An ocean circulation model
based upon operator-splitting, Hamiltonian brackets
and the inclusion of sound waves

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ABSTRACT

This paper offers a simple, entirely prognostic, ocean circulation model based upon the separation of the complete dynamics, including sound waves, into elementary Poisson brackets. For example, one bracket corresponds to the propagation of sound waves in a single direction. Other brackets correspond to the rotation of the velocity vector by individual components of the vorticity, and to the action of buoyancy force. The dynamics is solved by Strang splitting of the brackets. Key features of the method are the assumption that the sound waves propagate exactly one grid distance in a time step, and the use of Riemann invariants to solve the sound-wave dynamics exactly. In these features the method resembles the lattice Boltzmann method, but the flexibility of more conventional methods is retained. As in the lattice Boltzmann method, very short time steps are required to prevent unrealistically strong coupling between the sound waves and the slow hydrodynamic motions of primary interest. However, the disadvantage of small time steps is more than compensated by the model’s extreme simplicity, even in the presence of very complicated boundaries, and by its massively parallel form. Numerical tests and examples illustrate the practicality of the method.

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1. Introduction

This paper offers a simple, completely prognostic, numerical ocean circulation model based upon the full equations of fluid dynamics, including sound waves. Simplicity is achieved by splitting the dynamics into a sequence of elementary steps. For example, one step corresponds to the propagation of sound waves in (say) the $x$-direction. Another step corresponds to the action of Coriolis force. Each step corresponds to a Hamiltonian bracket, hence important conservation laws survive. Because the model dynamics includes sound waves, there is no need to solve any elliptic boundary-value problems. All the dependent variables step forward in time. This greatly facilitates the coding, especially in complicated geometry. In fact, the complete model comprises a relatively large number of very short subroutines, which, because of directional splitting, are actually indifferent to the complexity of the ocean basin shape.

The sound speed is an adjustable parameter of the model and need not be as large as the actual sound speed. The only general requirement is that the Mach number be small compared to unity. In rotating flow, the acoustic deformation radius $c/\Omega$ (sound speed divided by Coriolis parameter) must be larger than the domain size. In stratified flow, the scale depth $c/N$ (sound speed divided by Vaisala frequency) must be larger than the ocean depth. By choosing the time step to be the grid-point separation divided by the sound speed, we solve the ‘sound wave split’ exactly, using the method of Riemann invariants. Thus small time steps correspond to large sound speed and to realistically incompressible flow.

Two earlier papers (Salmon, 1999a, 1999b) entertained the idea of solving ocean circulation models using the lattice Boltzmann method. Unfortunately, subsequent attempts to apply the lattice Boltzmann method to ocean basins with realistic bathymetry failed for reasons connected with the inflexibility of the method and with the highly anisotropic nature of ocean dynamics—the huge dissimilarity between leading-order dynamical balances in the horizontal and vertical directions. At the same time, more recent work (Salmon, 2004, 2005, 2007) has re-emphasized the importance of retaining conservation laws in numerical algorithms. The present method, which attempts to combine the advantages of the lattice Boltzmann method with the need to maintain conservation laws, represents a marriage of these two philosophies. In the lattice Boltzmann method, fluid particles hop from one grid point to the next grid point in a time step. In the method of this paper, Riemann invariants propagate exactly one grid distance in a time step. However, whereas lattice Boltzmann particles relax irreversibly (and hence diffusively) toward a local equilibrium state that represents the entire dynamics, our propagation of Riemann invariants is but one component—one split—of the complete dynamics. We solve each of the splits by an algorithm that is designed especially for that split, and which maintains as many conservation laws as possible.

Of course, no matter what the method, viscosity and diffusion must be present to represent sub-gridscale physics. However, the viscous and diffusive splits should not affect the conservative nature of the others. Unlike in the lattice Boltzmann method—or, for that matter, the more widely used ‘semi-Lagrangian’ advection schemes—our viscosity and diffusion are independent, separately controllable components of the whole dynamics.
The plan of the paper is as follows. Section 2 introduces the basic physics in Hamiltonian form and explains the fundamental analytical approximation, a modification of the Hamiltonian that simplifies the dynamics but still accurately corresponds to the Boussinesq equations in the limit $c \to \infty$ of infinite sound speed. Since we make no approximation to the Poisson bracket, the analytical equations preserve all the exact conservation laws. A partition of the Poisson bracket into three parts produces rotation sub-dynamics, sound-wave sub-dynamics, and buoyancy sub-dynamics. Each of these sub-dynamics is applied separately to the flow using Strang-splitting. We solve the sound-wave sub-dynamics (Section 3) by directional splitting and the use of Riemann invariants. Buoyancy sub-dynamics (Section 4) also uses directional splitting, with the precise formulation selected for its compatibility with the sound-wave sub-dynamics and with the conservation of mass, buoyancy, and buoyancy-squared. The latter may be especially important because it, along with total energy conservation, guarantees the conservation of the sum of kinetic, internal and available potential energy. Rotation sub-dynamics (Section 5) does not use directional splitting, but instead splits the $\mathbf{\omega} \times \mathbf{v}$ term into pieces proportional to the three components of the vorticity $\mathbf{\omega}$. Section 6 compares solutions of our model in two horizontal dimensions to solutions obtained using more conventional methods.

All ocean circulation models face fundamental difficulties associated with the huge disparity between the horizontal and vertical spatial resolution. Section 7 explains how one solution to this problem—the ‘aspect ratio trick’ proposed by Browning et al. (1990)—is ideally suited to the present method; it corresponds to an adjustment to the sound speed that equates the time required for sound waves to propagate a vertical grid distance to the time required to propagate a horizontal grid distance. Section 8 presents solutions of ocean convection over realistic topography that incorporate this aspect-ratio trick. Section 9 concludes.

2. Analytical approximations

We begin with the perfect-fluid equations in the form

$$\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} & = \mathbf{v} \times \mathbf{\omega} - \nabla \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + c^2 \frac{\rho}{\rho_0} \right) + \theta \mathbf{k} \\
\frac{\partial \rho}{\partial t} & = -\nabla \cdot (\rho \mathbf{v}) \\
\frac{\partial \theta}{\partial t} & = -\mathbf{v} \cdot \nabla \theta
\end{align*}$$

