6
Geostrophic Turbulence

Strongly nonlinear, rapidly rotating, stably stratified flow is called *geostrophic turbulence*. This subject, which blends ideas from Chapters 2, 4 and 5, is relevant to the large-scale flow in the Earth’s oceans and atmosphere. The quasigeostrophic equations form the basis. We view the quasigeostrophic equations as a generalization of the vorticity equation for two-dimensional turbulence, to include the important effects of stratification, bottom topography and varying Coriolis parameter. Thus the theory of geostrophic turbulence represents an extension of the theory of two-dimensional turbulence. However, its richer physics and greater applicability to real geophysical flows make geostrophic turbulence a much more interesting and important subject. This chapter offers a very brief introduction to the theory of geostrophic turbulence. We illustrate the principal themes and ideas by separately considering the effects of bottom topography, varying Coriolis parameter and density stratification on highly nonlinear, quasigeostrophic flow. We make no attempt at a comprehensive review.\(^1\)

In every case, the theory of geostrophic turbulence relies almost solely on two now-familiar components: a *conservation principle* that energy and potential vorticity are (nearly) conserved, and an *irreversibility principle* in the form of an appealing assumption that breaks the time-reversal symmetry of the exact (inviscid) dynamics. This irreversibility assumption takes a great many superficially dissimilar forms, fostering the misleading impression of a great many competing explanations for the same phenomena. However, broad-minded analysis inevitably reveals that these competing explanations are virtually equivalent.

We begin by considering the quasigeostrophic flow of a single layer of homogeneous fluid over a bumpy bottom. No case better illustrates how diverse forms of the irreversibility principle lead to the same conclusions.

1. Quasigeostrophic flow over topography

Consider a single layer of rotating homogeneous fluid in a closed, simply connected domain with a rigid lid at \(z=0\) and a bumpy bottom at \(z=-H_0+\Delta(x,y)\), where \(H_0\) is the mean fluid depth, and \(|\Delta|<H_0\). Let the Coriolis parameter \(f_0\) be a constant. The governing quasigeostrophic equation is

\[
\frac{\partial q}{\partial t} + J(\psi, q) = 0 ,
\]

where

\[
q = \nabla^2 \psi + h \tag{1.2}
\]

is the potential vorticity.
\[ h(x,y) \equiv \frac{f_0}{H_0} \Delta(x,y), \]  

(1.3)

and \( J \) is the Jacobian operator. The boundary condition is \( \psi = 0 \). We shall call \( h \) the bottom topography. In the northern hemisphere, \( h \) is positive on seamounts and negative on seafloor troughs.

Now suppose that the flow begins, at \( t=0 \), with random initial conditions. By this we mean that the initial velocity is uncorrelated with the bottom topography. More specifically, the average initial relative vorticity \( \zeta \equiv \nabla^2 \psi \) vanishes at every location. We shall show that, as time increases, the flow acquires a nonzero average that is strongly correlated with the bottom topography.

The emergence of this non-vanishing mean flow is a simple consequence of the conservation of potential vorticity (1.2) on fluid particles. Consider an arbitrary curve enclosing the top of a seamount (Figure 6.1). At any fixed time after \( t=0 \), some of the fluid particles inside the curve will have originated at points outside the curve. Since all the outside points are at depths greater (i.e., at values of \( h \) less) than the points inside the curve, all of these fluid particles will have experienced an increase in \( h \). Then, since each fluid particle conserves its \( \zeta + h \), and all fluid particles start with zero (average) \( \zeta \), it follows that, at \( t>0 \), the fluid particles inside the seamount-curve in Figure 6.1 will have a net negative relative vorticity \( \zeta \). Therefore, by the circulation theorem, the circulation around the seamount-curve must be clockwise, as shown in Figure 6.1. That is, the seamount induces an anticyclonic circulation. By similar reasoning, the flow around an isolated trough (where \( h<0 \)) becomes cyclonic.

Now suppose that the fluid depth is relatively small near the boundary of the domain. (In quasigeostrophic theory, it cannot actually vanish!) By the same reasoning as in the preceding paragraph, one can show that the flow along the continental slope near the boundary must be cyclonic, that is, counter-clockwise around the basin in the northern hemisphere. To see this, we simply apply our argument to the “boundary region” in Figure 6.1, the region inside the closed curve that includes the boundary. If the fluid depth is small near the boundary, then some of the fluid particles inside this boundary region must have come from much deeper water, and, consequently, have a negative relative vorticity. The circulation around the boundary region in Figure 6.1 must therefore be anticyclonic. But since (1.1) conserves

\[ Q_i \equiv \iint dx \ q \]  

(1.4)

and hence

\[ V \equiv \iint dx \ \zeta, \]  

(1.5)

where the integration runs over the whole domain, the circulation around the boundary itself vanishes at all time. It follows that the flow around the seaward edge of the “boundary region” must be as shown in Figure 6.1.
These arguments depend critically on our hypothesis of an initial state in which the average relative vorticity vanishes everywhere. This assumption seems unrealistic, because it gives the time $t=0$ a special dynamical significance. However, we can avoid the assumption about initial conditions by introducing a dissipation that *gradually diminishes* the relative vorticity. If, to take the simplest example, we adopt Rayleigh friction, and replace (1.1) by

$$ \frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi + h) = -\varepsilon \nabla^2 \psi, $$

then the potential vorticity of each fluid particle decays toward the local value of $h$. Again, fluid particles in shallow water will most recently have been in deeper water, where the dissipation adjusted their potential vorticity toward the smaller value of $h$ in deep water. Then the increase in $h$ experienced by fluid particles moving onto the seamount, while (now) *nearly* conserving their $\zeta+h$ induces a negative $\zeta$, and the argument proceeds as before. Of course, flow governed by (1.6) *eventually* comes to rest unless a forcing term is also added to (1.6). However, the inclusion of forcing does not really disturb the argument unless the forcing is correlated with the flow or topography.

By these two qualitative arguments, quasigeostrophic flow develops a correlation between the flow and the topography, in the sense of anticyclonic flow over seamounts. The first argument is based on the *exact* conservation of potential vorticity on fluid particles, and on the assumption of vanishing average initial relative vorticity. The second argument assumes *approximate* potential vorticity conservation, with the relative vorticity continually nudged toward zero by the dissipation.

These two arguments are obviously very similar, and both contain the two essential ingredients: a *conservation principle*, that potential vorticity is conserved (or nearly conserved) on fluid particles, and an *irreversibility principle* that breaks the time-reversal symmetry of the exact dynamics (1.1). In the first argument, the irreversibility principle corresponds to the assumption of a highly constrained initial condition. The assumption of vanishing average initial vorticity is analogous to the assumption of a narrow spectral peak invoked in Chapter 4 to explain the transfer of energy to large spatial scales in two-dimensional turbulence. In the second argument, the irreversibility principle corresponds more simply to the introduction of dissipation, which imposes a direction in time.

In this and the following section, we consider various alternative explanations for the correlation between rapidly rotating flow and bottom topography. Although these explanations are distinct and superficially quite different, they all rely upon a linkage between conservation and irreversibility.

As a third explanation, we consider the *principle of potential vorticity homogenization*, an idea used in the theory of the wind-driven ocean circulation. Since (1.1) conserves the potential vorticity on fluid particles, an initially smooth distribution of $q$ ought gradually to acquire a very filamentary, *fine-grained* structure. Let the average, large-scale potential vorticity correspond to a *coarse-grained* smoothing of this filamentary $q$. If the fluid-particle trajectories are space-filling in the sense that each fluid particle eventually comes arbitrarily close to any point in the domain, then this average $q$-distribution will become uniform. If (for example) the total potential vorticity (1.4) vanishes initially, then the final average $\zeta$ is given by
\[ \zeta = -h, \quad (1.7) \]

which implies a correlation between flow and topography with the same sign as previously predicted.

The prediction (1.7) is more quantitative than the two previous arguments, but it cannot generally be correct. (It fails because the potential vorticity is not a \textit{passive} tracer; hence fluid particles with different values of potential vorticity spend different fractions of their time in a particular locality.) To see that (1.7) cannot be correct, consider the limiting case of an arbitrarily weak initial flow. Because flow governed by (1.1) also conserves (twice) the energy,

\[ E \equiv \int \int dx \nabla \psi \cdot \nabla \psi, \quad (1.8) \]

the fluid cannot generally reach the state (1.7) (whose amplitude is determined solely by the prescribed topography \( h(x,y) \)) without violating energy conservation.

To incorporate the constraint imposed by energy conservation, we consider the \textit{principle of minimum potential enstrophy}.\(^3\) According to this, the coarse-grained potential vorticity distribution is that which minimizes the \textit{potential enstrophy},

\[ Q_2 \equiv \int \int dx \, q^2, \quad (1.9) \]

a measure of the spatial nonuniformity in \( q \), subject to the requirement that the energy (1.8) and total vorticity (1.5) be conserved. In other words, we minimize

\[ \int \int dx \left\{ (\nabla^2 \psi + h)^2 + \mu \nabla \psi \cdot \nabla \psi - 2\lambda \nabla^2 \psi \right\}, \quad (1.10) \]

where \( \mu \) and \( \lambda \) are the Lagrange multipliers corresponding to the constraints (1.8) and (1.5), respectively. Suppose we minimize (1.10) (approximately) by first expanding

\[ (\psi, h) = \sum_i (\psi_i, h_i) \phi_i(x,y) \quad (1.11) \]

in the orthonormal eigenfunctions \( \{ \phi_i \} \) defined by

\[ \nabla^2 \phi_i = -k_i^2 \phi_i, \quad \text{and} \quad \phi_i = 0 \quad \text{on the boundary.} \quad (1.12) \]

In a square domain, (1.11-12) corresponds to a Fourier-sine expansion of \( \psi \) and \( h \). Then (1.10) becomes

\[ \sum_i \left\{ (k_i^2 \psi_i - h_i)^2 + \mu k_i^2 \psi_i^2 + 2\lambda k_i^2 \psi_i \int \int dx \phi_i \right\}, \quad (1.13) \]

with minimum at
\[ \psi_i = \frac{h_i - \lambda \int \! d\mathbf{x} \phi_i}{\mu + k_i^2}. \]  

(1.14)

The transformation of (1.14) is

\[ \nabla^2 \psi + h = \mu \psi + \lambda. \]  

(1.15)

The values of \( \mu \) and \( \lambda \) are determined by the values of the energy \( E \) and total potential vorticity \( Q_1 \). If \( \mu = 0 \) (no energy constraint) and \( \lambda = 0 \) (vanishing \( Q_1 \)), then (1.15) reduces to (1.7). Let \( k_{\min} \) be the lowest eigenvalue in the set defined by (1.12); then \( k_{\min}^{-1} \) is the size of the domain. In general, \( \mu \) depends on the value of the energy \( E \), but clearly \( \mu > -k_{\min}^2 \) for all finite \( E \). Thus, like all the previous arguments, (1.14) predicts a positive correlation between the streamfunction and the bottom topography.

