89.

Charney, J. G. (1971) Geostrophic turbulence. *J. Atmos. Sci.* **28**, 1087–1095.

Cocke, W. J. (1969) Turbulent hydrodynamic line stretching: consequences of isotropy. *Phys. Fluids* **12**, 2488–2492.

Comte-Bellot, G. and Corrsin, S. (1966) The use of a contraction to improve the isotropy of grid-generated turbulence. *J. Fluid Mech.* **24**, 657–682.

Corrsin, S. (1962) Turbulent dissipation fluctuations. *Phys. Fluids* **5**, 1301–1302.

Corrsin, S. (1963) Turbulence: experimental methods. *Handbuch der Physik, Fluid Dynamics II*, vol. VIII/2, 524–590. Springer-Verlag.

Deardorff, J. W. (1970) A numerical study of three-dimensional turbulent channel flow at large Reynolds numbers. *J. Fluid Mech.* **41**, 453–480.

Ebin, D. G. and Marsden, J. (1970) Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math.* **92**, 102–163.

Edwards, S. F. (1964) The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239–273.

Edwards, S. F. and Taylor, J. B. (1974) Negative temperature states of two-dimensional plasmas and vortex fluids. *Proc. Roy. Soc.* **A336**, 257–271.

Ellison, T. H. (1962) The universal small-scale spectrum of turbulence at high Reynolds number. *Mécanique de la Turbulence*, 113–121. Paris: C.N.R.S.

Fox, D. G. and Orszag, S. A. (1973) Inviscid dynamics of two-dimensional turbulence. *Phys. Fluids* **16**, 169–171.

Frenkiel, F. N. and Klebanoff, P. S. (1967) Higher-order correlations in a turbulent fluid. *Phys. Fluids* **10**, 507–520.

Frisch, U. (1968) Wave propagation in random media. *Probabilistic Methods in Applied Mathematics*, 75–197. Academic Press.

Frisch, U. and Bourret, R. (1970) Parastochastics. *J. Math. Phys.* **11**, 364–390.

Frisch, U., Lesieur, M. and Brissaud, A. (1973) Qualitative problems in fully developed turbulence: a soluble model. To be published.

Gibson, C. H. and Masiello, P. J. (1972) Observations of the variability of dissipation rates of turbulent velocity and temperature fields. *Statistical Models and Turbulence*, 427–453. Springer-Verlag.

Gibson, C. H., Stegun, G. R. and Williams, R. B. (1970) Statistics of the fine structure of turbulent velocity and temperature fields measured at high Reynolds number. *J. Fluid Mech.* **41**, 153–167.

Goldstein, S. (1940) Three-dimensional vortex motion in a viscous fluid. *Phil. Magazine* **30**, 85–102.

Görtler, H. (1940) Über eine dreidimensionale Instabilität laminarer Grenzschichten an konkaven Wanden. *Nachr. Ges. Will. Gottingen* **2**, 1–26.

Grant, H. L., Stewart, R. W. and Moilliet, A. (1962) Turbulence spectra from a tidal channel. *J. Fluid Mech.* **12**, 241–268.

Gurvich, A. S. and Yaglom, A. M. (1967) Breakdown of eddies and probability distributions for small-scale turbulence. *Phys. Fluids* **10**, S59–S65.

Halmos, P. R. (1956) *Lectures on Ergodic Theory*. New York: Chelsea.

Hasselmann, K. (1962) On the nonlinear energy transfer in a gravity wave spectrum. *J. Fluid Mech.* **12**, 481–500.

Heisenberg, W. (1948) Zur statistischen Theorie der Turbulenz. *Z. Phys.* **124**, 628–657.

Herring, J. R. (1963) Investigation of problems in thermal convection. *J. Atmos. Sci.* **20**, 325–338.

Herring, J. R. (1965) Self-consistent-field approach to turbulence theory. *Phys. Fluids* **8**, 2219–2225.

Herring, J. R. (1966) Self-consistent-field approach to nonstationary turbulence. *Phys. Fluids* **9**, 2106–2110.

Herring, J. R. and Kraichnan, R. H. (1972) Comparison of some approximations for isotropic turbulence. *Statistical Models and Turbulence*, 148–194. Springer-Verlag.

Herring, J. R., Orszag, S. A., Kraichnan, R. H. and Fox, D. G. (1974) Decay of two-dimensional homogeneous turbulence. *J. Fluid Mech.* **66**, 417–444.

Hopf, E. (1952) Statistical hydromechanics and functional calculus. *J. Rational Mech. Anal.* **1**, 87–123.