(2.1a) \hspace{1cm} (2.1b) \hspace{1cm} (2.1c)

where $\mathbf{v}$ is the fluid velocity; $\mathbf{\omega} = \nabla \times \mathbf{v} + 2 \mathbf{\Omega}$ is the total vorticity, including the Earth’s rotation vector $\mathbf{\Omega}$; $\rho$ is the mass density with representative constant value $\rho_0$; $c$ is the sound speed (a prescribed constant); and $\theta$ is the buoyancy, which could be further subdivided into temperature and salinity. The ‘exact’ dynamics (2.1) are equivalent to the Hamiltonian bracket formulation,

$$\frac{dF}{dt} = \{F, H_e\}$$

(2.2)

where $F = F[\mathbf{v}, \rho, \theta]$ is any functional of the variables $\mathbf{v}(x,y,z,t)$, $\rho(x,y,z,t)$, and $\theta(x,y,z,t)$; $\{F, H_e\}$ is the Poisson bracket defined by
\[ \{F,H_\varepsilon\} = \iiint dx \, \frac{\omega}{\rho} \left( \frac{\delta F}{\delta v} \times \frac{\delta H_\varepsilon}{\delta v} \right) \]
\[ + \iiint dx \left( \nabla \frac{\delta F}{\delta \rho} \cdot \frac{\delta H_\varepsilon}{\delta v} - \nabla \frac{\delta H_\varepsilon}{\delta \rho} \cdot \frac{\delta F}{\delta v} \right) + \iiint dx \, \frac{\nabla \theta}{\rho} \left( \frac{\delta F}{\delta v} \frac{\delta H_\varepsilon}{\delta \theta} - \frac{\delta H_\varepsilon}{\delta \theta} \frac{\delta F}{\delta v} \right) \]
\[ (2.3) \]

and
\[ H_\varepsilon = \iiint dx \left( \frac{1}{2} \rho v \cdot v + \frac{1}{2} \frac{c^2}{\rho_0} \rho^2 - \rho \theta z \right) \]
\[ (2.4) \]
is the exact Hamiltonian. By successively setting \( F = v(x_0), \rho(x_0), \theta(x_0) \), where \( x_0 \) is an arbitrary fixed location, and making use of the functional derivatives
\[ \frac{\delta H_\varepsilon}{\delta v} = \rho v, \quad \frac{\delta H_\varepsilon}{\delta \rho} = \frac{1}{2} v \cdot v + c^2 \rho / \rho_0 - \theta z, \quad \frac{\delta H_\varepsilon}{\delta \theta} = -\rho z \]
\[ (2.5) \]
we recover the exact dynamics (2.1). Strictly speaking, the bracket (2.3) applies only to unbounded flow; the incorporation of boundary conditions into Poisson brackets is problematic. However, we use (2.3) only to infer the governing finite-difference equations within the fluid interior. We infer the corresponding boundary conditions by requiring that the appropriate fluxes vanish at boundaries.

Now we introduce approximations, both analytical and numerical, to the Poisson bracket (2.3) and the Hamiltonian (2.4). Our single analytical approximation will be to replace the exact Hamiltonian \( H_\varepsilon \) by
\[ H = \iiint dx \left( \frac{1}{2} \rho_0 v \cdot v + \frac{1}{2} \frac{c^2}{\rho_0} \rho^2 - \rho \theta z \right) \]
\[ (2.6) \]
in which the constant average density replaces the density factor in the kinetic-energy term. We make no analytical approximations to the bracket. Since now
\[ \frac{\delta H}{\delta v} = \rho_0 v, \quad \frac{\delta H}{\delta \rho} = c^2 \rho / \rho_0 - \theta z, \quad \frac{\delta H}{\delta \theta} = -\rho z \]
\[ (2.7) \]
we obtain the approximate dynamics
\[ \frac{\partial v}{\partial t} = \frac{\rho_0}{\rho} v \times \omega - \nabla \left( \frac{c^2}{\rho_0} \rho \right) + \theta \mathbf{k} \]
\[ (2.8a) \]
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho_0 v) \]
\[ (2.8b) \]
\[ \frac{\partial \theta}{\partial t} = -\frac{\rho_0}{\rho} v \cdot \nabla \theta \]
\[ (2.8c) \]
The advantage of (2.6) over (2.4) is that it leads to equations, (2.8), in which the sound waves move at an absolutely constant speed. This makes it possible to solve the sound-wave split exactly. In the limit \( c \to \infty \) of interest, both the exact dynamics (2.1) and the approximate dynamics (2.8) reduce to the Boussinesq equations, despite the difference between (2.1a) and (2.8a).

To pave the way for numerical approximations, we introduce the new variables
\[ \phi = c^2 \frac{\rho}{\rho_0} \quad \text{and} \quad \alpha = \frac{\rho}{\rho_0} \theta \]
\[ (2.9) \]
and transform the variables from \((v, \rho, \theta)\) to \((v, \phi, \alpha)\). The functional derivatives transform as
\[
\frac{\delta F}{\delta v} = \frac{\partial F}{\partial v}, \quad \frac{\delta F}{\delta \rho} = \frac{c^2}{\rho_0} \left( \frac{\partial F}{\partial \phi} + \frac{\alpha}{\phi} \frac{\partial F}{\partial \alpha} \right), \quad \frac{\delta F}{\delta \theta} = c^2 \frac{\partial F}{\partial \alpha}
\]  
(2.10)

and hence (2.3) and (2.6) take the forms

\[
\frac{dF}{dt} = \{ F, H \}_1 + \{ F, H \}_2 + \{ F, H \}_3
\]  
(2.11)

where

\[
\{ F, H \}_1 = \iiint dx \frac{c^2}{\phi} \cdot \left( \frac{\partial F}{\partial \phi} \times \frac{\delta H}{\delta v} \right)
\]  
(2.12a)

\[
\{ F, H \}_2 = c^2 \iiint dx \left( \nabla \frac{\delta F}{\delta \phi} \cdot \frac{\delta H}{\delta v} - \nabla \frac{\delta F}{\delta \phi} \cdot \frac{\partial F}{\partial v} \right)
\]  
(2.12b)

\[
\{ F, H \}_3 = c^2 \iiint dx \frac{\alpha}{\phi} \left( \nabla \frac{\delta F}{\delta \alpha} \cdot \frac{\partial H}{\partial v} - \nabla \frac{\delta F}{\delta \alpha} \cdot \frac{\partial F}{\partial v} \right)
\]  
(2.12c)

and

\[
H = \iiint dx \left( \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} \frac{\phi^2}{c^2} - \alpha z \right)
\]  
(2.13)

(Note that a factor of \( \rho_0 \) has been cancelled between the brackets and the Hamiltonian.)

The advantage of the transformation (2.9) is that (2.12c) takes a simpler form than the last term in (2.3).

The three brackets in (2.12) correspond, respectively, to the rotation sub-dynamics,

\[
\frac{\partial \mathbf{v}}{\partial t} = \{ \mathbf{v}, H \}_1 = \frac{c^2}{\phi} \mathbf{v} \times \omega
\]  
(2.14a)

\[
\frac{\partial \phi}{\partial t} = \{ \phi, H \}_1 = 0
\]  
(2.14b)

\[
\frac{\partial \alpha}{\partial t} = \{ \alpha, H \}_1 = 0
\]  
(2.14c)

the sound-wave sub-dynamics,

\[
\frac{\partial \mathbf{v}}{\partial t} = \{ \mathbf{v}, H \}_2 = -\nabla \phi
\]  
(2.15a)

\[
\frac{\partial \phi}{\partial t} = \{ \phi, H \}_2 = -c^2 \nabla \cdot \mathbf{v}
\]  
(2.15b)

\[
\frac{\partial \alpha}{\partial t} = \{ \alpha, H \}_2 = 0
\]  
(2.15c)

and the buoyancy sub-dynamics

\[
\frac{\partial \mathbf{v}}{\partial t} = \{ \mathbf{v}, H \}_3 = c^2 \frac{\alpha}{\phi} \mathbf{k} = \beta \mathbf{k}
\]  
(2.16a)

\[
\frac{\partial \phi}{\partial t} = \{ \phi, H \}_3 = 0
\]  
(2.16b)

\[
\frac{\partial \alpha}{\partial t} = \{ \alpha, H \}_3 = -c^2 \nabla \cdot \left( \frac{\alpha}{\phi} \mathbf{v} \right) = -\nabla \cdot (\mathbf{v} \beta)
\]  
(2.16c)