The foregoing general arguments ought to apply to the limiting case of a very weak flow. In this small-amplitude limit, we can check our conclusions by actually calculating the flow. First, rewrite the quasigeostrophic equation (1.1) in the flux form,

\[ \frac{\partial \zeta}{\partial t} = -\nabla \cdot \mathbf{F}, \quad \mathbf{F} = \mathbf{u}(h + \zeta), \quad \mathbf{u} = (-\psi_y, \psi_x), \]  

(1.16)

and suppose that

\[ \psi(x,t) = \varepsilon \psi_0(x,t) + \varepsilon^2 \psi_1(x,t) + \cdots \]  

(1.17)

where \( \varepsilon \) is small. At leading order, the streamfunction is determined by the linear equation,

\[ \frac{\partial}{\partial t} \nabla^2 \psi_0 + J(\psi_0, h) = 0, \]  

(1.18)

for topographic Rossby waves. Again we consider a closed curve around the top of the seamount, but now we require that the closed curve be an isobath. The circulation around the isobath is determined by the integral of (1.16) over the region inside. However, since the net mass flux across the isobath must vanish, the first (linear) term in \( \mathbf{F} \) makes no net contribution. Thus the rate of change in the circulation around the isobath is solely determined by the nonlinear part of the flux,

\[ \mathbf{F}_{nl} = \mathbf{u} \zeta = \varepsilon^2 \mathbf{u}_0 \zeta_0 + O(\varepsilon^3), \]  

(1.19)

whose leading term depends only on the solutions of the linear equation (1.18). Now, for the (expected) negative circulation to develop, \( \mathbf{F}_{nl} \) must be directed off the seamount, that is,
\[ \zeta_0 \mathbf{u}_0 \cdot \nabla h < 0. \quad (1.20) \]

But (1.18) implies that
\[ \frac{\partial Z_0}{\partial t} = -\zeta_0 \mathbf{u}_0 \cdot \nabla h, \quad \text{where} \quad Z_0 \equiv \frac{1}{2} \zeta_0^2. \quad (1.21) \]
is the enstrophy density, a measure of the activity in the topographic Rossby waves satisfying (1.18). If this activity is increasing (\( \partial Z_0/\partial t > 0 \)), then (1.21) implies (1.20), and \( \mathbf{F}_{nl} \) has the right direction to cause anticyclonic flow around the seamount. However, if the waves are transient, that is, if \( Z_0 \) builds up and then gradually decays as the waves propagate away, then the average value of the right-hand side of (1.21) must vanish, and there can be no average flux leading to a mean circulation. Similarly, transient waves can cause no permanent change in the value of a pre-existing circulation. This is an example of a non-acceleration theorem.\(^4\)

To obtain mean circulation, we must again introduce irreversibility. Suppose then that friction acts to dissipate the waves. To keep the waves from dying out, we suppose that a distant forcing continually excites new waves. If the whole system reaches the state of steady waves, then the average (over a wave-cycle) of (1.21) is
\[ 0 = -\langle \zeta_0 \mathbf{u}_0 \cdot \nabla h \rangle - D, \quad (1.22) \]
where \( D \) represents the loss of wave-activity to friction. (There is no forcing term, because, by hypothesis, the forcing acts far from the seamount.) Since \( D \) must be positive, (1.20) holds on average, and the vorticity flux induces anticyclonic flow.

2. The statistical mechanics of flow over topography

To all of these arguments, we now add another. In Chapter 5, we proposed that equilibrium statistical mechanics describes the final states towards which numerical models of equations like (1.1) (that is, model systems with a finite number of degrees of freedom) evolve when forcing and dissipation are absent.\(^5\) Let the streamfunction and topography be expanded in the orthonormal eigenfunctions defined by (1.12). Then the model equation is
\[ \frac{d\psi_i}{dt} = \sum_{j,m} A_{jm} \psi_j \psi_m, \quad (2.1) \]
where
\[ A_{jm} = k_i^{-2} \iint d\mathbf{x} \phi_i \mathbf{J}(\phi_j, \phi_m) \quad (2.2) \]
and
\[ q_m = -k_m^2 \psi_m + h_m. \]  

(2.3)

The sums in (2.1) run over a *truncated* set of \( N \) modes with minimum wavenumber \( k_{\text{min}} \) and maximum wavenumber \( k_{\text{max}} \). \( k_{\text{min}}^{-1} \) is the domain size, and \( k_{\text{max}}^{-1} \) is the smallest scale resolved by the model.

Since \( A_{ijm} \) vanishes whenever two of its indices are equal, the motion defined by (2.1) is nondivergent in the phase space spanned by \( \{ \psi_1, \psi_2, \ldots, \psi_N \} \). By the reasoning of Chapter 5 (and with the many qualifications explained there), an ensemble of systems obeying (2.1) eventually acquires the probability density function

\[ P(\psi_1, \psi_2, \ldots, \psi_N) = C \exp \left\{ -\alpha \sum_i k_i^2 \psi_i^2 - \beta \sum_i \left( k_i^2 \psi_i - h_i \right)^2 - 2\gamma \sum_i k_i^2 \psi_i \int d\phi \right\}, \]

(2.4)

where the constants \( \alpha, \beta, \gamma \), and \( C \) are determined by the average values of the energy,

\[ E = \sum_i k_i^2 \langle \psi_i^2 \rangle, \]

(2.5)

the potential enstrophy,

\[ Q_2 = \sum_i \langle (k_i^2 \psi_i - h_i)^2 \rangle, \]

(2.6)

and the vorticity,

\[ V = -\sum_i k_i^2 \langle \psi_i \rangle \int d\phi, \]

(2.7)

and by the normalization requirement on \( P \). We find that

\[ \langle \psi_i \rangle = \int \cdots \int \left( \prod_j d\psi_j \right) \psi_i P = \frac{\int_{-\infty}^{\infty} \psi_i P(\psi_i) d\psi_i}{\int_{-\infty}^{\infty} P(\psi_i) d\psi_i} = \frac{\beta h_i - \gamma \int d\phi}{\alpha + \beta k_i^2} \]

(2.8)

where

\[ P(\psi_i) \equiv \exp \left\{ -k_i^2 \left( \alpha + \beta k_i^2 \right) \left\{ \psi_i - \frac{\beta h_i - \gamma \int d\phi}{\alpha + \beta k_i^2} \right\}^2 \right\}. \]

(2.9)

Similarly,

\[ \langle \psi_i^2 \rangle = \langle \psi_i \rangle^2 + \frac{1}{2k_i^2(\alpha + \beta k_i^2)}. \]

(2.10)
Thus, the absolute-equilibrium flow,

\[ \psi(x) = \langle \psi(x) \rangle + \psi'(x), \]  

(2.11)

consists of a mean flow \( \langle \psi \rangle \) satisfying

\[ \nabla^2 \langle \psi \rangle + h = \frac{\alpha}{\beta} \langle \psi \rangle + \frac{\gamma}{\beta}, \]  

(2.12)

and a fluctuating flow \( \psi' \) with energy

\[ k_i^2 \langle (\psi'_i)^2 \rangle = \frac{1}{2(\alpha + \beta k_i^2)} \]  

(2.13)

in wavenumber \( k_i \). The equation (2.12) for the mean flow is formally identical to the equation (1.15) obtained from the principle of minimum potential enstrophy; however, its constants \( (\alpha/\beta \) and \( \gamma/\beta \) are differently determined. Similarly, the energy (2.13) associated with the fluctuating part of the flow is formally identical to the absolute-equilibrium spectrum for ordinary two-dimensional turbulence (to which (1.1) reduces if \( h \equiv 0 \); see Section 7 of Chapter 5), but, again, \( \alpha \) and \( \beta \) are determined differently in the two cases.

Now \( \{ \psi'_i \} \) (all \( i \)) and hence \( \{ \psi'(x) \} \) (all \( x \)) are jointly Gaussian with zero means. The physical-space covariance function is

\[ \langle \psi'(x) \psi'(x_0) \rangle = \sum_{i,j} \langle \psi'_i \psi'_j \rangle \phi_i(x) \phi_j(x_0) \]

\[ = \sum_i \langle (\psi'_i)^2 \rangle \phi_i(x) \phi_i(x_0) = \sum_i \frac{1}{2k_i^2(\alpha + \beta k_i^2)} \phi_i(x) \phi_i(x_0) \]  

(2.14)

because the \( \psi_i \) are also uncorrelated. But if the eigenfunctions \( \{ \phi_i(x) \} \) are complete,

\[ \sum_i \phi_i(x) \phi_i(x_0) = \delta(x - x_0), \]  

(2.15)

where \( \delta(\cdot) \) is (the truncated approximation to) Dirac’s delta-function. Thus (2.14) implies that

\[ 2(\alpha - \beta \nabla^2) \nabla^2 \langle \psi'(x) \psi'(x_0) \rangle = -\delta(x - x_0). \]  

(2.16)

Since jointly Gaussian random variables are determined by their means and covariances, (2.12) and (2.16) completely describe the absolute-equilibrium state. However, these two equations contain no trace of the eigenfunctions \( \{ \phi_i \} \). Thus the absolute-equilibrium
state is independent of the precise method of discretization; we could also obtain (2.12) and (2.16) by regarding (2.1) as a finite-difference approximation to (1.1).

