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Hamilton's Principle and the Vorticity Laws for a Relativistic Perfect Fluid

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The equations for a relativistic perfect fluid result from the requirement that the total mass-energy be stationary with respect to variations $\delta x^\alpha(a, b, c, s)$ in the space-time location of the fluid particle identified by Lagrangian labels (a, b, c) at the point s on its world-line. By considering variations of the Lagrangian labels that leave the specific volume and entropy unchanged, we obtain a general covariant statement of vorticity conservation. The conservation laws for circulation, potential vorticity, and helicity are simple corollaries. This Noether-theorem derivation shows that the vorticity laws have no analogues in particle mechanics, where the corresponding particle labels cannot be continuously varied.

KEY WORDS: Vorticity, Noether theorem, relativistic perfect fluid.

1. INTRODUCTION

In the simplest Hamiltonian formulation of ordinary nonrelativistic fluid mechanics, the independent variables include a set of Lagrangian labels $\mathbf{a}=(a, b, c)$, which vary continuously in space and time, and remain constant following the motion of the fluid. The dependent variables are the Cartesian locations $\mathbf{x}(\mathbf{a}, t)$ at time t of the fluid particle labeled by $\mathbf{a}$. The Lagrangian labels can be arbitrarily assigned. However, the choice

$$da\,db\,dc = d(\text{mass}) \tag{1.1}$$

is especially convenient; then

$$\rho = \partial(a, b, c)/\partial(x, y, z) \tag{1.2}$$

is the mass density, and time-differentiation of (1.2) yields the mass conservation equation

$$\partial\rho/\partial t = -\rho\mathbf{\nabla}\cdot\mathbf{u}. \tag{1.3}$$

Here, $\partial/\partial t$ (with $\mathbf{a}$ held fixed) is the convective derivative, $\mathbf{u} \equiv \partial\mathbf{x}/\partial t$ is the fluid velocity, and $\mathbf{\nabla} \equiv (\partial_x, \partial_y, \partial_z)$ at fixed t. Thus, mass conservation is implicit in the labeling of fluid particles.

The momentum equation results from Hamilton's principle in the form

$$\delta \int dt \iiint d\mathbf{a}\{\tfrac{1}{2}\partial\mathbf{x}/\partial t\cdot\partial\mathbf{x}/\partial t - E(\alpha, S(\mathbf{a}))\} = 0 \tag{1.4}$$

for arbitrary variations $\delta\mathbf{x}(\mathbf{a}, t)$ in the fluid particle locations. The internal energy $E(\alpha, S)$ per unit mass is a prescribed function of the specific volume $\alpha \equiv \rho^{-1}$ and the specific entropy S. External forces, including the effects of boundaries, are easily included.

The internal energy in (1.4) is unaffected by particle label variations $\delta\mathbf{a}(\mathbf{x}, t)$ that leave the Jacobian (1.2) and entropy $S(\mathbf{a})$ unchanged. This symmetry property corresponds, by Noether's theorem, to a general statement of vorticity conservation. For homentropic flow the general vorticity law is

$$(\partial/\partial t)[\mathbf{\nabla}_{\mathbf{a}} \times \mathbf{A}(\mathbf{a}, t)] = \mathbf{0}, \tag{1.5}$$

where $\mathbf{\nabla}_{\mathbf{a}} = (\partial_a, \partial_b, \partial_c)$ is the gradient operator in $\mathbf{a}$-space, and $\mathbf{A}$ is defined by

$$\mathbf{A}\cdot d\mathbf{a} = \mathbf{u}\cdot d\mathbf{x}. \tag{1.6}$$

The well-known conservation theorems for circulation, potential vorticity, and helicity follow directly from (1.5). However, the general vorticity law (1.5) cannot be stated without the use of the Lagrangian labels. For a review of the nonrelativistic theory, including the results quoted above, refer to Salmon (1988).

As shown in Section 2, the equations for a relativistic perfect fluid result from a generalization of (1.4) in which the independent variables include the same Lagrangian labels (a, b, c) as in the nonrelativistic limit. By considering variations in these labels that leave the specific volume and entropy unchanged, we obtain in Section 3 the covariant generalization of (1.5) and its counterpart for nonhomentropic flow. The relativistic conservation laws for circulation, potential vorticity, and helicity are simple corollaries. As explained in Section 4, these vorticity laws have no analogues in particle mechanics, because the discrete particle labels analogous to (a, b, c) cannot be varied continuously.