We obtain approximate dynamics equivalent to (2.8) by summing up the terms on the right-hand sides of (2.14), (2.15) and (2.16). However, we solve the dynamics (2.14-16)
by applying each of the three sub-dynamics—(2.14), (2.15), and (2.16)—successively, at each time step. In fact, we further split each sub-dynamics into its directional or vorticity components, as described in following sections. In the limit $c \to \infty$, both the exact dynamics (2.1) and approximate dynamics (2.14-16) reduce to the Boussinesq equations

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \boldsymbol{\omega} - \nabla \phi + \theta \mathbf{k}$$  \hspace{1cm} (2.17a)
$$\nabla \cdot \mathbf{v} = 0$$  \hspace{1cm} (2.17b)
$$\frac{\partial \theta}{\partial t} = -\mathbf{v} \cdot \nabla \theta$$  \hspace{1cm} (2.17c)

for incompressible flow. Thus our entire procedure is merely a device for solving (2.17).

We replace each sub-dynamics by finite-difference analogs that maintain as many conservation laws as possible. The energy (2.13) is conserved or semi-conserved so long as the spatial discretization maintains the antisymmetry property of each of the brackets in (2.12). By ‘semi-conservation’ we mean conservation except for errors in the discretization of the time derivative. Besides the energy, we also wish to conserve the mass

$$M = \iiint dx \phi \propto \iiint dx \rho,$$  \hspace{1cm} (2.18)

the buoyancy

$$B = \iiint dx \alpha \propto \iiint dx \rho \theta,$$  \hspace{1cm} (2.19)

and the squared buoyancy

$$Z = \frac{1}{2} \iiint dx \frac{\alpha^2}{\phi} \propto \iiint dx \rho \theta^2$$  \hspace{1cm} (2.20)

Additional conserved quantities include the momentum (disregarding the effects of solid boundaries and the buoyancy force),

$$P = \iiint dx \phi \mathbf{v} \propto \iiint dx \rho \mathbf{v},$$  \hspace{1cm} (2.21)

the integrated potential vorticity

$$Q = \iint dx \phi q \propto \iint dx \omega \cdot \nabla \theta,$$  \hspace{1cm} (2.22)

and the potential enstrophy

$$Q_z = \iiint dx \phi \frac{q^2}{\phi} \propto \iiint dx \left(\omega \cdot \nabla \theta\right)^2 \rho^{-1}$$  \hspace{1cm} (2.23)

where $q = \omega \cdot \nabla \theta / \rho$ is the Ertel potential vorticity conserved on fluid particles. The approximate dynamics (2.14-16) conserves all of (2.18-23) because the bracket (2.12) is exact, and because the conservation of (2.18-23) depends only on the form of the bracket. (Although momentum conservation requires translation-invariance of the Hamiltonian, the form of the conserved momentum depends only on the bracket. For a discussion of this point, see Shepherd (1990).)

### 3. Sound wave splitting

We discretize the sound-wave sub-dynamics by further splitting (2.15) into each direction. For example, the $z$-direction sound-wave dynamics is

$$\frac{\partial \mathbf{w}}{\partial t} = -\frac{\partial \phi}{\partial z}, \quad \frac{\partial \phi}{\partial t} = -c^2 \frac{\partial \mathbf{w}}{\partial z}, \quad \text{and} \quad \frac{\partial (\theta \mathbf{k})}{\partial t} = 0$$  \hspace{1cm} (3.1)

Eqns. (3.1a) and (3.1b) form a closed system equivalent to
Once (3.2) are solved, (3.1c) determines $\theta$ from the new value of $\phi$. Eqns. (3.2) can be solved exactly if the vertical grid-spacing $\Delta z$ equals $c \Delta t$, the sound velocity times the time step. In an unbounded domain, the exact solution is

$$
\left( w + \frac{\phi}{c} \right)^{n+1}_i = \left( w + \frac{\phi}{c} \right)^n_{i-1} \quad \text{and} \quad \left( w - \frac{\phi}{c} \right)^{n+1}_i = \left( w - \frac{\phi}{c} \right)^n_{i+1}
$$

(3.3)

where superscripts denote the time step and subscripts denote the vertical grid location. The two Riemann invariants are shifted right and left. If a solid, bottom boundary is present at (say) grid point 1, then the boundary condition $w=0$ there implies that

$$
\left( w + \frac{\phi}{c} \right)^n_0 = -\left( w - \frac{\phi}{c} \right)_2^n
$$

(3.4)

This provides the recipe for reflecting one Riemann invariant into another. An analogous condition holds at the upper rigid boundary. At interior grid points, (3.3) correspond to the finite-difference formulae

$$
\begin{align*}
\phi_i^{n+1} &= \frac{1}{2} \left( \phi_{i-1}^n + \phi_{i+1}^n \right) + \frac{1}{2c} \left( \phi_{i-1}^n - \phi_{i+1}^n \right) \\
\phi_i^{n+1} &= \frac{1}{2} \left( \phi_{i-1}^n + \phi_{i+1}^n \right) + \frac{c}{2} \left( w_{i-1}^n - w_{i+1}^n \right)
\end{align*}
$$

(3.5a) and (3.5b)

while at the lower boundary we have

$$
\begin{align*}
\phi_1^{n+1} &= \phi_2^n - c \ w_2^n \\
\phi_1^{n+1} &= \phi_2^n - \frac{c}{2} \ w_2^n
\end{align*}
$$

(3.6a) and (3.6b)

Since $c = \Delta z / \Delta t$, (3.5) and (3.6) are logical finite-difference approximations. It is easy to verify that (3.5-6) conserve an approximation to the mass in the form

$$
\frac{1}{2} \phi_1 + \phi_2 + \phi_3 + \cdots
$$

(3.7)

and an approximation to the energy in the form (cf. 2.13)

$$
0 + \frac{w_2^2}{2} + \frac{w_3^2}{2} + \cdots + \frac{\phi_1^2}{4c^2} + \frac{\phi_2^2}{2c^2} + \frac{\phi_3^2}{2c^2} + \cdots
$$

(3.8)

Note that the boundary points receive half the weight—represent half the volume—of the interior points.

Now let $S_z(\Delta t)$ correspond to the propagator for sound-waves split just described. That is, if $\psi(t)$ is any dependent variable, such as the value of $w$ at a particular grid point, then $\psi(t + \Delta t) = S_z(\Delta t) \psi(t)$ is its value after a time $\Delta t = \Delta z / c$, according to (3.5) and (3.6). (Note that $S_z$ includes the exact, small change in $\theta$ that arises from (3.1c) and the change in $\phi$.) Let $S_x(\Delta t)$ and $S_y(\Delta t)$ be the corresponding propagators for sound-wave propagation in the $x$- and $y$- direction, respectively. For the moment, we assume that the grid spacing is the same in all directions; an alternative will be considered in Section 7. Then the algorithm

$$
\psi(t + 2\Delta t) = S_x(\Delta t)S_y(\Delta t)S_z(\Delta t)\psi(t)
$$

(3.9)

corresponding to sound-wave propagation in the $x$-direction for a time $\Delta t$, followed by $y$-direction propagation for a time $\Delta t$, followed by $z$-direction propagation, etc., is a logical approximation to evolution under the full, three-dimensional sound-wave subdynamics.
(2.15) for a time $2 \Delta t$. The pyramidal arrangement (3.9) is called \textit{Strang splitting}, and is well-known to yield a result that is second-order accurate in time. (See, for example, Durran (1999), pp 130-132.) Even more accurate splitting methods are known (e.g. Suzuki (1992)), but these require time steps that are not multiples of one another, and cannot therefore take advantage of our Riemann-invariant method for exactly solving the directional splits. Since each directional split conserves energy, mass and momentum, the composite algorithm (3.9) also conserves these quantities.

