The equilibrium statistical mechanics of simple quasi-geostrophic models

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We have applied the methods of classical statistical mechanics to derive the inviscid equilibrium states for one- and two-layer nonlinear quasi-geostrophic flows, with and without bottom topography and variable rotation rate. In the one-layer case without topography we recover the equilibrium energy spectrum given by Kraichnan (1967). In the two-layer case, we find that the internal radius of deformation constitutes an important dividing scale: at scales of motion larger than the radius of deformation the equilibrium flow is nearly barotropic, while at smaller scales the stream functions in the two layers are statistically uncorrelated. The equilibrium lower-layer flow is positively correlated with bottom topography (anticyclonic flow over seamounts) and the correlation extends to the upper layer at scales larger than the radius of deformation. We suggest that some of the statistical trends observed in non-equilibrium flows may be looked on as manifestations of the tendency for turbulent interactions to maximize the entropy of the system.

1. Introduction

Since the discovery a quarter of a century ago that atmospheric eddies constitute the primary source of the kinetic energy in the zonally averaged flow, interest in nonlinear processes in the geophysical fluids has grown at an accelerating pace. The importance of turbulent interactions in maintaining the atmospheric general circulation is now fully appreciated, and turbulent error amplification has been recognized as the primary obstacle to extended-range numerical weather prediction. In oceanography, the discovery within the past two decades of large amplitude and apparently ubiquitous transient eddy motions has led to a critical re-appraisal of classical ocean-circulation theories and a gradual appreciation that the turbulent interaction terms in the equations of motion may be important or dominant in much of the world ocean.

In this paper we investigate the equilibrium statistical mechanics of some simple but fully nonlinear fluid models which are particularly relevant to oceanography and meteorology. Specifically, we derive the equilibrium statistical states towards which spectrally truncated representations of the equations of motion would evolve in the absence of forcing and viscosity. The simultaneous assumption of a spectral truncation and zero viscosity is unrealistic but necessary

if the classical methods are to apply. The equilibrium states are of interest in themselves, and may be especially important if realistic, non-equilibrium flows are 'close' to inviscid equilibrium in some of their properties.

The models considered in this paper are quasi-geostrophic and also quasi-two-dimensional in the sense that they conserve analogues of the kinetic energy and enstrophy of ordinary two-dimensional flow. We focus initially on two complementary models. The first model corresponds to flow in a single homogeneous layer bounded vertically by flat horizontal plates. The fluid is constrained by rotation about a vertical axis to move only horizontally, so that the governing equation of motion is

$$\partial \zeta / \partial t + J(\psi, \zeta) = 0, \tag{1.1}$$

where ψ is the stream function of the flow and $\zeta = \nabla^2 \psi$. The equilibrium spectrum associated with (1.1) has been discussed by Kraichnan (1967, 1975) and Thompson (1972).

The second model corresponds to quasi-geostrophic flow in a system comprised of two immiscible layers. The governing equations are a coupled set:

$$\partial \zeta_1 / \partial t + J(\psi_1, \zeta_1) = 0 \tag{1.2a}$$

and

$$\partial \zeta_2 / \partial t + J(\psi_2, \zeta_2) = 0, \tag{1.2b}$$

where ψ_1 is the stream function of the top layer, ψ_2 that of the bottom layer, and ζ_1 and ζ_2 are given by

$$\zeta_1 = \nabla^2 \psi_1 + F_1(\psi_2 - \psi_1) \tag{1.3a}$$

and

$$\zeta_2 = \nabla^2 \psi_2 + F_2(\psi_1 - \psi_2). \tag{1.3b}$$

The constants F_1 and F_2 are defined by $F_i = f_0^2/(g'D_i)$, where f_0 is twice the (constant) rotation rate, g' the reduced gravity and D_i the mean depth of the ith layer. Equations (1.2) express the conservation of the potential vorticity ζ_i of each layer. A detailed derivation of these equations is given by Pedlosky (1970). With a slightly different interpretation of the F_i, (1.2) and (1.3) are equivalent to the 'two-and-one-half-level baroclinic model' of the atmosphere (see, for example, Phillips 1956). The equilibrium state of (1.2) has not been previously discussed.

Both (1.1) and (1.2) have been used to model geophysical flows. They differ from the general quasi-geostrophic equations in their neglect of the spatial variation of f_0 and D_i. We defer discussion of these latter effects until later in this paper. The initial goal is to contrast the equilibrium states of (1.1) and (1.2), or, said another way, to study (1.2) in the discontinuous cases $F_i = 0$ and $F_i \neq 0$.

Both (1.1) and (1.2) are assumed to hold within simple closed horizontal boundaries Γ. Corresponding to these boundaries, we define eigenfunctions $\{\phi_i\}$ as the solutions to

$$\nabla^2 \phi_i + k_i^2 \phi_i = 0 \quad (k_i^2 > 0) \tag{1.4a}$$

and

$$\phi_i = 0 \quad \text{on} \quad \Gamma, \tag{1.4b}$$

with the normalization $\overline{\phi_i^2} = 1$, where the overbar denotes integration over the region enclosed by the boundaries. For simplicity we assume $k_i^2 \neq k_j^2$ when $i \neq j$.