The action principle for a relativistic perfect fluid has been given in many forms. See, for example, Taub (1954), Schutz (1971), Hawking and Ellis (1973), and Moncrief (1977). The particular form (2.15, 2.18) given below is not new (except in details), but the particle-relabeling symmetry property is strikingly evident in the form (2.15), and this provides the motivation for the general vorticity law. The fundamental new results are the general vorticity laws (3.10–3.11) or (3.17–3.18), the motivated way in which they are derived from (2.15), and their relationship to the more familiar conservation laws for potential vorticity, circulation, and helicity.

2. RELATIVISTIC HYDRODYNAMICS

We assume that spacetime is flat, and scale time such that light moves at unit speed. Let $(t, x, y, z) = (x^0, x^1, x^2, x^3)$ be arbitrary Lorentz coordinates. If (t', x', y', z') is any other set of Lorentz coordinates then

$$\eta_{\alpha\beta}\, dx^\alpha\, dx^\beta = dx_\alpha\, dx^\alpha = dx'_\alpha\, dx'^\alpha \tag{2.1}$$

and

$$\partial(t', x', y', z')/\partial(t, x, y, z) = 1, \tag{2.2}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$ is the metric in flat spacetime. Here and below, Greek indices take values from 0 to 3, Latin indices from 1 to 3, and repeated indices are summed.

Let $x^\alpha(s, a, b, c) = x^\alpha(a^0, a^1, a^2, a^3)$ be the spacetime location of the fluid particle identified by Lagrangian labels (a, b, c) at the point on its world-line parameterized by s. We assume that (s, a, b, c) depend smoothly on (t, x, y, z), and that all world-lines are time-like,

$$(\partial x_\alpha/\partial s)(\partial x^\alpha/\partial s) < 0. \tag{2.3}$$

As in the nonrelativistic case, the Lagrangian labels (a, b, c) can be arbitrarily assigned. However, the choice

$$da\, db\, dc = d(\text{rest mass}) \tag{2.4}$$

is again convenient; then

$$\rho = \partial(t, a, b, c)/\partial(t, x, y, z) \tag{2.5}$$

is the (rest) mass density measured in the (t, x, y, z) frame. Define

$$R_s \equiv \{-(\partial x_\alpha/\partial s)(\partial x^\alpha/\partial s)\}^{1/2} \tag{2.6}$$

and

$$J_s \equiv \partial(t, x, y, z)/\partial(s, a, b, c). \tag{2.7}$$

Then

$$\rho_0 = R_s/J_s \tag{2.8}$$

is the *proper mass density*—the mass density measured in a Lorentz frame in which the fluid is locally at rest. Note that the quotient in (2.8) has the same value no matter how s is defined. A sensible choice is $s = \tau$, where the proper time τ is defined as the integral

$$\tau \equiv \int [-dx_\alpha\, dx^\alpha]^{1/2} \quad (a, b, c \text{ fixed}) \tag{2.9}$$

along world-lines, from a three-dimensional hypersurface that all world-lines intersect exactly once. Then, since $R_\tau = 1$, we have

$$\rho_0 = \partial(\tau, a, b, c)/\partial(t, x, y, z). \tag{2.10}$$

However, it is often useful to leave the parameter s unspecified. By (2.10)

$$\rho_0 = [\partial(t,a,b,c)/\partial(t,x,y,z)][\partial(\tau,a,b,c)/\partial(t,a,b,c)] = \rho/\gamma, \quad (2.11)$$

where

$$\gamma = \{1 - \mathbf{u}\cdot\mathbf{u}\}^{-1/2} \quad (2.12)$$

represents the effect of a Lorentz contraction in the direction of the local fluid velocity **u**. Equations (2.11–2.12) establish the validity of (2.10) [and hence of (2.8)]. A direct application of $\partial/\partial\tau$ to (2.10) yields the familiar mass conservation equation

$$\partial(\rho_0 U^\alpha)/\partial x^\alpha = 0, \quad (2.13)$$

where

$$U^\alpha \equiv \partial x^\alpha/\partial\tau \quad (2.14)$$

is the fluid four-velocity. The covariance of all these results is manifest from (2.1) and (2.2).