Our complete numerical algorithm will be a generalization of (3.9) that contains splits representing the remaining sub-dynamics (2.14) and (2.16) (as well as splits containing the forcing and dissipation). However, the errors caused by Strang splitting of (2.14) and (2.16) are small compared to the errors in (3.9), because the rotation and buoyancy subdynamics evolve much more slowly than the sound waves. Thus it makes sense to analyze (3.9) by itself.

To see how (3.9) works in the presence of boundaries, consider the two-dimensional domain shown in Figure 1. The solid lines represent solid boundaries, and the dashed lines are \textit{propagation lines} along which either $S_x$ or $S_y$ is applied. The propagation lines lie wholly within the fluid; nowhere is a propagation line tangent to a boundary. The boundary conditions, analogous to (3.6), assume that each propagation line intersects the boundary in a locally normal direction. Thus the boundary is poorly resolved at sharp corners where two, perpendicular propagation lines intersect. In the inviscid case, segments of the boundary with many sharp corners generate grid-scale sound waves. However, if the Mach number is sufficiently small, then the energy converted to sound waves in this way is negligible. The resulting small-scale oscillations, chiefly visible in the pressure field, have little effect on the much slower, “hydrodynamic” motions of primary interest.

The introduction of viscosity and no-slip boundary conditions further reduces the generation of grid-scale sound waves at rough boundaries. In the viscous case, the boundary conditions described above are closely analogous to the ‘bounce back’ boundary conditions corresponding to no-slip boundary conditions in lattice-Boltzmann theory. Experiments show that the spuriously generated sound waves largely disappear if the viscous boundary-layer thickness is at least several grid-distances wide. Of course, one could also overcome boundary roughness by either rounding or shaving the corners in Figure 1, but this would couple the sound waves propagating in perpendicular directions, greatly complicating our simple scheme.

For a more quantitative analysis of the sound-wave splitting, we consider \textit{unbounded} flow in two spatial dimensions. Letting $u(x,y,t) = U(t) \exp[i(kx + ly - \omega t)]$, etc., we define the state vector $\Psi(t) = [U(t), V(t), \Phi(t)/c]^T$. Then the sound-wave propagators $S_x(t)$ and $S_y(t)$ correspond to the matrices

$$
S_x(t) = \begin{bmatrix}
\cos(ckt) & 0 & -i\sin(ckt) \\
0 & 1 & 0 \\
-i\sin(ckt) & 0 & \cos(ckt)
\end{bmatrix}, \quad S_y(t) = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(clt) & -i\sin(clt) \\
0 & -i\sin(clt) & \cos(clt)
\end{bmatrix}
$$

(3.10)

The approximation

$$
\Psi(t) = S_x(t/2)S_y(t)S_x(t/2) \Psi(0)
$$

(3.11)
is the two-dimensional analog of (3.9). On the other hand, the exact solution of (2.15) in two dimensions is $\Psi(t) = S(t) \Psi(0)$, where

$$S(t) = \frac{1}{K^2} \begin{bmatrix} k^2 \cos(cKt) + l^2 & kl(\cos(cKt) - 1) & -iK \sin(cKt) \\ kl(\cos(cKt) - 1) & k^2 + l^2 \cos(cKt) & -iK \sin(cKt) \\ -iK \sin(cKt) & -iK \sin(cKt) & K^2 \cos(cKt) \end{bmatrix}$$  \hspace{1cm} (3.12)

is the exact propagator. Here $K = \sqrt{k^2 + l^2}$. All of the matrices in (3.10) and (3.12) are unitary matrices, corresponding to the fact that the energy is conserved.

Now if either $k=0$ or $l=0$ then (3.11) is exact, because $S_x$ and $S_y$ are exact. To investigate the worst possible case we therefore set $k=l$, corresponding to a wave vector directed at a 45-degree angle to the coordinate directions. In this case

$$S_x(\Delta t) = \begin{bmatrix} \cos(\gamma) & 0 & -i \sin(\gamma) \\ 0 & 1 & 0 \\ -i \sin(\gamma) & 0 & \cos(\gamma) \end{bmatrix} \quad \text{and} \quad S_y(\Delta t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -i \sin(\gamma) \\ 0 & -i \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$  \hspace{1cm} (3.13)

where $\gamma = ck\Delta t = k \Delta t$ ranges between 0 and $\pi$. For this same wave, the exact propagator (3.12) is

$$S(2\Delta t) = \frac{1}{2} \begin{bmatrix} \cos(2\sqrt{2} \gamma) + 1 & \cos(2\sqrt{2} \gamma) - 1 & -i\sqrt{2}\sin(2\sqrt{2} \gamma) \\ \cos(2\sqrt{2} \gamma) - 1 & \cos(2\sqrt{2} \gamma) + 1 & -i\sqrt{2}\sin(2\sqrt{2} \gamma) \\ -i\sqrt{2}\sin(2\sqrt{2} \gamma) & -i\sqrt{2}\sin(2\sqrt{2} \gamma) & 2\cos(2\sqrt{2} \gamma) \end{bmatrix}$$  \hspace{1cm} (3.14)

Thus, in this worst-case situation, the approximation (3.11) replaces (3.14) by the product

$$\begin{bmatrix} \cos(\gamma) & 0 & -i \sin(\gamma) \\ 0 & 1 & 0 \\ -i \sin(\gamma) & 0 & \cos(\gamma) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(2\gamma) & -i \sin(2\gamma) \\ 0 & -i \sin(2\gamma) & \cos(2\gamma) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & 0 & -i \sin(\gamma) \\ 0 & 1 & 0 \\ -i \sin(\gamma) & 0 & \cos(\gamma) \end{bmatrix}$$  \hspace{1cm} (3.15)

For well-resolved sound waves ($\gamma \to 0$), (3.14) and (3.15) differ by terms of $O(\gamma^3)$. However, for the worst-resolved waves near $\gamma = \pi$, the difference between (3.14) and (3.15) is significant. The exact propagator (3.14) corresponds to the two-dimensional form of (2.15), namely

$$\frac{\partial u}{\partial t} = -\frac{\partial \Psi}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Psi}{\partial t} = -c^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$  \hspace{1cm} (3.16)

The eigenvalues of (3.14) are 1 and $e^{\pm i\sqrt{2} \gamma}$, corresponding, respectively, to steady nondivergent flow and to sound waves propagating at the speed $c$ in opposite directions. The approximate propagator (3.15) has one unit eigenvalue—corresponding to steady motion—at all values of the non-dimensional wavenumber $\gamma$. However, the corresponding static eigenvector differs very significantly from the exact static eigenvector

$$\Psi = [-1, +1, 0]^T$$  \hspace{1cm} (3.17)

near the highest resolved wavenumber.