(When this condition fails in practice it can always be restored by infinitesimal perturbations of the boundaries.) It then follows that $\overline{\phi_i \phi_j} = \delta_{ij}$. The functions $\{\phi_i\}$ are assumed to be complete in the sense that for any regular function $f(x, y)$

$$f = \sum_{i=1}^{\infty} f_i \phi_i(x, y) \quad \text{where} \quad f_i = \overline{f \phi_i},$$

except perhaps at boundary points. We then expand the dependent variables in terms of the eigenfunctions:

$$\psi = \sum_i (x_i/k_i) \phi_i \qquad (1.5a)$$

and

$$(\psi_1, \psi_2) = \sum_i (a_i, b_i) \phi_i. \qquad (1.5b)$$

The choice of notation is a matter of convenience.

The coefficients $\{x_i\}$ and $\{a_i, b_i\}$ comprise generalized co-ordinates for the systems (1.1) and (1.2). In the phase spaces spanned by these co-ordinates each point represents a possible state of the fluid system, and the evolution of the fluid system is described by a trajectory that is specified by (1.1) and (1.2) in the form

$$dx_i/dt = \sum_j \sum_l k_l/(k_i k_j)\, \beta_{ijl} x_j x_l, \qquad (1.6a)$$

with

$$\beta_{ijl} = -\overline{\phi_i J(\phi_j, \phi_l)}; \qquad (1.6b)$$

and a similar equation for the other system. Let

$$\mathscr{P}(x_1, x_2, \ldots, x_n, t) \quad \text{and} \quad \mathscr{P}(a_1, \ldots, a_n, b_1, \ldots, b_n, t)$$

be the probability distribution functions for states in the phase spaces, which we must assume to be *finite-dimensional*. The time evolution of the finite-dimensional systems is assumed to be governed by (1.6) with the summations truncated to run from 1 up to n. The probability functions obey Liouville's equation,

$$\partial \mathscr{P}/\partial t + \sum_{i=1}^{n} \partial(\mathscr{P} \dot{x}_i)/\partial x_i = 0, \qquad (1.7)$$

and a similar equation for the other system. Equation (1.7) may be thought of as a continuity equation for motion in phase space with the 'velocity' $\dot{x}_i$ given by (1.6). From (1.6) and the fact that β_{ijl} is zero whenever two of its indices are equal, it follows that $\partial \dot{x}_i/\partial x_i = 0$, so that (1.7) becomes

$$\partial \mathscr{P}/\partial t + \sum_i \dot{x}_i\, \partial \mathscr{P}/\partial x_i \equiv D\mathscr{P}/Dt = 0. \qquad (1.8)$$

At equilibrium $(\partial \mathscr{P}/\partial t = 0)$, $\mathscr{P}(x_1, x_2, \ldots, x_n)$ is constant along trajectories in phase space. If a trajectory eventually passes arbitrarily close to every point within a given volume V of phase space, then it follows from (1.8) that $\mathscr{P}$ is a constant over V. The conventional assumption of equal *a priori* probability distributions in phase space is then equivalent to the assumption that V is determined only by a few general constraints on the motion.

Equation (1.1) conserves the quantities $\overline{\nabla \psi . \nabla \psi}$ and $\overline{\zeta^m}$, where m is any

integer.† However, the spectrally truncated form of (1.6) conserves only the spectrally truncated forms of $\overline{\nabla \psi . \nabla \psi}$ and $\overline{\zeta^2}$ and we take these for our general integral invariants. Similar remarks apply to (1.2) with quadratic invariants $\overline{\zeta_1^2}$, $\overline{\zeta_2^2}$ and $\overline{\nabla \psi_1 . \nabla \psi_1 / F_1 + \nabla \psi_2 . \nabla \psi_2 / F_2 + (\psi_1 - \psi_2)^2}$. In terms of the generalized Fourier coefficients, the constraints are

$$\sum_i x_i^2 = E_0, \quad \sum_i k_i^2 x_i^2 = Z_0, \tag{1.9a, b}$$

where E_0 and Z_0 are constants proportional to the constant total kinetic energy and enstrophy, and

$$\sum_i [(k_i^2 + F_1)\, a_i - F_1 b_i]^2 = Z_a, \tag{1.10a}$$

$$\sum_i [(k_i^2 + F_2)\, b_i - F_2 a_i]^2 = Z_b, \tag{1.10b}$$

$$\sum_i [k_i^2 a_i^2 / F_1 + k_i^2 b_i^2 / F_2 + (a_i - b_i)^2] = E_{ab}, \tag{1.10c}$$

where Z_a, Z_b and E_{ab} correspond to the potential enstrophies in the top and bottom layer and to the sum of the total kinetic and available potential energy.