According to (2.12) the mean flow is locked to the topography and hence generally anisotropic, but (2.16) shows that the fluctuating flow is isotropic to the extent allowed by boundary conditions.

To determine the absolute equilibrium state, we must solve (2.5), (2.6) and (2.7) for $\alpha$, $\beta$, and $\gamma$. As in Chapter 5, the interesting case is the limiting case of infinite spatial resolution, $k_{\text{max}} \to \infty$, with $E$, $Q$, $V$ and $k_{\text{min}}$ held fixed. In this limit, the mean flow determined by (2.12) absorbs as much of the energy as possible. If the lowest wavenumber $k_{\text{min}}$ has non-zero bottom topography (the generic case with bottom topography), then the mean flow absorbs all of the energy. (As $\alpha/\beta \to -k_{\text{min}}^{-2}$, the mean-flow energy increases without bound.) If, on the other hand, there is no topography in $k_{\text{min}}$ (as in the special case of a flat bottom), then the mean-flow energy is upper-bounded by its value at $\alpha/\beta = -k_{\text{min}}^{-2}$. If the given energy exceeds that amount, then the excess energy appears in a (basin-filling) eddy at $k_{\text{min}}$. As $k_{\text{max}} \to \infty$, all of the potential enstrophy not required by the topographic mean flow and the basin-filling eddy (if present) goes to infinite wavenumbers.

In the cases where the mean flow determined by (2.12) absorbs all of the energy, the mean flow must be identical to that predicted by the minimum potential enstrophy principle (1.15). That is, $\alpha/\beta \to \mu$ and $\gamma/\beta \to \lambda$ as $k_{\text{max}} \to \infty$. The equilibrium state is a nearly steady state, locked to the topography, in which the only fluctuations occur at very small spatial scales. Once again, we predict anticyclonic flow around seamounts and cyclonic flow around seafloor troughs.

Numerical experiments confirm this prediction. Figure 6.2 shows the streamfunction $\psi$ in a solution of the equation (1.1-2) for quasigeostrophic flow over topography in a bounded basin. The initial conditions (at $t=0$) are the same as those in the two-dimensional-turbulence ($h=0$) solution described in Chapter 4; see Figure 4.9. The boundary conditions are $\psi=0$. In the experiment summarized in Figure 6.2, the rms fractional topography $\Delta/H_0$ is about 25 times larger than the Rossby number based upon the basin width. This corresponds to $\Delta/H_0 \approx 0.02$ in the mid-latitude ocean. Thus, the topography is relatively small, but it still controls the dynamics. The time-unit is the time required for a fluid particle to cross the basin at the rms speed of the fluid. Figure 6.2 shows the streamfunction $\psi$ at times $t=1.0$, $t=2.0$ and $t=3.0$, the time-average streamfunction between $t=2.0$ and $3.0$, and the topography $h$. By $t=2.0$, the flow is nearly steady and locked to the topography. (However, the streamfunction must obey the boundary condition $\psi \neq 0$, whereas $h \neq 0$ at boundaries.)

Now, which of the various explanations for the strong correlation between flow and topography in this and the preceding section is the most satisfactory and convincing? I believe that, from a suitably broad-minded point of view, they are all really the same explanation. All invoke the conservation of potential vorticity on fluid particles and some form of the irreversibility principle. Their main differences lie in the particular choice of the latter.

In the explanation of this section, the irreversibility principle is the principle of maximum entropy — the fundamental principle of equilibrium statistical mechanics.
Although the principles of uniform potential vorticity and minimum potential enstrophy have undeniable appeal, the entropy principle is a general principle of physics. This gives the statistical-mechanical explanation a pleasing generality.

However, the statistical-mechanical explanation certainly demands a cautious interpretation. First, it really only applies to unforced, inviscid numerical models (that is, models with a finite number of degrees of freedom). Thus it gives an exact prediction but for an artificial situation that is not of primary interest. For realistic systems with forcing and dissipation, the absolute equilibrium states are of interest only to the extent to which they represent the general direction towards which the nonlinear terms in the equations of motion would, acting by themselves, drive the flow.

Second, even granted the ideal system, the prediction of equilibrium statistical mechanics rests on an irreversibility principle — the principle of uniform probability density in phase space — which is after all a guess. In the systems typically encountered in thermodynamics, this guess works so well that it has come to be regarded as law. However, discrete approximations to the differential equations governing macroscopic systems sometimes behave quite differently.

3. Flow on the beta-plane

Now consider the case of bounded, single-layer, quasigeostrophic flow in which the bottom is flat, but in which the Coriolis parameter,

\[ f = f_0 + \beta y, \]

increases in the northward direction, in the usual beta-plane approximation. The governing potential-vorticity equation,

\[ \frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad q = \nabla^2 \psi + f, \]

is equivalent to (1.1-2), with \( f \) replacing the bottom topography \( h \). However (3.2), unlike (1.1-2), holds under the weaker assumptions of shallow-water theory for an homogeneous fluid of uniform depth. In particular, \( f \) can change by a large amount over the fluid domain.

Since (3.1-2) is a case of (1.1-2), all the reasoning of Sections 1 and 2 also applies to (3.1-2). However, the special dependence of \( f \) on \( y \) leads to distinctive behavior that is best considered as a separate case.

Equilibrium statistical mechanics (or the principle of minimum potential enstrophy) predicts that numerical solutions of (3.1-2) approach a nearly steady state in which the streamfunction obeys

\[ \nabla^2 \psi + \beta(y - y_0) = \mu \psi, \]

and the constants \( y_0 \) and \( \mu \) are determined by the initial values of the energy,
\[ E = \iint d\mathbf{x} \, \nabla \psi \cdot \nabla \psi, \quad (3.4) \]

and the total vorticity,
\[ V = \iint d\mathbf{x} \, \nabla^2 \psi. \quad (3.5) \]

For simplicity, let the flow domain be a square box of side \( L \) on \( 0 < x, y < L \). Let the initial conditions correspond to a field of eddies with fluid velocity of size \( u_0 \). Let the initial total vorticity (3.5) be zero.

In all cases for which the energy is realistically small, the first term in the equation (3.3) for the final flow turns out to be insignificant outside thin inertial boundary layers at the sides of the domain. In these cases, (3.3), first considered by Fofonoff (1954), is a boundary-layer problem of the kind considered in Chapter 3. Let
\[ U \equiv \beta / \mu \quad (3.6) \]
replace the constant \( \mu \). Then the interior solution of (3.3) is
\[ \psi_I = U(y - y_0), \quad (3.7) \]
corresponding to a uniform eastward velocity of \(-U\).

The interior solution (3.7) does not satisfy the boundary condition \( \psi = 0 \). In the boundary layer near \( x = 0 \), the correction streamfunction, \( \hat{\psi} \equiv \psi - \psi_I \), obeys
\[ \hat{\psi}_{xx} = \frac{\beta}{U} \hat{\psi}, \quad (3.8) \]
with boundary condition
\[ \hat{\psi} = -\psi_I \quad \text{at} \quad x = 0. \quad (3.9) \]
Since \( \beta > 0 \), a decaying solution exists only if \( U > 0 \) (westward interior flow). Then
\[ \hat{\psi} = -\psi_I e^{-x/\delta}, \quad \text{where} \quad \delta \equiv \frac{\sqrt{U}}{\beta} \quad (3.10) \]
is the boundary-layer thickness. Similar boundary layers occur at the other three boundaries. Collecting results, we obtain the uniformly valid approximation,
\[ \psi = U(y - y_0) \left\{ 1 - e^{-x/\delta} - e^{-y/\delta} - e^{-(x-L)/\delta} - e^{-(y-L)/\delta} \right\}, \quad (3.11) \]
for absolute-equilibrium flow.
According to (3.11), the equilibrium state consists of a uniform westward interior flow closed by inertial boundary layers of thickness $\delta$. Since, in the limit of infinite spatial resolution, this equilibrium flow is steady, the solutions of (3.3) are also solutions of (3.2), and the streamlines are lines of constant potential vorticity. In the interior, the relative vorticity is negligible, and fluid particles move along lines of constant $f$. In the boundary layers, the northward- or southward-moving fluid particles acquire the large relative vorticities needed to balance the change in $f$.

The solution (3.11) generally comprises two gyres. The constant $y_0$ is the latitude that separates the northern anticyclonic gyre from the southern cyclonic gyre. If the initial total vorticity (3.5) is sufficiently positive or negative, then $y_0$ lies outside the domain, and the equilibrium flow has only a single gyre. If the total vorticity vanishes, then $y=y_0$ lies at mid-basin, and the two-gyre equilibrium flow is symmetric about the mid-latitude.

To determine $U$, we equate the initial energy, $u_0^2 L^2$, to the final energy. The interior flow makes a contribution $U^2 L^2$ to the final energy. However, this contribution is smaller than the contribution from the inertial boundary layers, by a factor of $\delta L$. Thus

$$u_0^2 L^2 = \frac{L}{\delta} U^2 L^2,$$

and hence

$$U = u_0 R^{1/3} \quad \text{and} \quad \delta = L R^{2/3},$$

(3.13)

where

$$R \equiv \frac{u_0}{\beta L^2}$$

(3.14)

is the Rossby number associated with the initial conditions. For typical oceanic values, $R \sim 10^{-4}$. Thus the boundary-layer limit is appropriate, and the final state is far from the state of uniform potential vorticity corresponding to $\mu=0$ in (3.3).

