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HAMILTON'S PRINCIPLE AND ERTEL'S THEOREM

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1. Introduction

Variation principles for the equations governing the motion of perfect fluids are of two types. In the first type, which corresponds to Hamilton's principle in particle mechanics, the positions of marked fluid particles are varied at fixed times.[1,2] In the second type of variation principle, appropriately chosen field variables are varied at fixed locations and times.[3,4] The field variables include a set of scalar potentials which represent the fluid velocity. It has recently been shown that these two types of variation principle are really the same: they are related by canonical transformations.[5,6]

This note has two objectives. The first is to demonstrate a simple and particularly illuminating connection between variation principles of the two types mentioned above. The second objective is to give a new and direct derivation of Ertel's theorem of hydrodynamics based upon a symmetry property of the fluid Lagrangian.[10] The results reported here were noticed in the course of an application of Hamiltonian methods to a study of the ocean's main thermocline.

2. Hamilton's Principle

Consider first a classical system composed of N discrete particles. Let i be a subscript index which identifies the particle, and let m_i and $\vec{x}_i(\tau)$ be the mass and the Cartesian position of the i-th particle at time τ. Let $V(\vec{x}_1,\ldots,\vec{x}_N)$ be the potential energy of the system. Then the Lagrangian is

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$$L = \left(\sum_{i=1}^{N} \tfrac{1}{2} m_i \dot{\vec{x}}_i \cdot \dot{\vec{x}}_i\right) - V(\vec{x}_1, \ldots, \vec{x}_N) \tag{2.1}$$

and the dynamical equations result from Hamilton's principle in the form

$$\delta \int L \, d\tau = 0 \tag{2.2}$$

where δ corresponds to arbitrary variations $\delta\vec{x}_i(\tau)$ in the particle trajectories, and $\delta\vec{x}_i(\pm\infty)=0$. Alternatively, one can define the conjugate momenta,

$$\vec{p}_i = \partial L / \partial \dot{\vec{x}}_i \ , \tag{2.3}$$

and invoke Hamilton's principle in the "extended" form,

$$\delta \int d\tau \left\{\sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{x}}_i - H(\vec{p}_j, \vec{x}_j)\right\} = 0 \ , \tag{2.4}$$

where

$$H = \sum_{i=1}^{N} \vec{p}_i \cdot \dot{\vec{x}}_i - L \tag{2.5}$$

and δ now corresponds to arbitrary independent variations $\delta\vec{p}_i(\tau)$, $\delta\vec{x}_i(\tau)$ in the momenta and positions of the particles.

Consider next the fluid continuum. Let the positions $\vec{x}=\vec{x}(\vec{a},\tau)$ of marked fluid particles be considered as functions of curvilinear labeling coordinates $\vec{a}$, amd the time τ. The labeling coordinates remain constant following fluid particles, and they are analogous to the subscript i above. It is convenient to assign these labeling coordinates so that equal volumes in $\vec{a}$-space contain equal masses. Then

$$\rho = \frac{\partial(a_1, a_2, a_3)}{\partial(x_1, x_2, x_3)} \tag{2.6}$$

is the mass density of the fluid. It follows directly from (2.6), and the fact that $\vec{a}$ is conserved on particles, that

$$\partial\rho/\partial\tau + \rho\nabla_x \cdot \vec{u} = 0 \tag{2.7}$$

where

$$\vec{u} = \partial\vec{x}/\partial\tau \tag{2.8}$$

and

$$\nabla_x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) \tag{2.9}$$

is the gradient operator in $\vec{x}$-space. Equation (2.7) is the usual equation of mass conservation.