Howard, L. N. (1972) Bounds on flow quantities. *Ann. Rev. Fluid Mech.* **4**, 473–494.

Joyce, G. and Montgomery, D. (1973) Negative temperature states for the two-dimensional guiding-centre plasma. *J. Plasma Phys.* **10**, 107–121.

Julian, P. R., Washington, W. M., Hembree, L. and Ridley, C. (1970) On the spectral distribution of large-scale atmospheric kinetic energy. *J. Atmos. Sci.* **27**, 376–387.

Kelvin, Lord (Sir William Thomson) (1887) On the propagation of laminar motion through a turbulently moving inviscid liquid. *Phil. Magazine* **24**, 342–353.

Khinchin, A. I. (1949) *Mathematical Foundations of Statistical Mechanics*. New York: Dover.

Kolmogorov, A. N. (1941a) The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *C. R. Acad. Sci., U.R.S.S.* **30**, 301–305. [Also *Soviet Physics-Uspekhi* **10**, 734–736 (1968).]

Kolmogorov, A. N. (1941b) On degeneration of isotropic turbulence in an incompressible viscous liquid. *C. R. Acad. Sci., U.R.S.S.* **31**, 538–540.

Kolmogorov, A. N. (1941c) Dissipation of energy in locally isotropic turbulence. *C. R. Acad. Sci., U.R.S.S.* **32**, 16–18.

Kolmogorov, A. N. (1962) A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **12**, 82–85.

Kovasznay, L. S. G. (1948) Spectrum of locally isotropic turbulence. *J. Aero. Sci.* **15**, 745–753.

Kraichnan, R. H. (1958) Irreversible statistical mechanics of incompressible hydromagnetic turbulence. *Phys. Rev.* **109**, 1407–1422.

Kraichnan, R. H. (1959) The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497–543.

Kraichnan, R. H. (1961) Dynamics of nonlinear stochastic systems. *J. Math. Phys.* **2**, 124–148. Erratum: 3, 205 (1962).

Kraichnan, R. H. (1962) Turbulent thermal convection at arbitrary Prandtl number. *Phys. Fluids* **5**, 1374–1389.

Kraichnan, R. H. (1964a) Decay of isotropic turbulence in the direct-interaction approximation. *Phys. Fluids* **7**, 1030–1048.

Kraichnan, R. H. (1964b) Kolmogorov's hypotheses and Eulerian turbulence theory. *Phys. Fluids* **7**, 1723–1734.

Kraichnan, R. H. (1965a) Inertial-range spectrum of hydromagnetic turbulence. *Phys. Fluids* **8**, 1385–1387.

Kraichnan, R. H. (1965b) Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575–598.

Kraichnan, R. H. (1966a) Invariance principles and approximation in turbulence dynamics. *Dynamics of Fluids and Plasmas*, 239–255. Academic Press.

Kraichnan, R. H. (1966b) Isotropic turbulence and inertial-range structure. *Phys. Fluids* **9**, 1728–1752.

Kraichnan, R. H. (1966c) Dispersion of particle pairs in homogeneous turbulence. *Phys. Fluids* **9**, 1937–1943.

Kraichnan, R. H. (1967) Inertial ranges in two-dimensional turbulence. *Phys. Fluids* **10**, 1417–1423.

Kraichnan, R. H. (1968a) Lagrangian-history statistical theory for Burgers' equation. *Phys. Fluids* **11**, 266–277.

Kraichnan, R. H. (1968b) Small-scale structure of a scalar field convected by turbulence. *Phys. Fluids* **11**, 945–953.

Kraichnan, R. H. (1970a) Convergents to turbulence functions. *J. Fluid Mech.* **41**, 189–217.

Kraichnan, R. H. (1970b) Turbulent diffusion: evaluation of primitive and renormalized perturbation series by Pade approximants and by expansion of Stieltjes transforms into contributions from continuous orthogonal functions. *The Pade Approximant in Theoretical Physics*, 129–170. Academic Press.

Kraichnan, R. H. (1971a) An almost-Markovian Galilean-invariant turbulence model,. *J. Fluid Mech.* **47**, 513–524.

Kraichnan, R. H. (1971b) Inertial-range transfer in two- and three-dimensional turbulence. *J. Fluid Mech.* **47**, 525–535.

Kraichnan, R. H. (1972) Test-field model for inhomogeneous turbulence. *J. Fluid Mech.* **56**, 287–304.