Figure 6.3 shows a numerical solution of (3.1-2) in a square domain. The prescribed $\beta$ corresponds to a Rossby number (3.14) of $R=0.02$, and an inertial boundary layer thickness (3.10) of $\delta=0.074 \; L$. The solution begins from the same random initial conditions as the numerical solution of two-dimensional turbulence (with $\beta=0$) described in Section 9 of Chapter 4; see Figure 4.9. Again, the time-unit is the time required for a fluid particle to traverse the domain at the rms fluid speed. Figure 6.3 shows the streamfunction at $t=1.0$, 3.0 and 7.0, and the average of the streamfunction between $t=5.0$ and 7.0. The numerical solution is approaching the predicted steady state (3.11), but the adjustment time is relatively long, even at this unrealistically large Rossby number.10

The growing negative correlation between vorticity and latitude is the means by which bounded $\beta$-plane flow increases its enstrophy,
\[ Z = \iint dx \, \zeta^2, \quad (3.15) \]

while conserving its potential enstrophy,

\[ Q_2 = \iint dx \, \zeta^2 + 2\iint dx \, \zeta f + \text{irrelevant constant.} \quad (3.16) \]

Recall that inviscid three-dimensional turbulence seeks the state of energy equipartition in wavevector, a high-entropy state unattainable by two-dimensional turbulence, because the conservation of (3.15) locks most of the energy in low wavenumbers. However, in the beta-plane box, enstrophy can increase if the second term on the right-hand side of (3.16) becomes negative. By (3.2), the rate of enstrophy increase is

\[
\frac{d}{dt} \iint dx \, \zeta^2 = -2\iint dx \, \zeta J(\psi, \zeta + \beta y) = -2\iint dx \, \zeta J(\psi, \beta y)
\]

\[
= -2\beta \int_0^L dx \left( v v_x - v u_y \right) = \beta \left[ \int_0^L v^2 dy \right]_{x=L}^{x=0}
\]

after some manipulation and the use of the boundary condition \( \psi = 0 \). Thus enstrophy increases whenever the mean-square velocity along the western boundary exceeds the mean-square velocity along the eastern boundary, that is, whenever the flow exhibits westward intensification. And, while the mechanism for the development of a negative correlation between \( \zeta \) and \( f \) is the same as in the case of topography (namely, that most of the fluid particles at high latitude originated at a lower latitude, and vice versa), (3.17) suggests that \( \beta \)-plane turbulence would behave very differently if the eastern and western walls were removed.

Suppose that the no-normal-flow boundary conditions at \( x=0, L \) are replaced by the boundary condition that the flow be periodic in the \( x \)-direction,

\[ \psi(x, y) = \psi(x + L, y). \quad (3.18) \]

It is easy to show that

\[ \iint dx \, \zeta^2 \quad \text{and} \quad \iint dx \, \zeta f \]

are then separately conserved. On account of the latter, no correlation between vorticity and latitude can arise (in freely decaying turbulence) unless it is present initially.

The splitting of the potential enstrophy (3.16) into the two separate invariants (3.19) corresponds to the appearance of a new conservation law: the conservation of momentum in the \( x \)-direction. In a fully bounded domain, the boundaries destroy all the translation-symmetries, and hence momentum is not conserved. However, when boundaries are missing in one direction, that component of momentum is conserved. We shall see that the conservation of linear or angular momentum has a profound effect upon the flow.
Consider, for example, the depth-independent flow in a shallow, homogeneous fluid on the surface of a rotating sphere. This is a simple model of the Earth’s atmosphere. In this case,

\[ f = 2 \left( \frac{2\pi}{\text{day}} \right) \sin \theta, \quad (3.20) \]

where \( \theta \) is the latitude. The dynamics (3.2) with (3.20) is mathematically equivalent to the case of constant (positive) \( f \), but with a seamount at the North Pole and a seafloor depression at the South Pole. By the reasoning in Sections 1 and 2 (which however ignores the conservation of angular momentum), we might anticipate that the absolute-equilibrium flow is clockwise around the North Pole and counterclockwise around the South Pole, that is, westward at all latitudes. However, flow on the rotating sphere also conserves its angular momentum about the rotation axis, and we must take this additional conservation law into account.

Consider the nearly equivalent case of a fluid governed by (3.1-2) between parallel, rigid boundaries at \( y = \pm L/2 \) (Figure 6.4). This re-entrant channel is a simple analogue of the flow on a rotating sphere. The flow is \( L \)-periodic in the \( x \)-direction, and \( \psi \) is a (different) prescribed constant at the two boundaries of the channel. The difference between the two prescribed boundary values of \( \psi \) determines the (conserved) value of the momentum,

\[ \int \int dx \; u, \quad (3.21) \]

where \( u \) is the \( x \)-direction velocity, and the integration runs over a periodic cycle in \( x \). By (3.2) and the boundary conditions, the flow conserves the energy (3.4) and the potential enstrophy (3.16). But if \( f \) is given by (3.1), then (3.19) are separately conserved.

Suppose that the flow is initially at rest. At \( t = 0 \), an external forcing commences at the mid-basin location shown in Figure 6.4. Let the forcing be random in the sense that it contributes no net momentum to the fluid. The forcing sets the fluid into motion, but all of the moving fluid particles not directly affected by the localized forcing keep the value of potential vorticity they had in the initial state of rest. By considering the motion of fluid particles across an arbitrary latitude line away from the latitude of the forcing, we conclude, by the same reasoning as in Section 1, that the flow is westward at all latitudes away from the forcing. (To reproduce the argument of Section 1, one must use the fact that \( \int u \, dx \) is conserved along each of the two channel boundaries.) However, the total momentum (3.21) must vanish, because the forcing contributes no average momentum. It follows that an intense eastward jet must develop at the latitude of the forcing. The momentum in this eastward jet exactly cancels the momentum in the relatively slow westward drift at other latitudes.

The same reasoning applies to flow on the rotating sphere, where the eastward jet becomes a crude model of the atmospheric jet stream. The midlatitude stirring force (in both hemispheres) represents the effect of baroclinically unstable eddies at the latitude of greatest instability.\(^{11} \) (In the single-layer model under present consideration, we must
regard baroclinic instability as an external forcing, but in the following two sections we consider the stratified form of the quasigeostrophic equations, which incorporate baroclinic motions.) The same ideas may explain the relatively large eastward abyssal velocity below the separated Gulf Stream and the weak westward flow on its flanks.\(^{12}\)

In all cases, the presence of Rossby waves inhibits the evolution of beta-plane turbulence towards absolute equilibrium. Consider beta-plane turbulence in the channel, and suppose that forcing (or initial conditions) introduces energy at a relatively high wavenumber. In ordinary ($\beta=0$) two-dimensional turbulence, an eddy of size $k^{-1}$ transfers its energy to other, generally lower, wavenumbers on the turn-over timescale $T_U$ defined by

$$\frac{1}{T_U} = \sqrt{\int_0^\infty k^2 E(k)dk} = Uk.$$\(^{(3.22)}\)

$T_U$ is the time required for larger-scale straining motions to distort the eddy. We can regard the last equality in (3.22) as the definition of $U$, or, with $U$ taken to be the rms velocity of the flow, as a cruder estimate of the distortion time than the preceding (integral) expression. Now, if $\beta\neq0$, then the eddy also feels the influence of Rossby-wave propagation on the timescale of its Rossby-wave period $T_\omega$,

$$\frac{1}{T_\omega} = |\omega| = \frac{\beta k_x}{k^2},$$\(^{(3.23)}\)

where $\omega$ is the frequency corresponding to wavevector $\mathbf{k}=(k_x,k_y)$, and $k=|\mathbf{k}|$. If $T_\omega<T_U$, then the energy transfer is ineffective, because Rossby waves alter the eddy before it can undergo significant distortion. If, on the other hand,

$$\frac{\beta k_x}{k^2} < Uk,$$

then $\beta$ has little effect on the energy transfer to wavevector $\mathbf{k}$. Since the left-hand side of (3.24) can be no bigger than $\beta/k$, $\beta$ is unimportant at all wavenumbers larger than

$$k_\beta \equiv \sqrt{\frac{\beta}{U}}.$$\(^{(3.25)}\)

The lengthscale $k_\beta^{-1}$ is often called the Rhines scale.\(^{13}\) Evidently, the dumbbell-shaped curve

$$\frac{\beta k_x}{k^2} = Uk$$\(^{(3.26)}\)
in the \((k_x,k_y)\) plane constitutes a barrier to the transfer of energy to low wavenumbers in beta-plane turbulence. Because \(\omega\) vanishes with \(k_x\), this transfer can proceed furthest along the \(k_y\)-axis, corresponding to the formation of zonal jets with \(k_x=0\). However, numerical experiments show that the energy seldom reaches \(k_y\) much smaller than \(k_\beta\), probably because the nearly zonal jets interact only weakly.\(^{14}\)

4. Stratified quasigeostrophic flow

The previous three sections considered quasigeostrophic turbulence in a single homogeneous layer. Now we consider stratified quasigeostrophic turbulence governed by the three-dimensional quasigeostrophic equation,

\[
\frac{\partial q}{\partial t} + J(\psi,q) = 0. \tag{4.1}
\]

Once again,

\[
J(\psi,q) = \frac{\partial \psi}{\partial x} \frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial q}{\partial x} \tag{4.2}
\]

is the horizontal Jacobian,

\[
q = \nabla^2 \psi + f + \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2(z)} \frac{\partial \psi}{\partial z} \right) \tag{4.3}
\]

is the potential vorticity, \(\nabla \equiv (\partial_x, \partial_y)\) is the horizontal gradient operator, and \(N(z)\) is the prescribed Vaisala frequency. The first term on the right-hand side of (4.3) represents the relative vorticity; the last term represents the potential vorticity stored in isopycnal displacements.