The ratio of the sound wave speed implied by the approximation (3.15) to the exact value $c$ implied by (3.14) is shown by curve A in Figure 2. Although the sound speed of well-resolved waves with $\gamma \ll 1$ is close to $c$, poorly resolved sound waves with $\gamma > 1$ propagate at a speed that vanishes in the limit $\gamma = \pi$ of the smallest resolved
wavelength. Thus, all three eigenvalues of (3.15) approach unity as $\gamma \to \pi$, and the Strang-splitting approximation (3.15) misrepresents the smallest resolved sound waves as entirely static. Curve B in Figure 2 is the fraction of energy in steady, nondivergent flow represented by the exact static eigenvector (3.17) that is lost after two time steps of evolution by the approximate dynamics (3.15). At two poorly resolved wavenumbers, the energy in (3.17) is entirely drained in this short time.

Once again, this simple, linear analysis applies to the worst possible case of sound waves propagating at a 45-degree angle to the coordinate axes; waves propagating in the directions of the axes experience no errors at all. In the complete model, which must include viscosity, the most poorly resolved waves lie inside the dissipation range. Nevertheless, our analysis suggests that Strang-splitting of sound waves suffers from errors that spuriously couple the slow hydrodynamic motions of primary physical interest to unphysical small-scale motions with relatively large pressure fluctuations. The fully nonlinear numerical experiments described in Section 6 support this suggestion, but, the experiments show that the errors are small and easy to control. These errors, which result solely from the directional splitting, are the price to be paid for the stark simplicity of our method.

It is important to emphasize that one would never use our model to study sound waves any more than one would use the lattice-Botzmann method for that purpose. In both models the sound waves are fast modes that serve merely as a device for (approximately) enforcing the incompressibility condition while avoiding elliptic boundary-value problems. Although we incorporate the remaining physics by a splitting method similar to that used for the sound waves, the remaining physics comprises slow, hydrodynamic modes that are well resolved by the short time steps. As long as the sound speed is sufficiently large, there is no significant coupling between the sound waves and the slow hydrodynamic motions of primary physical interest. Of course the most convincing assessment of our method is a direct comparison to more conventional methods with proven accuracy. This we do in Section 6. But first we consider the additional splits corresponding to buoyancy and rotation sub-dynamics.

4. Buoyancy splitting

In the buoyancy sub-dynamics (2.16), $v$ and $\alpha$ evolve, but $\phi$ remains fixed. In contrast to (3.1), the directional splits of (2.16) cannot be solved exactly. Instead we seek an algorithm that semi-conserves energy, buoyancy (2.19) and its square (2.20). Since (2.20) involves $\phi$ as well as $\alpha$, we must take the results of the previous section into account. That is, the conservation of buoyancy-squared must be coordinated with the sound-wave split.

Again considering only the vertical direction, we first note that the previously derived sound-wave algorithm (3.5) corresponds to the spatial discretizations

$$\{F,H\} = \frac{c^2}{2\Delta z} \sum_i \left[ \left( \frac{\partial F}{\partial \phi_i} - \frac{\partial F}{\partial \phi_{i+1}} \right) \frac{\partial H}{\partial w_i} + \frac{\partial H}{\partial w_{i+1}} + \left( \frac{\partial H}{\partial \phi_i} - \frac{\partial H}{\partial \phi_{i+1}} \right) \frac{\partial F}{\partial w_i} + \frac{\partial F}{\partial w_{i+1}} \right]$$

and

$$H = \frac{1}{2} \sum_{i=1}^{N-1} \left( \frac{w_i^2}{2} + \frac{w_{i+1}^2}{2} \right) + \left( \frac{\phi_i^2}{2c^2} + \frac{\phi_{i+1}^2}{2c^2} \right) - \left( \alpha_i z_i + \alpha_{i+1} z_{i+1} \right)$$
of (2.12b) and (2.13), where \( i = 1 \) and \( i = N \) correspond to solid boundaries. Note that (4.2) gives boundary points half the weight of interior points. As previously stated, the bracket (4.1) applies only to unbounded flow. Thus (4.1) and (4.2) imply the semi-discrete form

\[
\begin{align*}
\frac{d w_i}{dt} &= -\frac{1}{2\Delta z}(\phi_{i+1} - \phi_{i-1}) \\
\frac{d\phi_i}{dt} &= -c^2 \frac{1}{2\Delta z}(w_{i+1} - w_{i-1}) \\
\frac{d(\phi_i \theta_i)}{dt} &= 0
\end{align*}
\]

(4.3a)

(4.3b)

(4.3c)

of the interior equations (3.5), but not the semi-discrete form

\[
\begin{align*}
\frac{d w_i}{dt} &= 0, \quad \frac{d\phi_i}{dt} = -c^2 \frac{w_2}{\Delta z}, \quad \frac{d(\phi_i \theta_i)}{dt} = 0
\end{align*}
\]

(4.4)

of the boundary conditions (3.6).

For the discretization of the buoyancy bracket (2.12c), we take

\[
\{F,H\}_3 = \frac{c^2}{4\Delta z} \sum_i \left( \alpha_{i+1} \phi_{i+1} + \alpha_i \phi_i \right) \left( \frac{\partial F}{\partial \alpha_{i+1}} - \frac{\partial F}{\partial \alpha_i} \right) \left( \frac{\partial H}{\partial w_i} + \frac{\partial H}{\partial w_{i+1}} \right) - \left( \frac{\partial H}{\partial \alpha_{i+1}} - \frac{\partial H}{\partial \alpha_i} \right) \left( \frac{\partial F}{\partial w_i} + \frac{\partial F}{\partial w_{i+1}} \right)
\]

(4.5)

Setting \( F = w_i, \phi_i, \alpha_i \) in (4.5), and making use of (4.2), we obtain the semi-discrete buoyancy sub-dynamics

\[
\begin{align*}
\frac{d w_i}{dt} &= \frac{1}{4}(\theta_{i+1} + 2\theta_i + \theta_{i-1}) \\
\frac{d\phi_i}{dt} &= 0 \\
\frac{d}{dt} \left( \frac{\phi_i \theta_i}{c^2 \theta_i} \right) &= \frac{1}{4\Delta z} \left[ (\theta_{i+1} + \theta_i)(w_{i+1} + w_i) - (\theta_i + \theta_{i+1})(w_i + w_{i+1}) \right]
\end{align*}
\]

(4.6a)

(4.6b)

(4.6c)

where we have used the definition \( \alpha_i = \phi_i \theta_i / c^2 \). Like (4.3), the sub-dynamics (4.6) apply only at interior gridpoints \( 1 < i < N \). However, it is straightforward to show that (4.6) and the boundary equations

\[
\begin{align*}
w_1 &= 0, \quad \frac{d\phi_1}{dt} = 0, \quad \frac{d}{dt} \left( \frac{\phi_1 \theta_1}{c^2 \theta_1} \right) = \frac{1}{2\Delta z} \left[ -2(\theta_i + \theta_{i+1})w_2 \right]
\end{align*}
\]

(4.7)

semi-conservate the energy (4.2) and the buoyancy

\[
\frac{1}{2} \phi_i \theta_i + \phi_i \theta_i^2 + \cdots
\]

(4.8)

It is only slightly harder to show that the combined dynamics (4.3-4) and (4.6-7) together semi-conservate the buoyancy-squared

\[
\frac{1}{2} \phi_i \theta_i^2 + \phi_i \theta_i^2 + \cdots
\]

(4.9)

The discrete bracket (4.5) was obtained using the Nambu-bracket method developed by the author in two previous papers (Salmon, 2005, 2007). For present purposes it suffices merely to verify that (4.6) and (4.7) have the claimed conservation properties.