Kraichnan, R. H. (1974a) On Kolmogorov's inertial-range theories. *J. Fluid Mech.* **62**, 305–330.

Kraichnan, R. H. (1974b) Statistical dynamics of two-dimensional flow. To be published.

Kraichnan, R. H. and Spiegel, E. A. (1962) Model for energy transfer in isotropic turbulence. *Phys. Fluids* **5**, 583–588.

Kubo, R. (1959) Some aspects of the statistical-mechanical theory of irreversible processes. *Lectures in Theoretical Physics*, vol. 1. Interscience.

Kuo, A. Y. and Corrsin, S. (1971) Experiments on internal intermittency and fine-structure distribution functions in fully turbulent fluid. *J. Fluid Mech.* **50**, 285–319.

Kuo, A. Y. and Corrsin, S. (1972) Experiment on the geometry of the fine-structure regions in fully turbulent fluid. *J. Fluid Mech.* **56**, 447–479.

Ladyzhenskaya, O. A. (1963) *The Mathematical Theory of Viscous Incompressible Flow* (1st ed.) New York: Gordon and Breach [2nd ed., 1969].

Landau, L. D. and Lifschitz, E. M. (1959) *Fluid Mechanics*. Addison-Wesley.

Lee, T. D. (1952) On some statistical properties of hydrodynamical and magneto-hydrodynamical fields. *Q. Appl. Math.* **10**, 69–74.

Leith, C. E. (1967) Diffusion approximation to inertial energy transfer in isotropic turbulence. *Phys. Fluids* **10**, 1409–1416.

Leith, C. E. (1968) Diffusion approximation for two-dimensional turbulence. *Phys. Fluids* **11**, 671–673.

Leith, C. E. (1971) Atmospheric predictability and two-dimensional turbulence. *J. Atmos. Sci.* **28**, 145–161.

Leith, C. E. and Kraichnan, R. H. (1972) Predictability of turbulent flows. *J. Atmos. Sci.* **29**, 1041–1058.

Lighthill, M. J. (1958) *Fourier Analysis and Generalized Functions.* Cambridge University Press.

Lilly, D. K. (1971) Numerical simulation of developing and decaying two-dimensional turbulence. *J. Fluid Mech.* **45**, 395–415.

Lin, C. C. (1948) Note on the law of decay of isotropic turbulence. Proc. Nat. Acad. Sci., U.S.A. **34**, 230–233.

Lin, C. C. and Reid, W. H. (1963) Turbulent flow, theoretical aspects. *Handbuch der Physik, Fluid Dynamics II*, vol. VIII/2, 438–523. Springer-Verlag.

Loitsiansky, L. G. (1939) Some basic laws of isotropic turbulent flow. *Rep. Cent. Aero Hydodyn. Inst. (Moscow)*, no. 440, 3–23. (Translated as *Tech. Memor. Nat. Adv. Comm. Aero., Wash.*, no. 1079.)

Lorenz, E. N. (1972) Low-order models representing realizations of turbulence. *J. Fluid Mech.* **55**, 545–563.

Lumley, J. L. (1964) The spectrum of nearly inertial turbulence in a stably stratified fluid. *J. Atmos. Sci.* **21**, 99–102.

Lumley, J. L. (1970) *Stochastic Tools in Turbulence.* Academic Press.

Lumley, J. L. (1972) Application of central limit theorems to turbulence problems. *Statistical Models and Turbulence*, 1–26. Springer-Verlag.

Lundgren, T. S. (1967) Distribution functions in the statistical theory of turbulence. *Phys. Fluids* **10**, 969–975.

Mandelbrot, B. B. (1974) Intermittent turbulence in self-similar cascades. *J. Fluid Mech.* **62**, 331–358.

Martin, P. C., Siggia, E. D. and Rose, H. A. (1974) Statistical dynamics of classical systems. *Phys. Rev.* **A8**, 423–437.

Millionshtchikov, M. (1941) On the theory of homogeneous isotropic turbulence. *C. R. Acad. Sci., U.R.S.S.* **32**, 615–618.

Moffatt, H. K. (1976) Lectures on the dynamo problem. This volume.

Monin, A. C. and Yaglom, A. M. (1967) *Statistical Fluid Mechanics*, vol. 2 (in Russian). Moscow: Nauka Press.

Monin, A. C. and Yaglom, A. M. (1971) *Statistical Fluid Mechanics*, vol. 1. Cambridge: MIT Press.

Morton, J. B. and Corrsin, S. (1970) Consolidated expansions for estimating the response of a randomly driven nonlinear oscillator. *J. Stat. Phys.* **2**, 153–194.