Let the fluid Lagrangian be

$$L = \iiint d\vec{a} \; \{ \tfrac{1}{2} \, \partial\vec{x}/\partial\tau \cdot \partial\vec{x}/\partial\tau - V(\rho) \} \tag{2.10}$$

where the potential V is a specified function of the density ρ. In (2.10) and below, the symbol ρ is merely an abbreviation for the Jacobian in (2.6). Hamilton's principle now states that

$$\delta \int L \, d\tau = 0 \tag{2.11}$$

where L is given by (2.10) and δ corresponds to arbitrary variations $\delta\vec{x}(a,\tau)$ in the position of particle $\vec{a}$ at time τ. Assume for convenience that the fluid is infinite and that $\delta\vec{x}$ vanishes at large $|\vec{a}|$ and $|\tau|$. Then by the ordinary rules of variational calculus,

δx_1: $\quad 0 = \delta \int L \, d\tau =$

$$= \int d\tau \iiint d\vec{a} \; \{ \partial x_1/\partial\tau \; \partial\delta x_1/\partial\tau + V'(\rho)\rho^2 \frac{\partial(\delta x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} \}$$

$$= \int d\tau \iiint d\vec{a} \; \{ -\partial^2 x_1/\partial\tau^2 - \frac{\partial(\rho^2 V', x_2, x_3)}{\partial(a_1, a_2, a_3)} \} \; \delta x_1 \tag{2.12}$$

implies that

$$\partial^2 x_1/\partial\tau^2 = - \frac{\partial(x_1, x_2, x_3)}{\partial(a_1, a_2, a_3)} \frac{\partial(\rho^2 V', x_2, x_3)}{\partial(x_1, x_2, x_3)} = - \frac{1}{\rho} \, \partial(\rho^2 V')/\partial x_1 \tag{2.13}$$

plus similar equations for δx_2 and δx_3. The definition

$$p = \rho^2 \, dV/d\rho \tag{2.14}$$

brings (2.13) and its counterparts into the familiar form

$$\partial\vec{u}/\partial\tau = -\frac{1}{\rho}\ \nabla_x\, p\ . \tag{2.15}$$

If p is required to be the thermodynamic pressure, and the flow is isentropic, then by (2.14) $V(\rho)$ must be the internal energy per unit mass. However, no such interpretation is actually required, because the laws of particle mechanics do not depend on the axioms of thermodynamics. The extended principle analogous to (2.4) is

$$\delta\int d\tau\{\iiint d\vec{a}\ \vec{u}\cdot\partial\vec{x}/\partial\tau - H\} = 0 \tag{2.16}$$

where

$$H = \iiint d\vec{a}\{\ \tfrac{1}{2}\vec{u}\cdot\vec{u} + V(\rho)\}\ . \tag{2.17}$$

Independent variations $\delta\vec{u}(\vec{a},\tau)$, $\delta\vec{x}(\vec{a},\tau)$ yield (2.8) and (2.15).

From a slightly different point of view, the fluid motion is a time-dependent map,

$$\vec{x} = \vec{x}(\vec{a},\tau)\ , \tag{2.18}$$

from $\vec{a}$-space into $\vec{x}$-space, and Hamilton's principle requires that $\int L\, d\tau$ be stationary for arbitrary variations in this map. Since each forward map (2.18) uniquely determines an inverse map,

$$\vec{a} = \vec{a}(\vec{x},t)\ , \tag{2.19}$$

from $\vec{x}$-space into $\vec{a}$-space, it is obvious that variations in this inverse map would serve as well. Here $t=\tau$, but $\partial/\partial\tau$ implies that $\vec{a}$ is held constant, while $\partial/\partial t$ implies constant $\vec{x}$. Rewrite (2.10) as

$$L = \iiint d\vec{x}\rho\ \{\tfrac{1}{2}\ \vec{u}\cdot\vec{u} - V(\rho)\} \tag{2.20}$$

and substitute for $\vec{u}$ from the identities

$$\rho\ u_1 = -\frac{\partial(a_1,a_2,a_3)}{\partial(t,x_2,x_3)}\quad ,\ \text{etc.} \tag{2.21}$$