Let the flow be bounded by flat rigid plates at \(z=0\) and \(H\), where the (no-normal-flow) boundary conditions are

\[
\frac{\partial}{\partial t} \frac{\partial \psi}{\partial z} + f \left( \psi, \frac{\partial \psi}{\partial z} \right) = 0. \tag{4.4}
\]

As explained in Chapter 2, (4.4) is an approximation to the equation \(D\theta/Dt=0\) for the temperature \(\theta = f_0 \partial \psi / \partial z\); the term \(w \partial \theta / \partial z\) is missing because \(w=0\) at the boundaries. The horizontal boundary conditions are \(L\)-periodicity,

\[
\psi(x,y,z,t) = \psi(x+L,y,z,t) = \psi(x,y+L,z,t), \tag{4.5}
\]

in both horizontal directions.

With boundary conditions (4.4) and (4.5), the dynamics (4.1-3) conserve the energy,
\[ E = \iiint \, dx \left\{ \nabla \psi \cdot \nabla \psi + \frac{f_0^2}{N^2(z)} \left( \frac{\partial \psi}{\partial z} \right)^2 \right\}, \] (4.6)

and every integral of the form,

\[ \iint dx \, dy \, F(q(x,y,z,t)), \] (4.7)

where \( F(\ ) \) is an arbitrary function. The integral in (4.6) runs over an \( L \times L \times H \) periodic volume. The integral in (4.7) is a function of \( z \). The first term on the right-hand side of (4.6) is the kinetic energy (in horizontal motion). The last term in (4.6) represents the available potential energy. As for (4.7), we are again primarily interested in the average potential vorticity,

\[ Q_1(z) = \iint dx \, dy \, q, \] (4.8)

and potential enstrophy,

\[ Q_2(z) = \iint dx \, dy \, q^2, \] (4.9)

at level \( z \). For further review of three-dimensional quasigeostrophic dynamics and its conservation properties, refer to Chapter 2.

Suppose, for simplicity, that \( f \) and \( N \) are constants. Then, defining the new vertical coordinate

\[ \tilde{z} = \frac{N}{f_0} z, \] (4.10)

we obtain the quasigeostrophic dynamics in the \textit{nearly} isotropic form,

\[ \frac{\partial}{\partial t} \tilde{\nabla}^2 \psi + \tilde{J}(\psi, \tilde{\nabla}^2 \psi) = 0, \] (4.11)

where

\[ \tilde{\nabla} \equiv \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \tilde{z}} \right) \] (4.12)

is the \textit{three-dimensional} gradient operator in the new, \((x,y,\tilde{z})\) coordinates. The boundary conditions (4.4) and (4.5) take the same form in these new coordinates. The energy (4.6) becomes
\[ E = \iiint dx \tilde{\nabla} \psi \cdot \tilde{\nabla} \psi. \]  

(4.13)

Since \( f \) is a constant, conservation of (4.8) and (4.9) implies conservation of

\[ Z(z) = \iint dx dy \left( \tilde{\nabla}^2 \psi \right)^2. \]  

(4.14)

It follows that

\[ Z = \int dz Z(z) = \iiint dx \left( \tilde{\nabla}^2 \psi \right)^2 \]  

(4.15)

is also conserved.

In some cases, it may be sufficient to consider (4.15) instead of (4.14). Suppose, for example, that the initial \( Z(z) \) is the same at each \( z \), and that forcing, if present, is statistically \( z \)-independent. Then the conservation laws (4.14) plausibly contain no more information than their vertical integral (4.15).\(^{15}\)

The invariants (4.13) and (4.15) differ from the energy and enstrophy of two-dimensional turbulence only in the presence of vertical derivatives in \( \tilde{\nabla} \), and in the fact that (4.13) and (4.15) are integrals over three-dimensional space. Since much of the theory of two-dimensional turbulence depends only on the form of its energy- and enstrophy-invariants, we anticipate a strong resemblance between three-dimensional quasigeostrophic turbulence and ordinary two-dimensional turbulence.

In the Earth’s atmosphere, the sun produces energy and potential enstrophy at a very low wavenumber corresponding to the radius of the Earth. By the same (essentially dimensional) reasoning as in the theory of two-dimensional turbulence (see Chapter 4), the potential enstrophy created by the sun moves toward higher (three-dimensional) wavenumbers in \((x,y,\tilde{z})\) space, in an inertial-range spectrum of the form\(^{16}\)

\[ E(k_x, k_y, k_\tilde{z}) = C \eta^{2/3} \tilde{k}^{-3}, \]  

(4.16)

where

\[ \tilde{k} = \left( k_x^2 + k_y^2 + k_\tilde{z}^2 \right)^{1/2} \]  

(4.17)

is the total wavenumber in \((x,y,\tilde{z})\) space, \( C \) is a universal constant, and \( \eta \) is the rate of potential enstrophy-dissipation at very small scales. The spectrum (4.16) takes the same form as the enstrophy inertial range in two-dimensional turbulence because the corresponding symbols have the same dimensions. The inertial-range eddies are isotropic in \((x,y,\tilde{z})\) coordinates. It follows that the energy is equipartitioned between the kinetic energy in each horizontal direction and the available potential energy, and that each of these has a spectrum of the form (4.16). Atmospheric observations generally support these predictions. In fact, atmospheric spectra seem to agree better with inertial-range theory than do numerical simulations of ideal two-dimensional turbulence.\(^{17}\)
Suppose that the flow is indeed isotropic in \((x,y,\tilde{z})\) coordinates. Then, in \((x,y,z)\) coordinates, eddies with horizontal scale \(a\) have a vertical scale of \(fa/N\). Refer to Figure 6.5. However, only eddies with a vertical scale smaller than the distance \(H\) between boundaries can possibly be isotropic. Those eddies whose vertical scale just equals \(H\) have a horizontal scale of

\[ \lambda = NH / f . \]  

(4.18)

Readers of Chapter 2 will recognize (4.18) as the \textit{internal Rossby deformation radius}. Thus, only eddies with a horizontal scale larger than the deformation radius can be isotropic in the sense defined above. Recalling that \(\lambda\) is about 1000 km in the Earth’s atmosphere, and that direct solar forcing occurs at the longer scale of the Earth’s radius (6400 km), we see that a potential-enstrophy-cascading inertial range of the form (4.16) is possible. However, the Earth’s curvature and rotation impose anisotropy on all scales, and introduce a dimensional constant, \(\beta\), whose presence allows possibilities other than (4.16). Moreover, the range of scales on which atmospheric motions are approximately quasi-geostrophic is relatively small, covering no more than two decades of wavenumber, and very different physics governs smaller scale flow. It has even been suggested that some of the energy in large-scale atmospheric motions originates in the vigorous thermal convection at scales of only a few kilometers and moves toward larger scales of motion, perhaps in an inertial range resembling the \(k^{-5/3}\) energy-cascading range in two-dimensional turbulence.18

At horizontal scales larger than the internal deformation radius (4.18), we must take the boundaries at \(z=0,H\) into account. Suppose that the temperature is \textit{uniform} on both boundaries, that is, that \(\psi_z=0\) at \(z=0,H\). Then the eigenfunction expansion

\[ \psi(x,y,z,t) = \sum_{n=0}^{\infty} \psi_n(x,y,t) \cos \left( \frac{n\pi z}{H} \right) = \sum_{k,n} \psi_{k,n}(t) e^{ik \cdot x} \cos \left( \frac{n\pi z}{H} \right) \]  

(4.19)

is appropriate. Here, \(k=(k_x,k_y)\) is the \textit{horizontal} wavevector, and \(n\) denotes the vertical mode. The energy and total potential enstrophy now take the forms

\[ E = \sum_{k,n} \left( k_x^2 + k_y^2 + k_n^2 \right) |\psi_{k,n}|^2 \equiv \sum_{k,n} E_{k,n} \]  

(4.20)

and

\[ Z = \sum_{k,n} \left( k_x^2 + k_y^2 + k_n^2 \right) E_{k,n} , \]  

(4.21)

where

\[ k_n = \frac{1}{\lambda_n} = \frac{n\pi f}{NH} , \]  

(4.22)
and \( \lambda_n \) is the deformation radius associated with the \( n \)-th mode. The first internal deformation radius \( \lambda_1 \) corresponds (roughly) to (4.18). Although (three-dimensional) inertial-range theory cannot apply to the large scales of motion that feel the boundaries, the conservation of (4.20) and (4.21) suggests, by itself, that in quasigeostrophic turbulence, energy moves to the lowest total wavenumber,

\[
\left( k_x^2 + k_y^2 + k_n^2 \right)^{1/2}.
\]  

(4.23)

The argument is essentially the same as that given in Chapter 4 for ordinary two-dimensional turbulence. In the atmosphere, solar heating corresponds to energy input at very small \((k_x, k_y)\) and \(n=1\) (predominantly), and quasigeostrophic turbulence transfers this energy to a still lower (4.23) corresponding to \( n=0 \) (barotropic motion). We shall consider this large-scale transfer more closely in the following section.