The Lagrangian for a relativistic fluid of fixed composition is simply

$$L = \iiiint ds\, da\, db\, dc \quad R_s W(J_s/R_s, S(a,b,c)), \quad (2.15)$$

where $W(\alpha_0, S)$, the *total proper energy per unit mass*, is a prescribed function of the proper specific volume $\alpha_0 \equiv \rho_0^{-1}$ and the specific entropy S. The entropy $S(a,b,c)$ depends only on the particle identity, in a manner determined by initial conditions. Thus

$$\partial S/\partial\tau = 0. \quad (2.16)$$

By definition

$$W(\alpha_0, S) = 1 + E(\alpha_0, S), \quad (2.17)$$

where $E(\alpha_0, S)$, also a prescribed function, is the internal energy in a comoving frame. Hamilton's principle states that

$$\delta L = 0 \tag{2.18}$$

for arbitrary variations $\delta x^\alpha(s, a, b, c)$ in the spacetime locations of fluid particles. In the presently considered case of an infinite fluid, all variations vanish at infinity. Choosing $s = \tau$ we find that

$$\begin{aligned}\delta L &= \iiiint d\tau\, da\, db\, dc\{[W - (\partial W/\partial \alpha_0)J_\tau/R_\tau]\delta R_\tau + (\partial W/\partial \alpha_0)\delta J_\tau\} \\ &= \iiiint d\tau\, da\, db\, dc\{h\, \delta R_\tau - p\, \delta J_\tau\},\end{aligned} \tag{2.19}$$

where

$$p \equiv -\partial E(\alpha_0, S)/\partial \alpha_0 \tag{2.20}$$

is the pressure and

$$h \equiv W + p/\rho_0 \tag{2.21}$$

is the enthalpy. Note that $R_\tau = 1$, but we do not set $R_\tau = 1$ until after the variations. Now

$$\delta R_\tau = -U_\alpha \delta U^\alpha = -U_\alpha\, \partial \delta x^\alpha/\partial \tau \tag{2.22}$$

and

$$\delta J_\tau = \partial(\delta t, x, y, z)/\partial(\tau, a, b, c) + \partial(t, \delta x, y, z)/\partial(\tau, a, b, c) + \cdots \tag{2.23}$$

plus two similar terms. Substituting (2.22–2.23) into (2.19) and integrating by parts, we finally obtain

$$0 = \iiiint d\tau\, da\, db\, dc[\partial(hU_\alpha)/\partial \tau + \alpha_0\, \partial p/\partial x^\alpha]\delta x^\alpha. \tag{2.24}$$

Since δx^α is arbitrary

$$\partial(hU_\alpha)/\partial \tau + \alpha_0\, \partial p/\partial x^\alpha = 0. \tag{2.25}$$

But

$$\partial/\partial \tau = (\partial x^\alpha/\partial \tau)(\partial/\partial x^\alpha) = U^\alpha\, \partial/\partial x^\alpha. \tag{2.26}$$

Thus, after multiplication by ρ_0 and use of (2.13), (2.25) is equivalent to

$$(\partial/\partial x^\beta)[\rho_0 h U^\alpha U^\beta + p\eta^{\alpha\beta}] = 0. \tag{2.27}$$

Equations (2.13), (2.16), (2.20), (2.27) are the standard equations for a relativistic perfect fluid. The present derivation emphasizes that, apart from definitions, these equations result solely from Hamilton's principle (2.15, 2.18) and the choice of $E(\alpha_0, S)$.