The result is

$$L = \iiint d\vec{x}\ \frac{1}{\rho}\ \Big\{\ \tfrac{1}{2}\Big[\frac{\partial(a_1,a_2,a_3)}{\partial(t,x_2,x_3)}\Big]^2 + \tfrac{1}{2}\Big[\frac{\partial(a_1,a_2,a_3)}{\partial(x_1,t,x_2)}\Big]^2$$

$$+ \tfrac{1}{2}\Big[\frac{\partial(a_1,a_2,a_3)}{\partial(x_1,x_2,t)}\Big]^2 - \rho^2\ V(\rho)\ \Big\}\ . \qquad (2.22)$$

Hamilton's principle now requires that $\int L\, d\tau$ be stationary with respect to variations $\delta\vec{a}(\vec{x},t)$ in the labeling coordinates. This variation principle is Eulerian in the sense that $\vec{x}$ and t are the independent variables. To obtain the extended form, define momenta conjugate to $\vec{a}$, viz.

$$\vec{\Pi} = \delta L/\delta(\partial\vec{a}/\partial t)\ , \qquad (2.23)$$

and eliminate $\partial\vec{a}/\partial t$ from (2.21). The result is

$$\rho\ u_1 = -\ \Pi_1\ \partial a_1/\partial x_1 - \Pi_2\ \partial a_2/\partial x_1 - \Pi_3\ \partial a_3/\partial x_1\ ,\ \text{etc.} \qquad (2.24)$$

and the extended principle is therefore

$$\delta \int dt\{\iiint d\vec{x}\ \ \vec{\Pi}\partial\vec{a}/\partial t - H\} = 0 \qquad (2.25)$$

where now δ corresponds to independent variations $\delta\vec{\Pi}(\vec{x},t)$ and $\delta\vec{a}(\vec{x},t)$. It is convenient to define

$$\vec{A} = -\vec{\Pi}/\rho \qquad (2.26)$$

which can be freely varied in place of $\vec{\Pi}$. Then (2.25) takes the form

$$\delta \int dt\{\ \iiint d\vec{x}\ \rho\vec{A}\cdot\partial\vec{a}/\partial t + H\} = 0 \qquad (2.27)$$

where

$$H = \iiint d\vec{x}\ \rho\ \{\tfrac{1}{2}\ \vec{u}\cdot\vec{u} + V(\rho)\}\ \ , \qquad (2.28)$$

$$\vec{u} = A_1\ \nabla_x a_1 + A_2\ \nabla_x a_2 + A_3\ \nabla_x a_3\ \ , \qquad (2.29)$$

and δ corresponds to variations $\delta\vec{A}(\vec{x},t)$ and $\delta\vec{a}(\vec{x},t)$. The variation principle (2.27-2.29) was obtained by Seliger and Whitham[4] by a rather different approach. The present derivation emphasizes the close connection between (2.27) and the Hamilton's principle of particle mechanics, and it puts a clear interpretation on $\vec{a}$ and $\vec{A}$. The $\vec{a}$ are mass labeling coordinates which can be assigned in numerous ways to satisfy (2.6). But once the a_i have been chosen, the A_i are uniquely determined from (2.29) as the projections of $\vec{u}$ on the curvilinear basis vectors $\nabla_x a_i$. The $\nabla_x a_i$ form a basis provided only that ρ is nonzero. Thus $\vec{a}$ and $\vec{A}$ are always single-valued. I note for future use that the reciprocal of (2.29) is

$$\vec{A} = u_1 \nabla_a x_1 + u_2 \nabla_a x_2 + u_3 \nabla_a x_3 \qquad (2.30)$$

where

$$\nabla_a = (\partial/\partial a_1 , \partial/\partial a_2 , \partial/\partial a_3) \qquad (2.31)$$

is the gradient operator in $\vec{a}$-space.