2. The single-layer system

The assumption of equal *a priori* probability distributions in phase space subject only to the constraints (1.9) leads to the microcanonical distribution for the total probability at equilibrium:

$$\mathscr{P}(x_1, x_2, \ldots, x_n) = C\, \delta(E - E_0)\, \delta(Z - Z_0), \tag{2.1a}$$

where

$$E(x_1, x_2, \ldots, x_n) = \sum_i E_i \equiv \sum_i x_i^2, \tag{2.1b}$$

$$Z(x_1, x_2, \ldots, x_n) = \sum_i Z_i \equiv \sum_i k_i^2 x_i^2, \tag{2.1c}$$

and C is a constant of normalization. We obtain the expression for the distribution of a single component x_i by integrating (2.1) over the remaining $n-1$ variables. Asymptotic methods (appendix A) yield

$$\mathscr{P}_i(x_i) \sim \pi^{-\frac{1}{2}} (\alpha + \beta k_i^2)^{\frac{1}{2}} (1 + G_i)^{\frac{1}{2}} \exp\{-\tfrac{1}{2} x_i^2 / \langle x_i^2 \rangle - \tfrac{1}{4}(x_i^2 / \langle x_i^2 \rangle - 1)^2 G_i\}, \tag{2.2a}$$

with

$$\langle x_i^2 \rangle = \tfrac{1}{2}(\alpha + \beta k_i^2)^{-1}, \tag{2.2b}$$

which is valid for small

$$G_i = 2 \sum_l (k_l^2 - k_i^2)^2 \langle x_l^2 \rangle^2 \langle x_i^2 \rangle^2 \Big/ \sum_{l \neq i} \sum_{j \neq i} (k_l^2 - k_j^2)^2 \langle x_l^2 \rangle^2 \langle x_j^2 \rangle^2.$$

The constants α and β are the solutions to the coupled equations

$$\sum_i \langle E_i \rangle \equiv \tfrac{1}{2} \sum_i (\alpha + \beta k_i^2)^{-1} = E_0 \tag{2.3a}$$

and

$$\sum_i \langle Z_i \rangle \equiv \tfrac{1}{2} \sum_i k_i^2 (\alpha + \beta k_i^2)^{-1} = Z_0, \tag{2.3b}$$

† Thomson (1974) has shown that only $\overline{\zeta}$, $\overline{\nabla \psi . \nabla \psi}$ and $\overline{\zeta^2}$ are independent invariants.

subject to the constraints that $\alpha + \beta k_i^2 > 0$ for every i. In appendix B we show that such solutions exist and are unique for any E_0 and $Z_0 = k_*^2 E_0$ provided that $k_{\min}^2 < k_*^2 < k_{\max}^2$, where $k_{\min}$ and $k_{\max}$ are the minimum and maximum wavenumbers in the truncation. The proof is a generalization of a theorem of Fox & Orszag (1973).

For sufficiently small G_i, (2.2) reduces to Boltzmann's Law:

$$\mathscr{P}_i(x_i) \doteq \pi^{-\frac{1}{2}}(\alpha + \beta k_i^2)^{\frac{1}{2}} \exp\{-\alpha E_i - \beta Z_i\}. \tag{2.4}$$

When G_i is not small, which is the case when mode i contains an appreciable fraction of the total energy, then the distribution function for x_i contains a significant correction in the form of local maxima at the r.m.s. x_i. A novel feature of the two-invariant system is that Boltzmann's Law is *not* approached uniformly as n becomes large: it is always possible to choose E_0 and Z_0 such that most of the energy is trapped in the lowest or highest wavenumber. The extreme cases correspond to the negative-temperature states $\alpha \approx -\beta k_{\min}^2$ and $\alpha \approx -\beta k_{\max}^2$. In this paper we are chiefly interested in systems in which none of the modes contains an appreciable fraction of the total energy, and we use Boltzmann's Law without further qualification. For modes evenly distributed in wavenumber space, (2.4) predicts an equilibrium energy spectrum of the general form

$$E(k) = k/(a + bk^2), \tag{2.5}$$

first deduced by Kraichnan (1967). Equation (2.5) has been verified in numerical experiments by Fox & Orszag (1973) and Basdevant & Sadourny (1975).

It is interesting to note that the equilibrium circulation $\bar{\zeta}$ is zero for the truncated system. This is in contrast to the untruncated equations, in which $\bar{\zeta}$ is conserved (and thus non-zero in general), and it suggests that not all of our results will generalize to model equations that conserve the analogue of $\bar{\zeta}$ as well as $\overline{\zeta^2}$ and $\overline{\nabla \psi . \nabla \psi}$.

3. The two-layer system

We consider now the baroclinic model (1.2) with integral invariants given by (1.10). The mathematical development parallels the single-layer case rather closely, and we present only the results. In the two-layer model, the smallest component with definable energy and potential enstrophies consists of *two* co-ordinates a_i and b_i, and the Boltzmann Law approximation to their joint probability distribution is found to be

$$\mathscr{P}_i(a_i, b_i) = \pi^{-1}(Q_i R_i - P_i^2)^{\frac{1}{2}} \exp\{-Q_i a_i^2 - R_i b_i^2 + 2P_i a_i b_i\}, \tag{3.1a}$$

where

$$Q_i = \alpha(r_i + 1) + \beta_1(r_i + 1)^2 + \beta_2, \tag{3.1b}$$

$$R_i = \alpha(r_i/\delta + 1) + \beta_1 + \beta_2(r_i/\delta + 1)^2, \tag{3.1c}$$