Now let \( T_z \) be the \( z \)-direction propagator corresponding to (4.6) and (4.7). (Strictly speaking, \( T_z \) remains undefined until we choose a particular time-stepping algorithm for the buoyancy split. In the experiments described in Sections 6 and 8 we use second-order Runge-Kutta.) Let \( T_x \) and \( T_y \) be the corresponding propagators in the \( x\)-
and y-directions. These differ from $T_z$ only in that no buoyancy force occurs in the horizontal analogs of (4.6a). We apply these three propagators along the same propagation lines as the three sound-wave propagators. The composition

$$\psi(t + 2\Delta t) = S_x S_y S_z T_x T_y T_z T_x T_y T_z S_x S_y S_z \psi(t)$$  \hspace{1cm} (4.10)

represents the combined sound-wave and buoyancy dynamics. In (4.10) each propagator acts for a time $\Delta t$, the time required for a sound wave to propagate a distance equal to the grid-spacing. Each sound-wave propagator exactly conserves energy, mass, momentum and buoyancy. Each buoyancy propagator semi-conserves these same quantities. The full algorithm (4.10), which corresponds to the perfect-fluid dynamics without rotation or inertia, semi-conserves these quantities as well as the buoyancy-squared. To simulate the full Boussinesq dynamics, it only remains to devise the splits corresponding to rotation sub-dynamics (2.14).

5. Rotational splitting

The sub-dynamics (2.14), in which only the velocity $\mathbf{v}$ evolves, conserves energy and momentum. (The mass, buoyancy, and buoyancy-squared are also trivially conserved.) However, the most efficient algorithm for (2.14) is one that abandons exact conservation of momentum. First suppose that $\omega \equiv c^2$ and that the relative vorticity is omitted from (2.12a). Then, taking $\omega \equiv c^2$, we find that (2.14a) reduce to

$$\begin{align*}
\frac{\partial u}{\partial t} &= f v, \\
\frac{\partial v}{\partial t} &= -f u
\end{align*}$$  \hspace{1cm} (5.1)

which are exactly solvable as

$$\begin{align*}
u(t) &= v(0) \cos(ft) + v(0) \sin(ft) \\
u(t) &= v(0) \cos(ft) - u(0) \sin(ft)
\end{align*}$$  \hspace{1cm} (5.2)

In this case, the full dynamics resembles ‘planetary geostrophic dynamics’ except that it retains the local time derivative $\partial v / \partial t$. The solution (5.2) conserves energy because it corresponds to a length-preserving rotation of the velocity vector.

We solve the general case (2.14a) by three splits similar to (5.2). We define the vertical rotation propagator $R_z$ by

$$\begin{align*}
u(t) &= v(0) C_z + v(0) S_z \\
u(t) &= v(0) C_z - u(0) S_z
\end{align*}$$  \hspace{1cm} (5.3)

where $C_z$ and $S_z$ are the approximations

$$\begin{align*}
C_z &= 1 - \frac{1}{4} \gamma^2, \\
S_z &= \frac{\gamma}{1 + \frac{1}{4} \gamma^2},
\end{align*}$$  \hspace{1cm} \gamma = t \omega_z(0) c^2 / \phi$$  \hspace{1cm} (5.4)

to $\cos \left( t \omega_z(0) c^2 / \phi \right)$ and $\sin \left( t \omega_z(0) c^2 / \phi \right)$, respectively, and $\omega_z = f + \partial v / \partial x - \partial u / \partial y$ is the vertical component of the vorticity. The rational expressions (5.4) are faster to compute than the corresponding trigonometric functions. The solution (5.3) is an approximation because $\omega_z$ evolves with the flow. However, (5.3) conserves energy exactly because $C_z^2 + S_z^2 = 1$. The finite-difference approximation to $\omega_z(0)$ is arbitrary; in practice we use simple centered differences.
We solve the complete sub-dynamics (2.14) by a Strang composition of $R_z$ and
the propagators $R_x$ and $R_y$ corresponding to $\omega_x$ and $\omega_y$ respectively. Each rotation split
corresponds to a rotation of the velocity vector about a single component of the vorticity. Exact momentum conservation is lost because it involves a cancellation between the
terms in $R_x$, $R_y$, and $R_z$.

6. Two-dimensional experiments

In this section we compare solutions of the splitting algorithm described in
previous sections to solutions computed by more conventional methods. Our algorithm is
\[ S_x S_y R_z D_{xy} R_z S_y S_x \] (6.1)
where $S_x$ and $S_y$ are the sound-wave splits described in section 3, $R_z$ is the vertical
rotation split described in Section 5, and $D_{xy}$ is a split corresponding to the action of a
Navier-Stokes-type eddy-viscosity with viscosity coefficient $\nu$. All the splits except $D_{xy}$
act for the time $\Delta x/c$ required for a sound wave to propagate a horizontal grid distance
$\Delta x$ in either direction. $D_{xy}$, at the apex of the Strang-splitting pyramid, acts for time
$2\Delta x/c$. The complete sequence (6.1) corresponds to an evolution time of $2\Delta x/c$. For
$D_{xy}$ we use
\[ u_{ij}(t + 2\Delta x/c) = \frac{u_{ij} + \alpha(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})}{(1 + 4\alpha)} \] (6.2)
and similarly for $v_{ij}$, where subscripts denote grid locations, $\alpha = 2\nu/(c \Delta x)$, and all the
variables on the right hand side of (6.2) are evaluated at time $t$. In the limit $c \to \infty$, the
dynamics (6.1) corresponds to the two-dimensional Navier-Stokes equations, which are in
turn equivalent to the vorticity equation,
\[ \frac{\partial \zeta}{\partial t} + \frac{\partial(\psi \zeta + f)}{\partial(x,y)} = \nu \nabla^2 \zeta \] (6.3)
with
\[ \nabla^2 \psi = \zeta \] (6.4)
For comparison, we solve (6.3-4) using Arakawa’s (1966) energy- and enstrophy-
conserving Jacobian and a multi-grid elliptic solver.

First we consider non-rotating, infinitely periodic flow in a $2\pi \times 2\pi$ domain with
$512 \times 512$ grid points. The sound speed $c=1$. The initial condition is random flow with
rms velocity (i.e. Mach number) 0.01 and the enstrophy spectrum shown in Fig. 3a. Thus
the time required for fluid particles to traverse the domain at the rms velocity is about 600
time units. Figure 3a shows the enstrophy spectra in the splitting solution and in the
Arakawa solution at time $t=60$ for the case of vanishing viscosity. Such inviscid
solutions become increasingly unrealistic as time increases and as enstrophy piles up near
the highest resolved wavenumber. However, the intermediate-time comparison in Figure
3a is useful in that it confirms that the splitting algorithm spuriously accumulates
enstrophy at wavenumbers somewhat below the cutoff wavenumber, as suggested by the
discussion in Section 3.