The nonrelativistic limit of (2.15) is interesting. For this we choose $s = t$. Then, since $R_t = \gamma^{-1}$ and

$$E \ll 1, \quad \mathbf{u} \cdot \mathbf{u} \ll 1 \tag{2.28}$$

in nonrelativistic flow, we find that

$$\begin{aligned} L &= \iiiint dt\, d\mathbf{a}(1+E)(1-\mathbf{u}\cdot\mathbf{u})^{1/2} \\ &\approx \iiiint dt\, d\mathbf{a}\{1 + E(\alpha, S) - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\} \end{aligned} \tag{2.29}$$

so that (2.15) reduces to (1.4). Thus, as in particle mechanics, the Lagrangian is the *difference* between kinetic and potential energy only in the nonrelativistic limit. The exact relativistic Lagrangian is the exact *total* energy—an even simpler quantity.

3. RELATIVISTIC VORTICITY LAW

The fluid motion $x^\alpha(a^\beta)$ is a map from a-space into x-space, and Hamilton's principle (2.18) requires that the action be stationary for arbitrary variations $\delta x^\alpha(a^\beta)$ in this map. Since each forward map $x^\alpha(a^\beta)$ uniquely determines an inverse map $a^\alpha(x^\beta)$, Hamilton's principle is obviously equivalent to the statement

$$\delta \iiiint dx\, W(Q_s, S(a^1, a^2, a^3))/Q_s = 0, \qquad Q_s \equiv J_s/R_s \tag{3.1}$$

for arbitrary variations $\delta a^\alpha(x^\beta)$ in the inverse map.

The mass labels (a^1, a^2, a^3) enter (3.1) only through the Jacobian J_s and the entropy $S(a^1, a^2, a^3)$. We therefore consider variations $\delta a^i(x^\beta)$ for which $\delta J_s = 0$ and $\delta S = 0$. As in the nonrelativistic case, these variations correspond to a continuous relabeling of fluid particles that does not affect the specific volume or entropy. The resulting conservation law turns out to be the most general statement of vorticity conservation.

First suppose that the fluid is homentropic (i.e. $S=\text{constant}$). Then Hamilton's principle reduces to the simple form

$$\delta \iiiint dx \;\; W(J_s/R_s)R_s/J_s=0. \tag{3.2}$$

If $\delta J_s=0$, this is

$$\iiiint dx[(W/J_s)+p(J_s/R_s)/R_s]\delta R_s=0, \tag{3.3}$$

where

$$\delta R_s=-(\partial x_\alpha/\partial s)\delta(\partial x^\alpha/\partial s)=(\partial x_\alpha/\partial s)(\partial x^\alpha/\partial a^i)[\partial(\delta a^i)/\partial s]. \tag{3.4}$$

Again, repeated Greek indices are summed from 0 to 3, and Latin indices from 1 to 3. But $\delta J_s=0$ implies that

$$\partial(\delta a^i)/\partial a^i=0 \tag{3.5}$$

and thus that

$$\delta a^i=\varepsilon^{ijk}\,\partial T_k/\partial a^j, \tag{3.6}$$

where ε^{ijk} is the permutation symbol, and $T_i(a^\alpha)$ is arbitrary but vanishes at infinity. Setting $s=\tau$ and substituting (3.4) and (3.6) into (3.3), we obtain

$$\begin{aligned} 0&=\iiiint da \;\; hU_\alpha \;(\partial x^\alpha/\partial a^i)\,\partial(\delta a^i)/\partial\tau \\ &=\iiiint da \;\; hU_\alpha \;(\partial x^\alpha/\partial a^i)\varepsilon^{ijk}(\partial^2 T_k/\partial\tau\,\partial a^j) \\ &=-\iiiint (\partial Q^i/\partial\tau)T_i, \end{aligned} \tag{3.7}$$

where

$$Q^i=\varepsilon^{ijk}[\partial(hU_\mu)/\partial a^j](\partial x^\mu/\partial a^k). \tag{3.8}$$

Since T_i is arbitrary,

$$\partial Q^i/\partial\tau=0 \tag{3.9}$$

and Q^i is constant along world-lines. All of the vorticity laws for a relativistic perfect fluid are easy corollaries of (3.9) and its counterpart for nonhomentropic flow.