$$P_i = \alpha + \beta_1(r_i + 1) + \beta_2(r_i/\delta + 1), \tag{3.1d}$$

$r_i = k_i^2/F_1$ is the non-dimensional square wavenumber, $\delta = D_1/D_2$, and α, β_1 and β_2 are constants that depend on the specified total energy and potential enstrophies of the system. The conditions on α, β_1 and β_2 for the integrability of $\mathscr{P}_i(a_i, b_i)$ are

$$Q_i > 0 \quad \text{or} \quad R_i > 0 \tag{3.2a, b}$$

and

$$Q_i R_i - P_i^2 = (r+1+\delta)(r/\delta)[\alpha^2 + \beta_1\beta_2(r+1+\delta)r/\delta$$
$$+\alpha\beta_1(r+1) + \alpha\beta_2(r/\delta+1)] > 0 \qquad (3.2c)$$

for every i. For given values of the constants of motion, E_{ab}, Z_a and Z_b, we find α, β_1 and β_2 as the solutions to

$$\sum_i \langle E_{ab_i} \rangle = E_{ab}, \qquad (3.3a)$$

$$\sum_i \langle Z_{a_i} \rangle = Z_a, \quad \sum_i \langle Z_{b_i} \rangle = Z_b, \qquad (3.3b,c)$$

subject to the conditions (3.2). Equations (3.3) are the analogues of (2.3) in the single-layer case.

From (3.1) one may readily compute

$$\langle a_i^2 \rangle = \tfrac{1}{2}R_i/(Q_i R_i - P_i^2), \quad \langle b_i^2 \rangle = \tfrac{1}{2}Q_i/(Q_i R_i - P_i^2) \qquad (3.4a,b)$$

and

$$\langle a_i b_i \rangle = \tfrac{1}{2}P_i/(Q_i R_i - P_i^2), \qquad (3.4c)$$

and thereby obtain expressions for the following quantities of physical interest:

(i) The average kinetic energy per unit depth in mode i in the top layer, $K_T = \tfrac{1}{2}k_i^2 \langle a_i^2 \rangle$, and in the bottom layer, $K_B = \tfrac{1}{2}k_i^2 \langle b_i^2 \rangle$, and their ratio.

(ii) The average available potential energy in mode i, $\tfrac{1}{2}f_0^2 \langle (a_i - b_i)^2 \rangle /g'$, and its ratio to the total kinetic energy.

(iii) The correlation coefficient $\rho_i = \langle a_i b_i \rangle \langle a_i^2 \rangle^{-\frac{1}{2}} \langle b_i^2 \rangle^{-\frac{1}{2}}$ between the stream functions in the top and bottom layer for mode i.

We note that if the truncated system contains arbitrarily large wavenumbers ($k_{\max}^2 \to \infty$) then the conditions $Q_i, R_i > 0$ restrict β_1 and β_2 to positive values. In this circumstance it can easily be shown that P_i assumes only positive values, so that the stream functions in the two layers are positively correlated at all wavenumbers.

If $\beta_1, \beta_2 < 0$ then the layers may be negatively correlated. Such states are artificial in the sense that they exist only because of the finite truncation. They represent the relaxation states for fluids in which the energy is initially peaked near $k_{\max}$. As the system adjusts towards equilibrium, energy spreads towards lower wavenumbers, decreasing the enstrophy, and forcing a negative correlation between the layers to conserve potential enstrophy. For flow in bounded domains ($k_{\min}^2 > 0$) equilibrium states in which $\alpha < 0$ are possible. The latter states are non-artificial in the sense that boundedness is a property of real flows, but they show no discontinuous change in any property from equilibrium states in which $\alpha > 0$. For systems containing arbitrarily large and small scales, the conditions (3.2) become $\alpha, \beta_1, \beta_2 > 0$.

We delete the subscript from r_i and regard r as a continuously varying non-dimensional square wavenumber, with the value $r = 1+\delta$ corresponding to motion at the internal Rossby radius of deformation. In the two-layer system, the radius of deformation constitutes an important dividing scale. At scales of motion large compared with $r = 1, \delta$ the kinetic-energy spectra (per unit depth)

of the two layers are nearly equal and assume the same general form as the equilibrium spectrum (2.5) for single-layer flow:

$$r^{\frac{1}{2}}K_T \propto r^{\frac{1}{2}}[\alpha + (1+\delta)\delta^{-1}\beta_1\beta_2(\alpha+\beta_1+\beta_2)^{-1}r]^{-1}. \tag{3.5}$$

The kinetic-energy spectra also approach the single-layer limit at sufficiently small scales, where, for $r \gg 1 + \delta$,

$$r^{\frac{1}{2}}K_T \propto r^{\frac{1}{2}}(\alpha+\beta_1 r)^{-1}[1+\delta\beta_1/(\alpha r+\beta_2 r^2/\delta)] \tag{3.6a}$$

and $$r^{\frac{1}{2}}K_B \propto \delta r^{\frac{1}{2}}(\alpha+\beta_2 r/\delta)^{-1}[1+\beta_2/(\alpha r+\beta_1 r^2)]. \tag{3.6b}$$

As r becomes large, their ratio, top to bottom, tends to the value $\mathscr{R}^\infty = \beta_2/\beta_1\delta^2$. The spectrum of the available potential energy approaches zero at either extreme. For $r \ll 1, \delta$ the correlation coefficient between layers is nearly unity, but it falls abruptly towards zero near where r equals the larger of $\delta(\mathscr{R}^\infty)^{\frac{1}{2}}$ and $(\mathscr{R}^\infty)^{-\frac{1}{2}}$. In cases of interest, the latter expressions are both of order unity. Thus equilibrium two-layer flow resembles a single *barotropic* layer on scales larger than the radius of deformation and two *uncorrelated* single layers on smaller scales.