Equilibrium statistical mechanics predicts the absolute-equilibrium spectrum

\[
E_{k,n} = \frac{1}{\alpha + \beta \left( k_x^2 + k_y^2 + k_n^2 \right)},
\]  

(4.24)

which nonlinear interactions, acting alone, would drive the flow. Here, \( \alpha \) and \( \beta \) are constants determined by the values of \( E \) and \( Z \). Roughly speaking, (4.24) is as isotropic as the boundary conditions allow. According to (4.24), the absolute-equilibrium energy \( E(k,n) \) in horizontal wavenumber

\[
k \equiv \left( k_x^2 + k_y^2 \right)^{1/2}
\]  

(4.25)

and vertical mode \( n \), is equal to the barotropic (and thus purely kinetic) energy \( E(k_0,0) \) at the higher horizontal wavenumber \( k_0 \) given by

\[
k_0^2 = k^2 + \lambda_n^2.
\]  

(4.26)

If, as is usually the case, \( \alpha \) and \( \beta \) are such that (4.24) is a decreasing function of the total wavenumber (4.23), then the absolute-equilibrium state is one in which most of the energy resides in barotropic \((n=0)\) flow at very large horizontal scales. By the same reasoning as in Chapter 5, this always occurs as the maximum total wavenumber in the truncation tends toward infinity (and \( \alpha + \beta k_{\min}^2 \to 0 \)).

In quasigeostrophic theory, we normally take the \( f \) in \( k_n \) (arising from the last term in (4.3)) to be a constant, even when we allow the other \( f \) in (4.3) to vary. But consider what happens if \( k_n \) varies with latitude through its dependence on \( f \). In this WKB-like version of the theory, we only require that the lengthscales of the flow be small compared to the scale on which \( f \) varies. As \( f \to 0 \) at the equator, the \( k_n \) vanish there, removing the inhibition against energy in high vertical mode numbers. Moreover, since the total wavenumber (4.23) of each mode \( n \) is smaller than its value at higher latitude, the energy
density in each mode (and, thus, the total energy density) should increase toward the equator. This energy increase ceases at a distance

$$\left(\frac{NH}{n\pi\beta}\right)^{1/2}$$

(4.27)

from the equator, where the increase in $k_y$ begins to overcome the decrease in $k_n$. The distance (4.27) is the *equatorial deformation radius* associated with mode $n$. Thus the same logic that predicts energy transfer to low wavenumbers in ordinary two-dimensional turbulence suggests that quasigeostrophic turbulence transfers its energy toward the equator and into high vertical mode. Unfortunately, both quasigeostrophic theory and the WKB reasoning break down very near the equator, so we must view this prediction cautiously. However, current-meter measurements show the presence, at the equator, of strong, low-frequency ocean currents varying rapidly with $z$\textsuperscript{19}

Most reported numerical solutions of (4.11) consider flow that is periodic in all three directions and hence do not require the boundary conditions (4.4). Initially, the flow tends toward isotropy in $(x,y,z)$, with the formation of nearly spherical, isolated vortices of either sign. Like-signed vortices at the same level subsequently merge in much the same manner as in ordinary two-dimensional turbulence. However, like-signed vortices at different levels tend to align vertically into long but gappy vortex-columns that are highly anisotropic in $(x,y,z)$-space\textsuperscript{20}

So far we have considered quasigeostrophic turbulence in which the potential vorticity varies continuously throughout the fluid. The boundaries, if present, were boundaries of (horizontally) uniform temperature $\partial \psi / \partial z$. Now we consider the case of arbitrary boundary temperature, but $q$ uniform throughout the body of the fluid. In this case, the potential vorticity equation (4.1) is automatically satisfied, and the dynamics reduce to the temperature equations (4.4) at the boundaries.

Again we assume that $f$ and $N$ are constants. Let $q=f$ throughout the fluid. In the simplest case, there is a single boundary at $z=0$, and the fluid lies on $z>0$. The condition $q=f$ implies that

$$\nabla^2 \psi = 0.$$  

(4.28)

Equation (4.28) determines the surface-streamfunction $\psi_s(x,y,t) \equiv \psi(x,y,0,t)$ from the surface temperature

$$\theta_s(x,y,t) \equiv \frac{\partial \psi}{\partial z}(x,y,0,t).$$

(4.29)

That is, Laplace’s equation (4.28) and the Neumann boundary condition (4.29) determine $\psi(x,y,z,t)$ and hence $\psi_s(x,y,0,t)$ in terms of $\theta_s(x,y,t)$. Then the temperature equation

$$\frac{\partial \theta}{\partial t} + J(\psi_s, \theta_s) = 0$$

(4.30)
advances \( \theta_s \) to a new time.

The solution of (4.28) and (4.29), bounded as \( z \to \infty \), is

\[
\psi(x,y,z,t) = -\iiint d\mathbf{k} \ k^{-1} \theta_s(\mathbf{k},t) e^{-\ii z \cdot \mathbf{k} + k_x x},
\]

where \( \mathbf{k} = (k_x, k_y) \) is the horizontal wavevector, \( k = |\mathbf{k}|, \mathbf{x} = (x,y) \), and \( \theta_s(\mathbf{k}) \) is the Fourier transform of \( \theta_s(x) \). Note that motions with horizontal scale \( a \) are trapped within \( z = af/N \) of the boundary. From (4.31) it follows that

\[
\psi_s(x,y,t) = -\iiint d\mathbf{k} \ k^{-1} \theta_s(\mathbf{k},t) e^{k_x x}.
\]

Thus the Fourier transforms of \( \theta_s \) and \( \psi_s \) are related by

\[
\theta_s(\mathbf{k}) = -k \ \psi_s(\mathbf{k}).
\]

The vorticity equation governing two-dimensional turbulence has the same form as (4.30), but with \( \theta_s \) replaced by

\[
\zeta_s \equiv \nabla^2 \psi_s,
\]

where \( \nabla \) is the horizontal gradient operator. The Fourier transform of (4.34) is

\[
\zeta_s(\mathbf{k}) = -k^2 \ \psi_s(\mathbf{k}).
\]

Thus the only difference between surface quasigeostrophic turbulence governed by (4.30) and ordinary two-dimensional turbulence is the power of \( k \) in the equation relating the streamfunction to the quantity conserved on fluid particles. However, this difference turns out to be quite significant.\(^2\)

If, for example, the advected fields \( \theta_s(x,y) \) and \( \zeta_s(x,y) \) are proportional to one another, the streamfunction corresponding to \( \zeta_s \) is much smoother than the streamfunction corresponding to \( \theta_s \). Specifically, if \( \zeta_s \) and \( \theta_s \) are both proportional to the delta-function \( \delta(x) \), then the velocity field corresponding to \( \zeta_s \) varies as \( r^{-1} \) (where \( r = |x| \)), but the velocity field corresponding to \( \theta_s \) falls off much faster, as \( r^{-2} \). (These velocity fields correspond to the Green’s functions of (4.34) and (4.29)). This suggests that the interaction between distant eddies is much weaker in surface quasigeostrophic turbulence than in ordinary two-dimensional turbulence. That is, the interaction between flow structures in surface quasigeostrophic turbulence is more local, and also more violent, because the disruptive effect of large-scale straining is less important. For example, in ordinary two-dimensional turbulence, merging vortices strip off long ribbons of vorticity from one another, but these vortex-ribbons are passively strained into wispy filaments by the velocity field associated with the vortex cores. However, in surface quasigeostrophic
turbulence the vorticity-ribbons interact strongly with themselves, generating lines of secondary vortices.\(^{22}\)

The dynamics (4.30,4.33) governing surface quasigeostrophic turbulence conserve the energy

\[
E = \iiint dx \, \tilde{\nabla} \psi \cdot \tilde{\nabla} \psi = - \iiint dx \, \psi_s \theta_s = \iint dk \, k |\psi(k)|^2 \equiv \int_0^\infty E(k) \, dk, \tag{4.36}
\]

and the surface temperature variance,

\[
\Theta = \iiint dx \, \theta_s^2 = \iint dk \, k^2 |\psi(k)|^2 \equiv \int_0^\infty k \, E(k) \, dk. \tag{4.37}
\]

The latter is analogous to the enstrophy of two-dimensional turbulence; again the primary difference is the lower power of \(k\) in the last term in (4.37). Inertial-range theory analogous to that presented in Chapter 4 for two-dimensional turbulence predicts two inertial ranges for surface quasigeostrophic turbulence. At low wavenumber, energy cascades toward smaller \(k\) through an energy spectrum \(E(k) \sim k^{-2}\). At high wavenumber temperature variance cascades toward high \(k\) through a spectrum \(E(k) \sim k^{-8/3}\). However, the relevant comparison is between the corresponding temperature-variance spectra (respectively \(k^{-1}\) and \(k^{-5/3}\)) and the enstrophy spectra in the energy- and enstrophy-cascading ranges of ordinary two-dimensional turbulence (respectively \(k^{+1/3}\) and \(k^{-1}\)). The former are steeper, as might be expected from the fact that in surface quasigeostrophic turbulence only eddies of comparable size interact strongly. Numerical solutions of (4.30) generally support the predictions of inertial-range theory, but also show the formation of very intricate and possibly self-similar flow structures.

5. **Two-layer turbulence**

Finally, we consider two-layer quasigeostrophic turbulence with constant Coriolis parameter \(f\). The simplifying assumption of only two layers (or levels) permits us to draw conclusions about stratified quasigeostrophic flow that do not depend on the strong assumption of three-dimensional statistical isotropy invoked frequently in the preceding section. In particular, we shall be able to make general statements about the motion on spatial scales large enough to feel the top and bottom boundaries. Once again, the principles of conservation and irreversibility form the basis for the theory.