Nakano, T. (1972) A theory of homogeneous, isotropic turbulence of incompressible fluids. *Ann. Phys.* **73**, 326–371.

Novikov, E. A. (1971) Intermittency and scale similarity in the structure of a turbulent flow. *Appl. Math. Mech.* **35**, 231–241. (*Prikl. Math. Mech.* **35**, 266–277.)

Nyquist, H. (1928) Thermal agitation of electric charge in conductors. *Phys. Rev.* **32**, 110.

Obukhov, A. M. (1941) On the distribution of energy in the spectrum of a turbulent flow. *C. R. Acad. Sci., U.R.S.S.* **32**, 19–21.

Obukhov, A. M. (1962) Some specific features of atmospheric turbulence. *J. Fluid Mech.* **12**, 77–81. (Also *J. Geophys. Res.* **67**, 3011–3014.)

Ogura, Y. (1963) A consequence of the zero-fourth-cumulant approximation in the decay of isotropic turbulence. *J. Fluid Mech.* **16**, 33–40.

Onsager, L. (1945) The distribution of energy in turbulence. *Phys. Rev.* **68**, 286.

Onsager, L. (1949) Statistical hydrodynamics. *Nuovo Cimento* (Suppl.) **6**, 279–287.

Orszag, S. A. (1966) *Dynamics of Fluid Turbulence.* Princeton: Plasma Physics Laboratory Report PPL-AF-13.

Orszag, S. A. (1967) Approximate calculation of the Kolmogorov-Obukhov constant. *Phys. Fluids* **10**, 454–455.

Orszag, S. A. (1969) Representation of isotropic turbulence by scalar functions. *Stud. in Appl. Math.* **48**, 275–279.

Orszag, S. A. (1970a) Comments on "Turbulent hydrodynamic line stretching: consequences of isotropy." *Phys. Fluids* 13, 2203–2204.

Orszag, S. A. (1970b) Indeterminacy of the moment problem for intermittent turbulence. *Phys. Fluids* **13**, 2211–2212.

Orszag, S. A. (1970c) Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363–386.

Orszag, S. A. and Fateman, R. J. (1974) The Taylor-Green vortex.

Orszag, S. A. and Israeli, M. (1974) Numerical simulation of viscous incompressible flows. *Ann. Rev. Fluid Mech.* **6**, 281–318.

Orszag, S. A. and Kruskal, M. D. (1968) Formulation of the theory of turbulence. *Phys. Fluids* **11**, 43–60.

Orszag, S. A. and Patterson, G. S. (1972) Numerical simulation of three-dimensional homogeneous isotropic turbulence. *Phys. Rev. Letters* **28**, 76–79. (Also *Statistical Models and Turbulence*, 127–147. Springer-Verlag.)

Orszag, S. A. and Raila, D. S. (1973) Test of spectral energy transfer models and turbulence decay. *Phys. Fluids* **16**, 172–173.

Phillips, O. M. (1966) *The Dynamics of the Upper Ocean.* Cambridge University Press.

Phythian, R. (1969) Self-consistent perturbation series for stationary homogeneous turbulence. *J. Phys.* (*Gen. Phys.*) **A2**, 181–192.

Proudman, I. and Reid, W. H. (1954) On the decay of a normally distributed and homogeneous turbulent velocity field. *Phil. Trans. Roy. Soc.* **A247**, 163–189.

Reissner, E. (1938) Note on the statistical theory of turbulence. *Proc. 5th Int. Congr. Appl. Mech.*, 359.

Robertson, H. P. (1940) The invariant theory of isotropic turbulence. *Proc. Camb. Phil. Soc.* **36**, 209–223.

Ruelle, D. and Takens, F. (1971) On the nature of turbulence. *Comm. Math. Phys.* **20**, 167–192.

Saffman, P. G. (1967a) The large-scale structure of homogeneous turbulence. *J. Fluid Mech.* **27**, 581–593.

Saffman, P. G. (1967b) Note on decay of homogeneous turbulence. *Phys. Fluids* **10**, 1349.

Saffman, P. G. (1968) Lectures in homogeneous turbulence. *Topics in Nonlinear Physics*, 485–614. Springer-Verlag.

Saffman, P. G. (1970) Dependence on Reynolds number of high-order moments of velocity derivatives in isotropic turbulence. *Phys. Fluids* **13**, 2193–2194.

Saffman, P. G. (1971) A note on the spectrum and decay of random two-dimensional vorticity distributions at large Reynolds number. *Stud. in Appl. Math.* **50**, 377–383.