In extensive numerical experiments, Rhines (1975b) has studied the evolution of the two-layer model from random initial conditions. He finds that large-scale baroclinic currents decay rapidly to deformation-scale eddies, which quickly become barotropic and then gradually increase their scale. Rhines notes that the barotropic final state may be deduced by qualitative arguments based on potential-vorticity conservation for a variety of initial conditions. He also points out that the tendency towards grave horizontal *and* vertical modes is anticipated by Charney's (1971) theory of geostrophic turbulence.

If $\delta = 1$ and $\beta = \beta_1 = \beta_2$ then we are examining the algebraically simple case of 'equivalent layers'. If $\beta > 0$ then the realizability conditions (3.2) become simply $\alpha > -\beta r_{\min}$. Assume $r_{\min} \ll 1$ and $r_{\max} \gg 1$. The two kinetic-energy spectra are identical and their shape is controlled by the ratio β/α. If

$$0 < \beta/\alpha < 1/r_{\max}$$

then the spectra are increasing with r at all wavenumbers, but, if

$$|\beta/\alpha| > 1/r_{\min} \gg 1,$$

the spectra are red. For $1/r_{\max} < \beta/\alpha < 1/r_{\min}$ the spectra have a maximum between $r_{\min}$ and $r_{\max}$. The ratio of available potential to total kinetic energy is

$$[1+r+\beta(r+2)(\alpha+\beta r)^{-1}]^{-1} < 1,$$

which peaks near $r = \sqrt{2}$ at a value of 0.24 for $|\beta/\alpha| \gg 1$.

The ocean and atmosphere are strongly forced fluids whose high-wavenumber cut-offs are not arbitrarily chosen limits, but are determined by the 'viscosity' itself. Nevertheless, the model equations contain realistic turbulent interaction terms, and we may expect even strongly viscous flow to exhibit some of the characteristics of the model inviscid equilibrium if the turbulent interactions are efficient in moving the system towards the state of maximum entropy. In any case, the equilibrium states may indicate the direction in which the interactions will tend to drive the statistics of real flows. Figure 1 presents the equilibrium
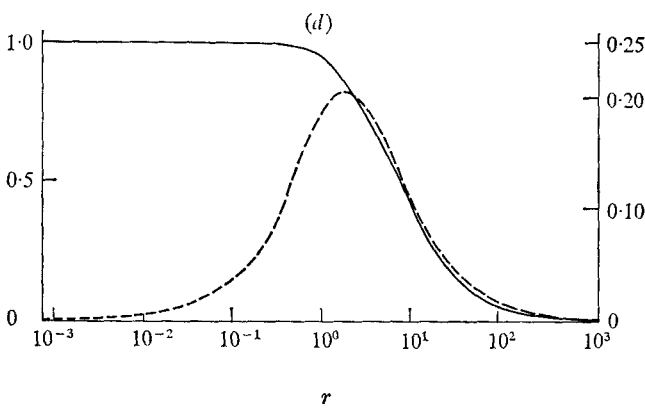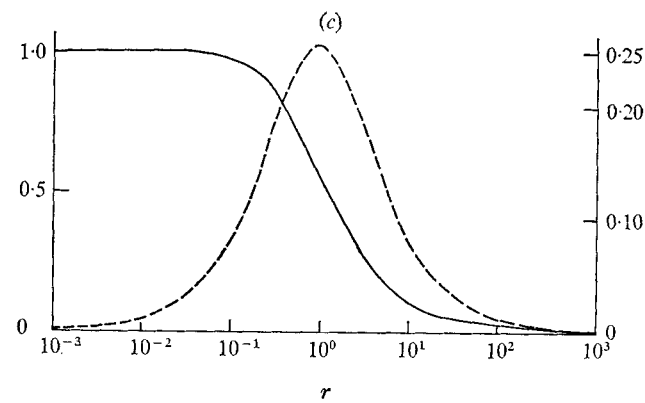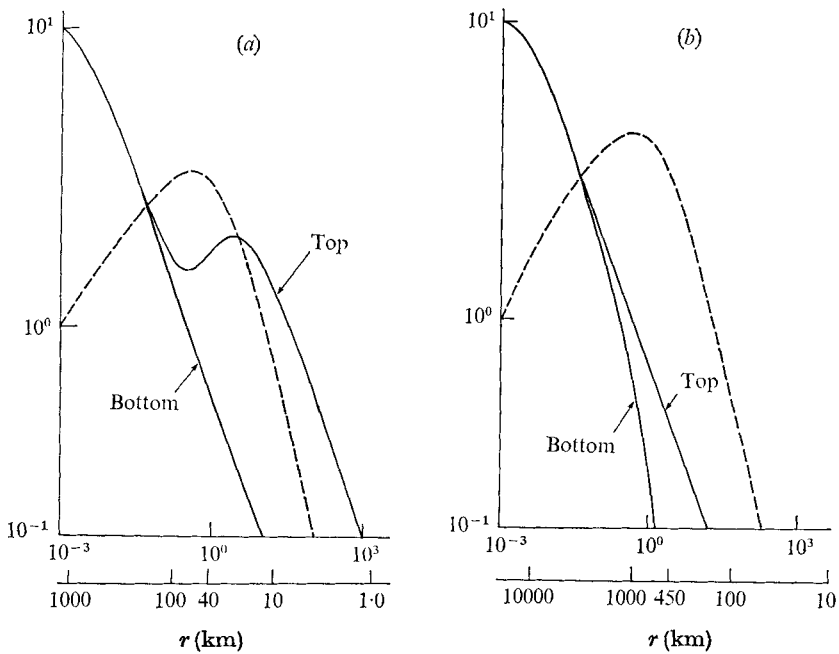
FIGURES 1 (a–d). For legend see facing page.