The governing potential vorticity equations are

\[
\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0, \quad i = 1, 2, \tag{5.1}
\]

where \(i=1\) corresponds to the top layer, with potential vorticity

\[
q_1 = \nabla^2 \psi_1 + \frac{1}{2} k_R^2 (\psi_2 - \psi_1), \tag{5.2}
\]

and \(i=2\) corresponds to the bottom layer, with potential vorticity
\[ q_2 = \nabla^2 \psi_2 + \frac{1}{2} k_R^2 (\psi_1 - \psi_2^2), \]  

(5.3)

and \( k_R^{-1} \) is the (first) internal deformation radius. As explained in Chapter 2, we can view (5.1-3) as the quasigeostrophic equations for two homogeneous, immiscible fluid layers; as the truncation of (4.1-4) to two vertical modes \( (n=0,1) \); or as a vertical-finite-difference analogue of (4.1). We shall regard (5.1-3) as a very simple model of the Earth’s atmosphere; then the last interpretation is probably the most appropriate. For further background, see Section 20 of Chapter 2.

As in Chapter 2, it is convenient to replace the dependent variables \( \psi_1(x,y,t) \) and \( \psi_2(x,y,t) \) by the modal variables

\[ \psi \equiv \frac{1}{2} (\psi_1 + \psi_2), \quad \tau \equiv \frac{1}{2} (\psi_1 - \psi_2). \]  

(5.4)

The \textit{barotropic streamfunction} \( \psi \) is the streamfunction for the vertically-averaged flow. We may also view \( \psi \) as the streamfunction for the velocity field at the mid-depth. The \textit{baroclinic streamfunction} \( \tau \) is the streamfunction for the vertical shear (the vertical derivative of the horizontal velocity). By the thermal wind relation, \( \tau(x,y,t) \) is proportional to the vertically averaged temperature anomaly (or to the thickness-anomaly of the top layer, in the homogeneous-layers interpretation of (5.1-3)).

Let the flow be periodic in both horizontal directions. This mimics the geometry of a spherical surface but avoids the inconvenience of spherical coordinates. Then the two-level dynamics (5.1-3) conserves energy in the form

\[ E = \frac{1}{2} \iint dx [\nabla \psi_1 \cdot \nabla \psi_1 + \nabla \psi_2 \cdot \nabla \psi_2 + \frac{1}{2} k_R^2 (\psi_1 - \psi_2)^2] \]

\[ = \iint dx [\nabla \psi \cdot \nabla \psi + \nabla \tau \cdot \nabla \tau + k_R^2 \tau^2] \]  

(5.5)

The terms proportional to \( k_R^2 \) represent the available potential energy. The potential enstrophies,

\[ Z_i = \iint dx q_i^2, \quad i = 1,2 \]  

(5.6)

are also conserved. In terms of \( \psi \) and \( \tau \), these take the forms

\[ Z_1 = \iint dx [\nabla^2 (\psi + \tau) - k_R^2 \tau]^2 \]  

(5.7)

and

\[ Z_2 = \iint dx [\nabla^2 (\psi - \tau) + k_R^2 \tau]^2. \]  

(5.8)

The \textit{average} potential enstrophy,
Z ≡ \frac{1}{2}(Z_1 + Z_2) = \int d\mathbf{x} \left[ (\nabla^2 \psi)^2 + (\nabla^2 \tau - k^2 \tau)^2 \right] \tag{5.9}

is obviously also conserved.

Since the flow is horizontally periodic, we introduce the spatial Fourier transformations,

\[ \psi(x,t) = \sum_k \psi_k(t)e^{ik \cdot x}, \quad \tau(x,t) = \sum_k \tau_k(t)e^{ik \cdot x}. \] \tag{5.10}

In terms of the Fourier amplitudes, the conserved quantities (5.5) and (5.9) are (proportional to)

\[ E = \sum_k \left[ k^2 |\psi_k|^2 + (k^2 + k_R^2)|\tau_k|^2 \right] \] \tag{5.11}

and

\[ Z = \sum_k \left[ k^4 |\psi_k|^2 + (k^2 + k_R^2)^2|\tau_k|^2 \right]. \] \tag{5.12}

Thus

\[ U_k \equiv k^2 |\psi_k|^2 \] \tag{5.13}

is the \textit{barotropic energy} — the kinetic energy in barotropic flow at wavenumber \( k \). The \textit{baroclinic energy},

\[ T_k \equiv (k^2 + k_R^2)|\tau_k|^2, \] \tag{5.14}

is the sum of the kinetic energy, \( k^2 |\tau_k|^2 \), in baroclinic flow, and the available potential energy, \( k_R^2 |\tau_k|^2 \). The latter exceeds the former at length scales larger than the deformation radius (that is, at \( k<k_R \)).

The conserved quantities (5.11) and (5.12) can now be written

\[ E = \sum_k (U_k + T_k) \] \tag{5.15}

and

\[ Z = \sum_k \left[ k^2 U_k + (k^2 + k_R^2)T_k \right]. \] \tag{5.16}

Eqns. (5.15) and (5.16) are the two-layer analogues of (4.20) and (4.21). Once again we see that the energy and potential-enstrophy invariants (5.15-16) have the same spectral
form as the energy and enstrophy invariants of ordinary two-dimensional turbulence except that the baroclinic modes have an effective squared wavenumber of $k^2 + k_R^2$ instead of $k^2$. Just as the energy in two-dimensional turbulence moves toward lower wavenumber, the energy in two-layer turbulence should move toward lower effective wavenumber, that is toward lower $k$ and into barotropic mode.

We get a more detailed picture of energy transfer in two-layer turbulence by considering the constraints imposed by these conservation properties on individual triads. First rewrite (5.1) as evolution equations for $\psi$ and $\tau$, namely

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) + J(\tau, \nabla^2 \tau) = 0,$$  \hspace{1cm} (5.17)

and

$$\frac{\partial}{\partial t} \nabla^2 \tau + J(\psi, \nabla^2 \tau) + J(\tau, \nabla^2 \psi) = k_R^2 \left[ \frac{\partial \tau}{\partial t} + J(\psi, \tau) \right].$$  \hspace{1cm} (5.18)

As explained in Chapter 2, we can view (5.17) as the vorticity equation for the vertically averaged flow. Vortex-stretching is absent because of the $w=0$ boundary conditions at the top and bottom.

Similarly, we can view (5.18) as the vorticity equation for the baroclinic flow. The whole right-hand side represents the vortex stretching (after use of the temperature equation to eliminate the vertical velocity). Alternatively, we can view (5.18) as the equation for the vertically averaged temperature; then the left-hand side represents the vertical advection of the mean temperature (after use of the vorticity equation to eliminate the vertical velocity). For further details, see Section 14 of Chapter 2.

From (5.17) and (5.18), we see that there are two types of triad in two-layer turbulence. The first type has three barotropic components,

$$\psi_p, \psi_q, \psi_k \quad (\text{barotropic triad}). \hspace{1cm} (5.19)$$

The second type of triad has one barotropic and two baroclinic components,

$$\psi_p, \tau_q, \tau_k \quad (\text{baroclinic triad}). \hspace{1cm} (5.20)$$

Only these two types of triad can occur, because the vertical wavenumbers (or mode numbers) must obey a selection rule analogous to the selection rule

$$p + q + k = 0$$  \hspace{1cm} (5.21)

for horizontal wavenumbers. The barotropic components all have vertical wavenumber zero. Thus, 0+0+0=0 in barotropic triads. The baroclinic components have vertical wavenumber $\pm 1$. Hence, 0+1-1=0 in baroclinc triads. We regard the whole dynamics as the sum of all possible triad interactions. Each triad is of type (5.19) or (5.20), and obeys (5.21).
Each triad conserves energy and potential enstrophy \textit{in detail}. (To see that this must be so, consider an initial condition in which only the three components of a particular triad are excited.) Thus, in barotropic triads,

\begin{align}
\dot{U}_k + \dot{U}_p + \dot{U}_q &= 0 \\
k^2 \dot{U}_k + p^2 \dot{U}_p + q^2 \dot{U}_q &= 0
\end{align}

(5.22)

where the time derivatives denote the change caused by interaction with the other two members of the triad \textit{only}. Similarly, in baroclinic triads,

\begin{align}
\dot{U}_k + \dot{T}_p + \dot{T}_q &= 0 \\
k^2 \dot{U}_k + (p^2 + k_R^2) \dot{T}_p + (q^2 + k_R^2) \dot{T}_q &= 0
\end{align}

(5.23)

Equations (5.22) are the same as the corresponding equations for ordinary two-dimensional turbulence. Hence the energy transfer in barotropic triads is governed by the same rules as in two-dimensional turbulence.

However, according to (5.23), energy transfer in baroclinic triads depends very much on the size of the wavenumbers. For triad members with lengthscales much smaller than the deformation radius (that is, \(k, p, q \gg k_R\)), (5.23) take the same form as (5.22). This is logical, because, on these small lengthscales, the vortex-stretching terms in (5.2-3) are negligible, and (5.1) reduce to the vorticity equations for \textit{uncoupled} layers. That is, small-scale motions see the interface between layers as a rigid barrier.

At scales of motion larger than the deformation radius (that is, \(k, p, q \ll k_R\)), (5.23) become

\begin{align}
\dot{U}_k + \dot{T}_p + \dot{T}_q &= 0 \\
k^2 \dot{U}_k + k_R^2 \dot{T}_p + k_R^2 \dot{T}_q &= 0
\end{align}

(5.24)

Equations (5.24) imply that

\[ \dot{U}_k = 0 \quad \text{and} \quad \dot{T}_p = -\dot{T}_q. \]

(5.25)

According to (5.25), the energy transfer in large-scale baroclinic triads is a pair-wise exchange between the two baroclinic components only. The barotropic component is an essential part of this interaction, but it neither gives nor receives energy. In contrast to two-dimensional turbulence, large-scale baroclinic energy can move toward higher or lower wavenumber.