3. Ertel's Theorem[10]

As remarked in section 2, the Lagrangian (2.10) is unaffected by any transformation of the labeling coordinates which leaves the Jacobian (2.6) unchanged. This symmetry property leads to a conservation law discovered by Ertel[7] using wholly different methods. The following derivation by way of Noether's theorem is considerably more direct. Suppose that $\delta\vec{a}(x,t)$ is indeed such that

$$\delta \frac{\partial(a_1,a_2,a_3)}{\partial(x_1,x_2,x_3)} = 0 \quad . \qquad (3.1)$$

This implies that

$$\partial\delta a_1/\partial a_1 + \partial\delta a_2/\partial a_2 + \partial\delta a_3/\partial a_3 = 0 \qquad (3.2)$$

provided that ρ is nonzero. Thus

$$\delta\vec{a} = \nabla_a \times \vec{T} \qquad (3.3) \cdot$$

for some $\vec{T}=\vec{T}(\vec{a},\tau)$. For such a variation,

$$\delta\int L\, d\tau = \delta\int d\tau \iiint d\vec{a}\ \tfrac{1}{2}\, \partial\vec{x}/\partial\tau \cdot \partial\vec{x}/\partial\tau$$
$$= -\int d\tau \iiint d\vec{a}\ \ \vec{A}\cdot \partial\vec{a}/\partial\tau \qquad (3.4)$$

since it can easily be shown that

$$\delta(\partial\vec{x}/\partial\tau) = -\frac{\partial\vec{x}}{\partial a_1}\frac{\partial\delta a_1}{\partial\tau} - \frac{\partial\vec{x}}{\partial a_2}\frac{\partial\delta a_2}{\partial\tau} - \frac{\partial\vec{x}}{\partial a_3}\frac{\partial\delta a_3}{\partial\tau} \qquad (3.5)$$

The vector $\vec{A}$ is that given by (2.30). Substitution from (3.3) and integrations by parts bring (3.4) into the form

$$\delta\int L\, d\tau = \int d\tau \iiint d\vec{a}\ \vec{T}\cdot\partial/\partial\tau\ \{\nabla_a \times \vec{A}\} \quad . \qquad (3.6)$$

But $\vec{T}$ is arbitrary and (3.6) must vanish by Hamilton's principle. It follows that

$$\partial\vec{Q}/\partial\tau = \vec{0} \qquad (3.7)$$

where

$$\vec{Q} = \nabla_a \times \vec{A} \quad . \qquad (3.8)$$

The vector $\vec{Q}$ is conserved on particles. Now let $\Phi=\Phi(a_1,a_2,a_3)$ be any quantity which is also conserved on particles. Then

$$q = \vec{Q}\cdot\nabla_a\Phi \qquad (3.9)$$

is also conserved. With help from (2.30), q may also be written

$$q = \frac{1}{\rho}\ \nabla_x\Phi\cdot(\ \nabla_x \times \vec{u}\ ) \quad . \qquad (3.10)$$

The statement

$$\partial q/\partial\tau = 0 \qquad (3.11)$$

is Ertel's theorem. Since Φ is arbitrary (3.11) and (3.7) are equivalent. With $\Phi=\rho$ (and assuming $\partial\rho/\partial\tau=0$) the quantity q is called the potential vorticity.

The foregoing procedure extends to include rotating coordinates and it provides an elegant unification for all forms of Ertel's theorem which is lacking in the conventional derivations: for any continuum system, Ertel's theorem is simply the conservation law which results from the most general transformation of labeling coordinates that leaves every term in the Lagrangian unchanged. This approach also provides a motivation for Ertel's theorem: the conservation law is known to exist as soon as inspection of the Lagrangian reveals a symmetry property. One need not depend on unguided manipulations.

If the potential in (2.10) is replaced by

$$V(\rho, S(a_1, a_2, a_3)) \ , \tag{3.12}$$

where S is a function of the labeling coordinates only (usually considered to be the specific entropy), then the results of section 2 generalize easily, but the general conservation law is destroyed. This is obvious because, if S is completely arbitrary, as it must be to accomodate arbitrary initial entropy distributions, then no general transformation of the labeling coordinates leaves V unchanged.

Eckart[2] derived the conservation law (3.9) using the energy-momentum tensor formalism, which is related to the procedure followed here, but he did not notice the connection with Ertel's theorem. See also Bretherton[8,9] for some closely related results.

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