statistical variables in model two-layer fluids corresponding to the ocean and atmosphere. We choose $\delta = \frac{1}{7}$ for the ocean and $\delta = 1$ for the troposphere. The 'inverse temperatures' α, β_1 and β_2 are chosen to make $\mathscr{R}^\infty = 10$, and the ratios β_1/α and β_2/α are near the minimum required to make the kinetic-energy spectra red at $r_{min} = 10^{-3}$. The secondary maximum in the theoretical oceanic spectrum is a novel consequence of the unequal layer depths and vertical energy densities.

4. Bottom topography and variable rotation rate

We now generalize (1.1) and (1.2) to cases where the lower bounding surface is not flat and the Coriolis parameter f varies linearly in the northward direction y by a small fraction of its average value in the flow region. For model (1.1) the new equation of motion is

$$\partial\zeta/\partial t + J(\psi, \zeta + H(x,y)) = 0, \tag{4.1}$$

where

$$H(x,y) = h(x,y) + \beta^*(y - y_0)$$

and

$$h(x,y) = f_0\, d(x,y)/D.$$

Here, D is the mean depth of the fluid, $d(x,y)$ is the elevation of the bottom above its average level (assumed small compared with D), f_0 is the value of f at $y = y_0$, and $\beta^* = df/dy$ at $y = y_0$.

Expanding $H(x,y)$ in terms of the eigenfunctions, $H(x,y) = \sum_i H_i \phi_i(x,y)$, we

obtain for the invariants of the motion in terms of the general coefficients

$$\sum_i x_i^2 = E_0 \tag{4.2a}$$

and

$$\sum_i k_i^2 x_i^2 - 2\sum_i k_i H_i x_i + \sum_i H_i^2 = Z_0. \tag{4.2b}$$

Methods similar to those used previously then give

$$\mathscr{P}_i(x_i) = \pi^{-\frac{1}{2}}(\alpha + \beta k_i^2)^{\frac{1}{2}} \exp\left\{-(\alpha + \beta k_i^2)\,(x_i - \langle x_i\rangle)^2\right\} \tag{4.3}$$

as a first approximation to the distribution of x_i, with

$$\langle x_i \rangle = \beta k_i(\alpha + \beta k_i^2)^{-1} H_i. \tag{4.4}$$

Again, α and β are constants determined from E_0 and Z_0 subject to the condition that $\alpha + \beta k_i^2 > 0$ for every i.

FIGURE 1. (a) The kinetic-energy spectra per unit depth (solid lines, arbitrary scale) and the spectrum of the available potential energy (dashed line, different arbitrary scale) for equilibrium two-layer flow in the case where $\delta = \frac{1}{7}$, $\mathscr{R}^\infty = 10$ and $\beta_1/\alpha = 10^3$. The kilometre scale gives the corresponding inverse wavenumber in an ocean with a radius of deformation of 35 km, the value on the earth in mid-latitude. (b) The same as (a) except $\delta = 1$. The kilometre scale gives the inverse wavenumber in an atmosphere with radius of deformation equal to 450 km. (c) The correlation coefficient ρ_i (solid line, left scale) between layers and the ratio of available potential to total kinetic energy (dashed line, right scale) for the case of (a). (d) The same as (c) except $\delta = 1$.

Equations (4.3) and (4.4) are a generalization of (2.4), which is recovered when $H_i = 0$. In the general case, the average stream function is, according to (4.4), an energy-weighted version of the topographic field. If the flow contains arbitrarily small scales of motion then $\beta > 0$ and the correlation between stream function and topography is positive at all wavenumbers (anticyclonic flow over seamounts, vorticity decreasing northwards). If $\beta < 0$ then the correlation is negative at all wavenumbers. Again, the latter states are the relaxation states for fluids in which the energy is initially peaked near k_{max}. As the energy spreads towards lower wavenumbers, the correlation between x_i and H_i must become negative to offset the resulting decrease in the total enstrophy $\overline{\zeta^2}$. For all $\beta \neq 0$ the energy,

$$\langle x_i^2 \rangle = \tfrac{1}{2}(\alpha + \beta k_i^2)^{-1} + \beta^2 k_i^2 (\alpha + \beta k_i^2)^{-2} H_i^2, \tag{4.5}$$

is enhanced on topographic scales. The second term in (4.5) represents the energy in the average contour current. The transient energy spectrum has the same functional form as in the case of no topography. In strongly damped numerical simulations of (4.1) Holloway & Hendershott (1974) report a positive correlation between topography and stream function and a topographic enhancement of the energy spectrum similar to (4.5). For large-scale topography, Rhines (1975b) has emphasized the appearance of contour currents from random initial conditions.