Figure 3b compares the Arakawa and splitting algorithms at time $t=100$ for the
case of a non-vanishing viscosity $\nu = 10^{-5}$ sufficient to fully resolve the dissipation range.
The Reynolds number based on the rms velocity and the domain size is about 5000. Viscosity destroys the spurious hump in Figure 3a, but the splitting solution still contains significantly more energy in the viscous dissipation range. The Arakawa and splitting solutions can be brought into closer agreement by minor adjustments to the algorithms—most simply by using a slightly larger viscosity in the splitting algorithm—but here we show only direct comparisons.

Figure 4 compares the vorticity in these same two algorithms for the case of initially overlapping Gaussian vortices of the same sign. Again we take $\nu = 10^{-5}$. At time $t=100$ (Figures 4a and 4b) the two vorticity fields are indistinguishable. At time $t=300$ (Figures 4c and 4d)—the time required for fluid particles to traverse half the periodic domain—small differences within the vortex cores become evident, but the two solutions are still remarkably similar. For example, the energies, enstrophies and maximum velocities differ by less than 1%.

All these solutions are for the case $f=0$ of non-rotating flow. If $f$ is a non-vanishing constant, it has no effect on the exactly incompressible case governed by (6.3-4). However, the splitting algorithm (6.1) corresponds to slightly compressible, two-dimensional flow governed by (2.8). By the analogy between rotating shallow-water dynamics and rotating two-dimensional compressible flow, we expect constant $f \neq 0$ to produce spurious effects whenever the acoustic deformation radius $c/f$ is less than the domain size. Repeating the experiments shown in figure 4a-d for the case $c/f = 2\pi$ of acoustic deformation radius equal to the domain size, we find this difference to be negligible. However, when $c/f$ is much less than the domain size, the Arakawa and splitting solutions differ significantly. For example, Figure 4e shows the vorticity at $t=100$ for a splitting-algorithm solution in which $c/f = 2\pi/10$. The contrast between Figure 4e and Figures 4a-b is striking. In the solution corresponding to Figure 4e, the fluid density departure $\phi'$ from the mean density $c^2$ is only about 2%, but the ratio of the ‘available internal energy’ $\frac{1}{2} \iint d\mathbf{x} (\phi')^2 / c^2$ to the kinetic energy is about 0.5. In all the previously described solutions this ratio was less than 0.01. Thus the requirement $c/f > L$, where $L$ is the domain size, is a stringent requirement of our method.

How does this requirement compare to our other requirement, that the Mach number be small? In basin-scale ocean modeling, the largest fluid velocities—the largest Mach numbers—occur in western boundary layers. The western-boundary-layer velocity typically scales as $U_L/l$ where $U_L$ is the scale for the interior fluid velocity and $l = \sqrt{U_L/\beta}$ is the inertial western-boundary-layer thickness. Assuming that $f$ is of size $L\beta$, the requirement that the Mach number be small in the western boundary layer becomes $c/f > l$; the acoustic deformation radius must exceed the western boundary-layer thickness. Thus the previous requirement $c/f > L$ is much more severe. For, say, $L = 4000$ km the sound speed must exceed 25,000 km per day. Although this is 5 times smaller than the actual sound speed, it forces the time step to be very small. If, for example, the grid-spacing $\Delta x = 4$ km, then $\Delta t = \Delta x/c$ is only about 15 sec. However, these short time steps are really analogous to the individual cycles of the elliptic solver in the more conventional approach. Compared to it, our splitting method offers the advantages of extreme simplicity and massively parallel form.
In the non-rotating experiments described above, the splitting and Arakawa methods seem about equally efficient. In these experiments the maximum velocity (maximum Mach number) never exceeds 0.03. Other experiments show that strong coupling between the hydrodynamic modes and the sound waves does not occur until the Mach number reaches 0.1–0.2. In the non-rotating experiments described above, the splitting-algorithm time step $\Delta x/c = 2\pi/512 = 0.012$ is about 5 times smaller than the time step $\Delta t = 0.05 \Delta x/u_{rms}$ used for the Arakawa algorithm. However, the later requires about 10% more CPU time per unit of simulated time. Most of the time required by the Arakawa algorithm goes to the solution of (6.4). In domains with a complicated, irregular shape, the only practical methods for solving elliptic equations like (6.4) are multi-grid methods. However, multi-grid algorithms are difficult to code in domains with an irregular shape. In contrast, the splitting algorithm is no more difficult to implement in complicated geometry than in simple geometry. One need only store the beginning and ending locations of each propagation line.

Figure 5 depicts a splitting-algorithm solution in a two-dimensional domain with an arbitrary shape. Again we take $c=1$ and $\nu = 10^{-5}$. The domain width is order unity, and there are $310^2$ interior gridpoints. The initial conditions (Figure 5a) correspond to counter-rotating vortices of opposite sign with maximum velocity 0.10. These vortices propel each other toward the irregular boundary, generating large values of vorticity in thin viscous boundary layers that enforce the no-slip condition. By time $t=10$ (Figure 5b) these viscous boundary layers dominate the vorticity field. To keep better track of the interior vorticity, we follow the windowed vorticity (Figure 5c), obtained by multiplying the window contoured in Figure 5d by the vorticity in Figure 5a. The window is a function that varies smoothly between zero on the solid boundary and unity in the interior of the domain. Figure 5e shows the windowed vorticity at $t=30$, by which time the maximum fluid velocity has fallen to 0.044. For comparison, Figure 5f shows a passive scalar field $\theta$ that is initially set equal to the vorticity. Although the scalar has a diffusivity equal to the fluid viscosity, Figures 5e and 5f differ significantly because of the huge difference in boundary conditions: no flux for the scalar versus no-slip for the vorticity. Nevertheless, the similarity between Figures 5e and 5f in the fluid interior shows that the vorticity there is carried on fluid particles (apart from the effects of viscosity). We solve for $\theta$ with the splitting sequence

$$T_xT_yD_xD_yD_xD_yT_x$$

(6.5)

where $T_x$ and $T_y$ are the buoyancy splits defined in Section 4, and $D_x, D_y$ are diffusion splits, similar to (6.2) but acting only in the $x$- and $y$-directions, respectively, and incorporating the boundary conditions of no scalar flux through the boundary. (Because of the no-flux boundary conditions, it is simplest to split the diffusion into its directional components.) Since neither $T_x$ nor $T_y$ contains buoyancy force, $\theta$ is a passive scalar, and the various splits in (6.5), each of which acts for a time $\Delta x/c$, need not be sandwiched within (6.1). In practice, we alternate between (6.1) and (6.5). Finally, in this same flow field, we consider a passive scalar with the random initial conditions shown in Figure 5g. The scalar field at $t=30$ (Figure 5h) shows how (6.5) accommodates the no-flux boundary conditions.
It remains to consider solutions containing buoyancy force. This we do in Section 8. But first we address a fundamental difficulty that affects ocean circulation models of all kinds.