The homentropic vorticity law (3.9) can also be written in the *apparently* more general form

$$\partial Q^{\alpha\beta}/\partial\tau=0, \tag{3.10}$$

where

$$Q^{\alpha\beta}\equiv\varepsilon^{\alpha\beta\gamma\delta}[\partial(hU_\mu)/\partial a^\gamma](\partial x^\mu/\partial a^\delta). \tag{3.11}$$

The form (3.10–3.11) is easily obtained by considering variations $\delta a^\alpha(x^\beta)$ of all *four* labeling coordinates for which $\delta J_s=0$ (still considering homentropic flow). In this case, $\delta J_s=0$ implies that

$$\partial(\delta a^\alpha)/\partial a^\alpha=0 \tag{3.12}$$

and thus that

$$\delta a^\alpha=\varepsilon^{\alpha\beta\gamma\delta}\,\partial T_{\gamma\delta}/\partial a^\beta \tag{3.13}$$

for some $T_{\gamma\delta}(a^\mu)$. Substitutions into (3.3) now lead to

$$0=\iiiint da(\partial Q^{\alpha\beta}/\partial\tau)T_{\alpha\beta} \tag{3.14}$$

and (3.10) follows. However, it is easily shown that $Q^i=Q^{0i}$ and [using (3.15)] that $Q^{ij}\equiv0$, so that (3.9) and (3.10) have precisely the same physical content. This is expected because the variations δa^α differ from δa^i only in that the former include world-line parameter variations δs, and, as previously noted, the Lagrangian (3.1) is unaffected by δs because J_s and R_s always occur as a quotient. However, the fully four-dimensional form of (3.10) is a convenience for transferring results out of label-space. Both (3.9) and (3.10) can be verified directly from the homentropic form of (2.25), viz.

$$\partial(hU_\mu)/\partial\tau + \partial h/\partial x^\mu = 0. \tag{3.15}$$

If the flow is nonhomentropic (i.e. $\nabla S \neq \mathbf{0}$), we let the entropy S be one of the particle labels ($a^3 = S$, say) and consider variations δa^1, δa^2 that leave the Jacobian (2.7) unchanged. These variations correspond to a relabeling of fluid particles within surfaces of constant entropy. Then

$$\delta a^1 = -\partial T/\partial a^2, \qquad \delta a^2 = \partial T/\partial a^1 \tag{3.16}$$

for some $T(a^\alpha)$ and, omitting easy steps, we find that

$$\partial Q[S]/\partial\tau = 0, \tag{3.17}$$

where the *potential vorticity* $Q[S]$ is defined by

$$Q[\theta] \equiv \varepsilon^{ijk}[\partial(hU_\alpha)/\partial a^i](\partial x^\alpha/\partial a^j)(\partial\theta/\partial a^k). \tag{3.18}$$

Choosing $a^0 \equiv s = t$, this is

$$Q[\theta] = \rho^{-1}\varepsilon^{ijk}[\partial(hU_m)/\partial x^i](\partial x^m/\partial x^j)(\partial\theta/\partial x^k)$$

$$= \rho^{-1}[\nabla \times (\gamma h\mathbf{u})] \cdot \nabla\theta. \tag{3.19}$$

Holm (1985, p. 15) derived the potential vorticity law in the form (3.17, 3.19) without the use of Noether's theorem.

Now let $\theta(a^1, a^2, a^3)$ be an *arbitrary* conserved scalar,

$$\partial\theta/\partial\tau = 0. \tag{3.20}$$

If the flow is homentropic, then (3.9) and (3.20) imply that

$$\partial Q[\theta]/\partial\tau = 0. \tag{3.21}$$

Since θ is arbitrary, (3.9) and (3.21) are actually equivalent. Thus all of our results are summed up by the general potential vorticity law

(3.21) in which θ is an arbitrary conserved scalar if the flow is homentropic, and θ is the entropy itself if the flow is nonhomentropic.