The average stream function (4.4) is an exact steady solution to the truncated spectral equations. It is also the spectral truncation of an exact steady solution of the untruncated equations (4.1) satisfying

$$\nabla^2 \psi + H = (\alpha/\beta)\, \psi. \tag{4.6}$$

An important special case is that of beta-plane flow, $H = \beta^*(y - y_0)$, in a bounded rectangular ocean, $x_1 \leqslant x \leqslant x_2$, $y_1 \leqslant y \leqslant y_2$. Fofonoff (1954) considered the case where $k_{min}^2 \beta/\alpha \ll 1$, in which the interior flow is a uniform westward current with return eastward flow occurring in side-wall boundary layers of thickness $(\beta/\alpha)^{\frac{1}{2}}$. The location of the boundary layers is controlled by y_0. If $y_0 = \tfrac{1}{2}(y_1 + y_2)$, then the flow is symmetric, with return currents of equal strength at the north and south boundaries. If $y_0 = y_1$ then there is no southern boundary layer. Thus the equilibrium average state depends on the choice of y_0: at equilibrum,

$$\overline{\zeta + H} = \sum_i \alpha(\alpha + \beta k_i^2)^{-1} H_i\, \overline{\phi}_i,$$

no matter what its initial value. For large β/α, and especially $\alpha \approx -\beta k_{min} < 0$, the regions of eastward flow need *not* be small. In such cases the integral constraints on the motion trap energy in scales too large to resolve the thin boundary layers. Bretherton & Haidvogel (1976) obtain steady solutions of the form (4.4) by quite a different approach, and also note the connexion with Fofonoff's problem.

Actual ocean bottom topography is distinctly inhomogeneous, ranging from extremely rugged regions to smooth abyssal plains, with $h_i \sim k^{-\frac{3}{2}}$ perhaps representative. However, for all h_i decreasing with k_i (and $\beta > 0$), both the proportion of energy in the average current and the correlation between x_i and h_i decrease with increasing k_i at equilibrium. For realistic non-equilibrium flow, the same conclusions need not apply. We anticipate that the largest scales of motion,

which are perhaps most subject to external forcing, may also require the longest adjustment times.

In the two-layer system, the effects of topography and variable rotation rate are non-equivalent in the sense that topography alters the expression for potential vorticity in the lower layer only. The governing equations are (1.2) with

$$\zeta_1 = \nabla^2 \psi_1 + F_1(\psi_2 - \psi_1) + \beta^*(y - y_0)$$

and

$$\zeta_2 = \nabla^2 \psi_2 + F_2(\psi_1 - \psi_2) + \beta^*(y - y_0) + h.$$

Let

$$h = \sum_i h_i \phi_i, \quad \beta^*(y - y_0) = \sum_i h_i^* \phi_i.$$

Then the joint distribution of a_i and b_i is $\mathscr{P}_i(a_i - \langle a_i \rangle, \, b_i - \langle b_i \rangle)$, where $\mathscr{P}_i$ is given by (3.1), and the new average flow field consists of both a topographic part and a β^*-part:

$$\langle a_i \rangle = S(r_i) \{\beta_1 h_i + [\alpha \beta_1 \delta / \beta_2 + \beta_1(r_i + 1 + \delta)] h_i^*\}$$

and

$$\langle b_i \rangle = S(r_i) \{[\alpha + \beta_1(r_i + 1)] h_i + [\alpha + \beta_1(r_i + 1 + \delta)] h_i^*\},$$

with

$$S(r) = (\beta_2 / F_2 \delta) r(r + 1 + \delta) [QR - P^2]^{-1}.$$

For large $|\beta_1 / \alpha|$ (red kinetic-energy spectra), the ratio of upper to lower mean topographic flow is nearly unity for $r \ll 1$ and decreases sharply at $r = 1$. Thus bottom topography affects the equilibrium upper-layer flow only on scales large compared with the radius of deformation. Smaller-scale topography traps energy preferentially in the lower layer. For red spectra, the ratio a_i^* / b_i^* is nearly unity and both layers are equally affected by the variable rotation rate.

5. Conclusions

We have derived the equilibrium statistical states towards which spectrally truncated representations of the equations of motion would evolve in the absence of forcing and viscosity. The theory gives no information about the speed of approach to inviscid equilibrium, which may depend both on the initial conditions and on the statistics being considered. We anticipate that waves (which occur in our systems if topography or variable rotation is present) will complicate the adjustment. Numerical experiments by Rhines (1975a) have shown that, on scales where the Rossby-wave phase speed approaches the r.m.s. particle speed, the transfer of energy by turbulent interactions is inhibited. Holloway & Hendershott (1975) have demonstrated such suppression of turbulence by waves in the context of a turbulence closure approximation. In both the problem of approach to inviscid equilibrium and the problem with forcing and dissipation, some form of turbulence closure theory is required. A promising avenue of understanding is the study of the class of closures (see, for example, Orszag 1970) for which, in the absence of forcing and dissipation, the equilibrium states derived above are the stable stationary solutions.