Shur, G. N. (1962) Experimental investigations of the energy spectrum of atmospheric turbulence. *Trudy T. S.*; *A.O.* **43**, 79–90.

Simmons, L. F. G. and Salter, C. (1934) Experimental investigation and analysis of the velocity variations in turbulent flow. *Proc. Roy. Soc.* **A145**, 212–234.

Smagorinsky, J. (1963) General circulation experiments with the primitive equations. I. The basic experiment. *Mon. Weath. Rev.* **91**, 99–165.

Stewart, R. W., Wilson, J. R. and Burling, R. W. (1970) Some statistical properties of small-scale turbulence in an atmospheric boundary layer. *J. Fluid Mech.* **41**, 141–152.

Takahasi, H. (1952) Generalized theory of thermal fluctuations. *J. Phys. Soc., Japan* **7**, 439–446.

Tanaka, H. (1969) 0–5th cumulant approximation of inviscid Burgers turbulence. *J. Meteor. Soc., Japan* **47**, 373–383.

Tatarskii, V. I. (1962) Application of quantum field theory methods to the problem of degeneration of homogeneous turbulence. *Soviet Physics-JETP* **15**, 961–967.

Tatsumi, T. (1957) The theory of decay process of incompressible homogeneous turbulence. Proc. Roy. Soc. **A239**, 16–45.

Tatsumi, T. (1960) Energy spectra in magneto-fluid dynamic turbulence. *Rev. Mod. Phys.* **32**, 807.

Taylor, G. I. (1923) Stability of a viscous liquid contained between two rotating cylinders. *Phil. Trans. Roy. Soc.* **A223**, 289–343.

Taylor, G. I. (1938) Production and dissipation of vorticity in a turbulent fluid. *Proc. Roy. Soc.* **A164**, 15–23.

Taylor, G. I. and Green, A. E. (1937) Mechanism of the production of small eddies from large ones. *Proc. Roy. Soc.* **A158**, 499–521.

Tennekes, H. (1968) Simple model for the small-scale structure of turbulence. *Phys. Fluids* **11**, 669–671.

Tennekes, H. and Wyngaard, J. C. (1972) The intermittent small-scale structure of turbulence: data-processing hazards. *J. Fluid Mech.* **55**, 93–103.

Thompson, P. D. (1972) Some exact statistics of two-dimensional viscous flow with random forcing. *J. Fluid Mech.* **55**, 711–717.

Thompson, P. D. (1973) The equilibrium energy spectrum of randomly forced two-dimensional turbulence. *J. Atmos. Sci.* **30**, 1593–1598.

Townsend, A. A. (1951) On the fine-scale structure of turbulence. *Proc. Roy. Soc.* **A208**, 534–542.

Van Atta, C. W. (1974) Sampling techniques in turbulence measurements. *Ann. Rev. Fluid Mech.* **6**, 75–91.

Van Atta, C. W. and Park, J. (1972) Statistical self-similarity and inertial subrange turbulence. *Statistical Models and Turbulence*, 402–426. Springer-Verlag.

von Kármán, T. (1948) Sur la théorie statistique de la turbulence. *C. R. Acad. Sci., Paris* **226**, 2108–2114.

von Kármán, T. and Howarth, L. (1938) On the statistical theory of isotropic turbulence. *Proc. Roy. Soc.* **A164**, 192–215.

von Weizsäcker, C. F. (1948) Das Spektrum der Turbulenz bei grossen Reynolds'schen Zahlen. *Z. Phys.* **124**, 614–627.

Whittaker, E. T. and Watson, G. N. (1927) *A Course of Modern Analysis* (4th ed.), Cambridge University Press.

Wiener, N. (1930) Generalized harmonic analysis. *Acta Math.* **55**, 117–258

Wyld, H. W. (1961) Formulation of the theory of turbulence in an incompressible fluid. *Ann. Phys.* **14**, 143–165.

Wyngaard, J. C. and Pao, Y. H. (1972) Some measurements of the fine structure of large Reynolds number turbulence. *Statistical Models and Turbulence*, 384–401. Springer-Verlag.

Wyngaard, J. C. and Tennekes, H. (1970) Measurements of the small-scale structure of turbulence at moderate Reynolds numbers. *Phys. Fluids* **13**, 1962–1969.

Yaglom, A. M. (1966) The influence of fluctuations in energy dissipation on the shape of turbulence characteristics in the inertial interval. *Soviet Physics-Doklady* **11**, 26–29.