The definition (3.18) can be rewritten as

$$Q[\theta]=\varepsilon^{\alpha\beta\gamma 0}[\partial(hU_\mu)/\partial a^\alpha](\partial x^\mu/\partial a^\beta)(\partial\theta/\partial a^\gamma). \tag{3.22}$$

Multiplying (3.22) by the constant $U_\rho U^\rho = U_\rho\, \partial x^\rho/\partial a^0$, and noting that (3.22) must be covariant with respect to transformations of the labeling coordinates, we obtain the four-dimensional equivalent,

$$\begin{aligned} Q[\theta] &= \varepsilon^{\alpha\beta\gamma\delta}[\partial(hU_\mu)/\partial a^\alpha](\partial x^\mu/\partial a^\beta)(\partial x^\rho/\partial a^\gamma)(\partial\theta/\partial a^\delta)U_\rho \\ &= [\partial(hU_\mu, x^\mu, x^\rho, \theta)/\partial(\tau, a, b, c)]U_\rho \\ &= \rho_0^{-1}[\partial(hU_\mu, x^\mu, x^\rho, \theta)/\partial(t, x, y, z)]U_\rho \\ &= \rho_0^{-1}\varepsilon^{\alpha\beta\gamma\delta}[\partial(hU_\beta)/\partial x^\alpha]U_\gamma(\partial\theta/\partial x^\delta). \end{aligned} \tag{3.23}$$

This is the covariant form of the potential vorticity discovered by Katz (1984). Katz showed that (3.21, 3.23) is related to Kelvin's theorem. We next show that Kelvin's theorem, like (3.21), follows immediately from (3.9) or (3.17).

For homentropic flow (3.8–3.9) can be rewritten

$$(\partial/\partial s)[\nabla_{\mathbf{a}} \times \mathbf{A}(\mathbf{a}, s)] = \mathbf{0}, \tag{3.24}$$

where

$$A_i = hU_\mu\, \partial x^\mu/\partial a^i \tag{3.25}$$

is the relativistic generalization of (1.6). Consider any loop fixed in **a**-space. By (3.24–3.25) and Stokes' theorem,

$$(d/ds)\oint \mathbf{A}\cdot d\mathbf{a} = (d/ds)\oint hU_\mu\, dx^\mu = 0. \tag{3.26}$$

This is Kelvin's theorem. For nonhomentropic flow, (3.17) can be rewritten

$$(d/ds)[\nabla_{\mathbf{a}} S \cdot \nabla_{\mathbf{a}} \times \mathbf{A}(\mathbf{a}, s)] = 0, \tag{3.27}$$

so that (3.26) holds only for loops within surfaces of constant entropy.

Next consider any fixed volume V in $\mathbf{a}$-space with boundary everywhere tangent to $\nabla_{\mathbf{a}}\times\mathbf{A}$. It follows from (3.24) that

$$\partial\mathbf{A}/\partial s=\nabla_{\mathbf{a}}\phi(\mathbf{a},s) \tag{3.28}$$

and thus that

$$(d/ds)\iiint_V d\mathbf{a}[\mathbf{A}\cdot\nabla_{\mathbf{a}}\times\mathbf{A}]=0. \tag{3.29}$$

This expresses the conservation of helicity (Carter, 1978). If $s=t$ then (3.29) is equivalent to

$$(d/dt)\iiint_{V'} d\mathbf{x}[h\mathbf{u}\cdot\nabla\times(h\mathbf{u})]=0, \tag{3.30}$$

where V', the corresponding volume in $\mathbf{x}$-space, is a material volume of closed vortex tubes.

4. CONNECTION WITH PARTICLE MECHANICS

The Lagrangian for a system of non-interacting particles is (Landau and Lifshitz, 1975)

$$L=\int ds\sum m_{(i)}R_{s(i)}, \tag{4.1}$$

where $m_{(i)}$ and $x^{\alpha}_{(i)}$ are the mass and location of the ith particle, and

$$R_{s(i)}=\{-(dx_{\alpha(i)}/ds)(dx^{\alpha}_{(i)}/ds)\}^{1/2}. \tag{4.2}$$

Clearly, (2.15, 2.17) is the generalization of (4.1) to the case of particles distributed continuously in spacetime. The proper energy W per unit mass includes a contribution E from the *motions* of "microscopic constituents" with respect to the local center of mass. This internal energy E depends on only two local properties of the macroscopic motion—the proper mass density ρ_0 and the comoving entropy S. This special dependence is the essence of the perfect fluid approximation.

In the analogy between (2.15) and (4.1), the Lagrangian labels (a, b, c) are analogous to the discrete particle labels i. The general vorticity laws are therefore a result of the continuum approximation. They have no analogue in particle mechanics, where the integers i cannot be continuously varied.

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