Appendix A

The expression sought is

$$\mathscr{P}_n(x_n) = C \iint_{-\infty}^{\infty} \dots \int \left[\prod_{i=1}^{n-1} dx_i \right] \delta(E - E_0) \, \delta(Z - Z_0),$$

where E and Z are given by (2.1). Let $\omega_i(E, Z) \equiv E^{-\frac{1}{2}}\delta(k_i^2 E - Z)$. Then the above integral may be expressed exactly as

$$\mathscr{P}_n(x_n) = C \iint_0^{\infty} \dots \int \left[\prod_{i=1}^{n-1} dE_i \, dZ_i \, \omega_i(E_i, Z_i) \right] \delta(E - E_0) \, \delta(Z - Z_0)$$

$$= C \iint_0^{\infty} \dots \int \left[\prod_{i=1}^{n-2} dE_i \, dZ_i \, \omega_i(E_i, Z_i) \right]$$

$$\times \omega_{n-1}\left(E_0 - E_n - \sum_{i=1}^{n-2} E_i, \, Z_0 - Z_n - \sum_{i=1}^{n-2} Z_i \right).$$

Now define

$$f_i(E, Z) \equiv \pi^{-\frac{1}{2}}(\alpha + \beta k_i^2)^{\frac{1}{2}} \, \omega_i(E, Z) \exp\{-\alpha E - \beta Z\},$$

such that

$$\iint_0^{\infty} dE \, dZ f_i = 1.$$

At this point, α and β can be any two constants such that the above integrals converge. Then

$$\mathscr{P}_n(x_n) = C' \exp\left(-\alpha E_n - \beta Z_n\right) \iint_0^{\infty} \dots \int \left[\prod_{i=1}^{n-2} dE_i \, dZ_i f_i(E_i, Z_i) \right]$$

$$\times f_{n-1}\left(E_0 - E_n - \sum_{i=1}^{n-2} E_i, \, Z_0 - Z_n - \sum_{i=1}^{n-2} Z_i \right)$$

$$= C' \exp\left(-\alpha E_n - \beta Z_n\right) f(E_0 - E_n, Z_0 - Z_n),$$

where $f(X, Y)$ is the same as the joint distribution for the sums of independent random variables

$$X = X_1 + X_2 + \dots + X_{n-1}$$

and

$$Y = Y_1 + Y_2 + \dots + Y_{n-1}$$

with joint distribution functions $f_i(X_i, Y_i)$. For large n and small G_i the central limit theorem applies and (2.2) results when α and β are chosen to be the solutions of (2.3). The above is essentially the method of Khinchin (1949, pp. 70–110). Further details are given in Salmon (1975).

Appendix B

This appendix shows that there exists a unique solution (α, β) to

$$\tfrac{1}{2} \sum_i (\alpha + \beta k_i^2)^{-1} = E_0 \tag{B 1a}$$

and

$$\tfrac{1}{2} \sum_i k_i^2 (\alpha + \beta k_i^2)^{-1} = k_*^2 E_0, \tag{B 1b}$$

subject to the constraint $\alpha + \beta k_i^2 > 0$ for every i, provided that $k_{\min}^2 < k_*^2 < k_{\max}^2$.

The system (B 1) is equivalent to

$$-\partial f/\partial \alpha = E_0, \quad -\partial f/\partial \beta = Z_0, \tag{B 2a, b}$$

where

$$f(\alpha, \beta) \equiv -\tfrac{1}{2} \sum_i \ln(\alpha + \beta k_i^2).$$

Thus $f(\alpha, \beta)$ is defined on the infinite sector $R_{\alpha\beta}$ of the α, β plane in which $\alpha + \beta k_i^2 > 0$ for every i. By direct computation, one can verify that on $R_{\alpha\beta}$

$$-f_\alpha > 0, \quad -f_\beta > 0, \quad f_{\alpha\alpha} f_{\beta\beta} - (f_{\alpha\beta})^2 > 0. \tag{B 3a–c}$$

We can therefore define curvilinear co-ordinates on $R_{\alpha\beta}$ by

$$\gamma = f_\beta / f_\alpha, \quad \delta = -f_\alpha.$$

In terms of the curvilinear co-ordinates, the region $R_{\alpha\beta}$ is described by

$$k_{\min}^2 < \gamma < k_{\max}^2, \quad 0 < \delta < \infty,$$

and (B 2) become

$$\delta = E_0, \quad \gamma = k_*^2, \tag{B 4a, b}$$

which always has a unique solution in $R_{\alpha\beta}$ because, from (B 3),

$$\partial(\gamma, \delta)/\partial(\alpha, \beta) = [f_{\alpha\alpha} f_{\beta\beta} - (f_{\alpha\beta})^2]/f_\alpha < 0$$

throughout $R_{\alpha\beta}$.

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