The baroclinic instability studied in Section 15 of Chapter 2 can be viewed as a baroclinic triad. The mean flow corresponds to a baroclinic component with zero horizontal wavenumber, say \(p=0\). Then, by (5.21), \(q=-k\). Thus (5.23) become
\[ \dot{U}_k + \dot{T}_0 + \dot{T}_k = 0 \]
\[ k^2 \dot{U}_k + k_R^2 \dot{T}_0 + (k^2 + k_R^2) \dot{T}_k = 0 \]

Eliminating \( \dot{T}_0 \) between (5.26) yields
\[ \dot{U}_k = \frac{k^2}{(k_R^2 - k^2)} \dot{T}_k. \]

It is easy to verify that the baroclinically unstable solutions found in Chapter 2 satisfy (5.27). The relation (5.27) shows why the waves with \( k > k_R \) were found to be stable. For these short waves \( \dot{U}_k \) and \( \dot{T}_k \) must, by (5.27), have opposite signs. Thus, an initially infinitesimal wave, in which both \( U_k \) and \( T_k \) are very small, cannot grow very far because \( U_k \) and \( T_k \) are always positive.

We can use these ideas to deduce the nature of the energy transfer in a two-layer flow with forcing and damping terms added to the right-hand sides of (5.1). We assume that the flow has reached statistical equilibrium, that is, that its statistics are steady. Let \( k_F \) be the minimum wavenumber in the system, and suppose that the stirring or heating force acts only near \( k_F \). In the atmosphere, \( k_F^{-1} \) is of the order of the Earth’s radius, and the external force is solar heating.

To have equilibrium, dissipation must also be present. We first suppose that this dissipation is present in only two regions of the spectrum: near \( k_F \), where it represents the loss of large-scale energy to Ekman-like boundary layers; and at \( k > k_D >> k_R \), where it represents the loss of energy to small-scale ageostrophic motions not explicitly described by the quasigeostrophic dynamics. On the range \( k_F < k < k_D \), neither forcing nor dissipation is significant, and the transfer of energy and potential enstrophy is governed by the triad rules (5.22) and (5.23).

If the flow evolves to a statistically steady state, then the transfer of energy and potential enstrophy past \( k \) must be the same for every \( k \) on \( k_F < k < k_D \). Refer to Figure 6.6. On lengthscales smaller than the deformation radius \( (k > k_R) \), the dynamics are those of uncoupled single layers, in which the invariants (5.15-16) reduce to ordinary kinetic energy and enstrophy. By hypothesis, the only energy source is at \( k_F \). Thus if \( k_D/k_R \) is large, the energy reaching \( k_D \) is negligible (by the same reasoning as in the enstrophy-cascading inertial range of two-dimensional turbulence; see Chapter 4). This means that the large scales must adjust to a state in which the net energy gain at \( k_F \) vanishes. That is,
\[ \left. \frac{d}{dt} \right|_{nc} U(k_F) + \left. \frac{d}{dt} \right|_{nc} T(k_F) = 0, \]

where the subscripts \( nc \) (for nonconservative) denote that part of the time derivative caused by forcing and dissipation.

Now, the net production of potential enstrophy at \( k_F \) is
\[ k_F^2 \frac{d}{dt} U(k_F) + (k_F^2 + k_R^2) \frac{d}{dt} T(k_F). \]  

(5.29)

However, by (5.28), (5.29) is

\[ k_R^2 \frac{d}{dt} T(k_F). \]  

(5.30)

But (5.30) must be positive, because there is only an enstrophy sink at \( k_D \). Thus

\[ \frac{d}{dt} T(k_F) > 0. \]  

(5.31)

Then, by (5.28),

\[ \frac{d}{dt} U(k_F) < 0. \]  

(5.32)

Thus, the net production of baroclinic energy at \( k_F \) must be positive, the net production of barotropic energy must be negative, and triad interactions must transfer large-scale energy from baroclinic to barotropic modes to maintain statistical equilibrium. However, by (5.25), large-scale baroclinic components can transfer energy only between themselves. Therefore, the transfer from baroclinic to barotropic mode on \( k_F < k < k_R \) must occur as a rightward (that is, toward higher \( k \)) transfer of baroclinic energy in baroclinic triads, and an equal and opposite leftward transfer of barotropic energy in barotropic triads.

The conversion from baroclinic to barotropic energy can occur near \( k_R \). Baroclinic instability triads (in which one baroclinic member has a very small horizontal wavenumber, and the other two members have nearly equal wavenumbers) can participate in the rightward transfer of baroclinic energy, but more general, ‘non-isosceles’ baroclinic triads can also be important. Figure 6.6 summarizes the equilibrium state.\(^{23}\)

Several points are worth emphasizing. First, our arguments have led to the somewhat surprising conclusion that the only statistical equilibrium possible with large-scale forcing is one in which there is net forcing of the large-scale baroclinic motion and net damping of the large-scale barotropic motion. Second, our arguments depend on the detailed (i.e. triad-) form of the conservation laws for energy (5.5) and total potential enstrophy (5.9), but do not make use of the potential enstrophy invariants (5.7) and (5.8) for each layer. These latter invariants contain further information about the flow, but they do not affect the argument given above. Finally, besides the conservation principles for energy and potential enstrophy, our deduction depends critically on the irreversibility principle that there is a sink of ordinary enstrophy at high horizontal wavenumbers. We have repeatedly emphasized the need for such an irreversibility principle to establish the direction of energy transfer at statistical equilibrium.
The worst deficiency of (5.1-3) as a model of atmospheric motion lies in the assumption of a constant Coriolis parameter $f$. However, our main conclusions survive the inclusion of beta because they depend upon conservation laws that are unaffected by the presence of beta (on account of the periodic boundary conditions). Of course, the beta-effect will affect the rate of energy transfer in triads (particularly at low wavenumbers, where the wave speeds exceed the fluid-particle speeds), thereby influencing the shape of the equilibrium spectrum. For instance, strong beta prevents the leftward transfer of barotropic energy from proceeding very far before it is wiped out by the Ekman friction. But strong beta does not change the important conclusion that, at equilibrium, the average energy transfer is from baroclinic to barotropic modes.

Numerical simulations of freely decaying two-layer turbulence show that an initially baroclinic flow rapidly becomes barotropic on horizontal scales larger than the internal deformation radius. The completeness of this barotropization is suggested by the two-layer absolute-equilibrium states (that is, the two-vertical-mode truncation of (4.24)). The occlusion of atmospheric storms apparently exemplifies this process. In the ocean, observations show that the strong low-frequency flow near the Gulf Stream sometimes exhibits a nearly depth-independent horizontal velocity.

Notes for Chapter 6.
1. For reviews of geostrophic turbulence, see Rhines (1977,1979), Salmon (1982) and Holloway (1986).
2. See Rhines and Young (1982a,b) and Young (1987).
3. See Bretherton and Haidvogel (1976).
4. For a review of mean flows driven by wave dissipation, see McIntyre and Norton (1990).
5. Salmon et al. (1976) considered the equilibrium statistical mechanics of quasigeostrophic models, including single-layer flow over topography. Herring (1977) and Holloway (1978) studied quasigeostrophic flow over topography with turbulence closure theories of the kind described in Chapter 5.
7. Bretherton and Haidvogel (1976) and Holloway and Hendershott (1977) reported the first numerical solutions of this kind. Treguier (1989) reports a very extensive set of numerical experiments with various flow and topography spectra. Cummins and Holloway (1994) describe solutions in which the topography consists solely of a continental slope. Zou and Holloway (1994) explore the connection between increasing entropy and the growing correlation between the flow and topography. Merryfield and Holloway (1996) perform a quantitative check on the statistical mechanical theory. According to Holloway and Sou (1996), it explains an observed correlation between ocean currents and bottom topography. Emery and Csanady (1973) had reported that the observed surface circulation in northern hemisphere lakes is almost always cyclonic.
8. The numerical solution of Figure 6.2, and all the other numerical solutions described in Chapter 6, employed the same numerical method (including the same implicit eddy viscosity arising from third-order upwind differencing) as the numerical solution described in Section 9 of Chapter 4. See the corresponding note in Chapter 4.
9. In a famous paper, Fermi, Pasta and Ulam (1955) presented numerical solutions of a one-dimensional differential equation that certainly did not approach the corresponding absolute-equilibrium state.

10. Griffa and Salmon (1989) found that the time to reach absolute equilibrium becomes very long as the Rossby number becomes small. They found that the relevance of absolute-equilibrium theory to solutions with forcing and dissipation depended very much on the specific wind stress. See also Wang and Vallis (1994).

11. See Held (1993) for an application of these ideas to the theory of atmospheric circulation.

12. See also Cessi (1990).

13. After Rhines (1975), who based much of his argument on resonant-interaction theory for small-amplitude Rossby waves. EDM theory (Chapter 5) offers a means of extending Rhines’s argument to the case of large amplitudes. The rate of energy transfer is proportional to $\theta_{ijk}$ given by eqn. (2.24) of Chapter 5, which is small when $c \gg \mu$. See Holloway and Hendershott (1977).

14. See Williams (1978), and Vallis and Maltrud (1993) and references therein. Williams (1979a,b) and Panetta (1993) describe the zonal jets that appear as a consequence of beta in the two-layer quasigeostrophic model.

15. The absence of vertical transport of potential vorticity and potential enstrophy might well be considered an artificial defect of the quasigeostrophic equation (4.11). This provides another reason for preferring (4.15) over (4.14).


17. See Boer and Shepherd (1983) and Gage and Nastrom (1986).


19. Luyten and Swallow (1976) discovered the deep equatorial jets. See also Muench et al. (1994) and references therein. For a slightly more detailed theoretical discussion of the concentration of high-vertical-mode energy at the equator, see Salmon (1982).


21. For a review of surface quasigeostrophic dynamics, see Held et al. (1995).

22. Constantin et al. (1994) suggest that the first derivative of $\theta_s$ develops discontinuities in a finite time.

23. For further details, see Salmon (1978,1980). See also Haidvogel and Held (1980), and Held and Larichev (1996).