7. The aspect-ratio trick

The typical horizontal resolution in ocean general circulation models has increased enormously; the horizontal grid spacing $\Delta x$ is now sometimes as small as a few kilometers. However, this is still much larger than the typical vertical grid spacing $\Delta z$ of a few tens of meters. The huge disparity between these scales is the source of many difficulties. To cite but one example, open ocean convection occurs in plumes that are no wider than they are deep. The models cannot resolve these plumes. Moreover, since the plumes occur on the scale of fastest growth for convective instability, the model convection occurs at the smallest resolved horizontal scales—a recipe for numerical noise. In practice, eddy viscosity damps out the convective instability, and an explicit—but quite arbitrary—vertical mixing relieves static instability.

Browning et al (1990) proposed that the vertical component of the Boussinesq momentum equation (2.17a) be replaced by

$$\frac{1}{\mu^2} \frac{Dw}{Dt} = -\frac{\partial \phi}{\partial z} + \theta$$

(7.1)

in which $\mu = \Delta z / \Delta x \ll 1$ is a small parameter. The standard Boussinesq equations correspond to $\mu = 1$. The modification (7.1) increases the vertical inertia by a vast amount—but not so much as to upset hydrostatic balance at large scales of motion. As pointed out by Newberger & Allen (1996) and Salmon (1999b), the modification (7.1) is equivalent to erasing the factor of aspect-ratio-squared that precedes the vertical acceleration when the Boussinesq equations are written in standard non-dimensional form. This ‘aspect ratio trick,’ which has been independently rediscovered by several authors, was originally designed only to allow bigger time steps. However, as one of several bonuses, this trick increases the horizontal scale of interior convection and the thickness of sidewall boundary layers to the point where they are resolved by the model. The aspect-ratio trick is close in spirit to our treatment of sound waves: Just as we loosen the incompressibility constraint—decrease the sound speed—to allow larger time steps, the modification $\mu \ll 1$ loosens the hydrostatic constraint to the same purpose.

Newberger and Allen (1996) compare numerical solutions using (7.1) with very small values of $\mu$ to solutions of the standard Boussinesq equations with $\mu = 1$. They find significant differences. However, their test cases have a relatively high horizontal resolution; the typical domain width is 60 km. In such cases, the standard Boussinesq equations can be accurately solved. In basin-scale calculations the choice is between severely under-resolving the exact physics on the one hand, and accurately solving the modified physics on the other. In this paper, we choose the latter. However, as in Salmon (1999b), our basic motivation is a desire to make sound waves to propagate one vertical grid-spacing in the same time required to propagate one horizontal grid-spacing. Thus if $c$ is the horizontal sound speed, we take the vertical sound speed to be $\mu c$.

The modification (7.1) is equivalent to the following changes to the algorithm described in previous sections. First, we replace the vertical part of the sound-wave bracket (2.12b) by
\[
\{F,H\} = \mu^2 c^2 \iiint d\mathbf{x} \left( \frac{\partial H}{\partial \omega} \frac{\partial F}{\partial \varepsilon} - \frac{\partial F}{\partial \omega} \frac{\partial H}{\partial \varepsilon} \right) \quad (7.2)
\]

Second, we replace the vertical part of the buoyancy bracket (2.12c) by

\[
\{F,H\} = \mu^2 c^2 \iiint d\mathbf{x} \frac{\alpha}{\phi} \left( \frac{\partial H}{\partial \omega} \frac{\partial F}{\partial \varepsilon} - \frac{\partial F}{\partial \omega} \frac{\partial H}{\partial \varepsilon} \right) \quad (7.3)
\]

Third, in the rotation bracket (2.12a) we replace the relative vorticity

\[
\omega = \nabla \times \mathbf{v} = (w_y - v_z, u_z - w_x, v_x - u_y)
\]

\[
\rightarrow (w_y - \mu^2 v_z, \mu^2 u_z - w_x, v_x - u_y) = (\omega^\ast, \omega^\ast, \omega^\ast)
\]

Finally, we replace the Hamiltonian (2.13) by

\[
H = \iiint d\mathbf{x} \left( \frac{1}{2} u^2 + \frac{1}{2} v^2 + \frac{1}{2} w^2 + \frac{1}{2} \frac{\phi^2}{c^2} - \alpha z \right)
\]

The effect of these changes is to replace the vertical sound-wave split by

\[
S_z: \quad \frac{\partial \omega}{\partial t} = -\mu^2 \frac{\partial \phi}{\partial z}, \quad \frac{\partial \phi}{\partial t} = -c^2 \frac{\partial \omega}{\partial z}, \quad \text{and} \quad \frac{\partial (\phi \theta)}{\partial t} = 0 \quad (7.6)
\]

the vertical buoyancy split by

\[
T_z: \quad \frac{\partial \omega}{\partial t} = \mu^2 \theta, \quad \frac{\partial \phi}{\partial t} = 0, \quad \text{and} \quad \frac{\partial (\phi \theta)}{\partial t} = -\frac{\partial}{\partial z} (w \theta) \quad (7.7)
\]

and the two horizontal rotation splits by

\[
R_x: \quad \frac{\partial u}{\partial t} = 0 \quad \frac{\partial v}{\partial t} = \frac{c^2}{\phi} \frac{\partial w}{\partial t} \quad \frac{\partial w}{\partial t} = -\frac{c^2}{\phi} \mu^2 \omega^\ast v
\]

\[
R_y: \quad \frac{\partial u}{\partial t} = -\frac{c^2}{\phi} \omega^\ast \frac{w}{\mu^2} \quad \frac{\partial v}{\partial t} = 0 \quad \frac{\partial w}{\partial t} = \frac{c^2}{\phi} \omega^\ast u
\]

where the modified vorticity components are defined in (7.4). The energy-conserving solutions of (7.8-9) analogous to (5.3) are

\[
v(t) = v(0) \cos(\gamma_x) + \frac{w(0)}{\mu} \sin(\gamma_x),
\]

\[
R_x: \quad \frac{w(t)}{\mu} = \frac{w(0)}{\mu} \cos(\gamma_x) - v(0) \sin(\gamma_x), \quad (7.10)
\]

\[
\gamma_x = \frac{\omega_x(0)}{\mu} t c^2 / \phi
\]

and

\[
u(t) = u(0) \cos(\gamma_y) - \frac{w(0)}{\mu} \sin(\gamma_y),
\]

\[
R_y: \quad \frac{w(t)}{\mu} = \frac{w(0)}{\mu} \cos(\gamma_y) + u(0) \sin(\gamma_y), \quad (7.11)
\]

\[
\gamma_y = \frac{\omega_y(0)}{\mu} t c^2 / \phi
\]
The vertical rotation split $R_z$ is unaffected. We approximate the rotation splits (7.10-11) in the same manner as (5.4). If $\mu < 1$, the split (7.10) rotates the vector $(v, w)$ in an ellipse of aspect ratio $\mu$. In the limit $c \to \infty$, the complete modified dynamics,

$$
\frac{\partial u}{\partial t} = c^2 \left[ -\mathbf{v} \cdot \nabla u + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2 + v^2 + w^2 / \mu^2 \right) \right] - \frac{\partial \phi}{\partial x} \tag{7.12a}
$$

$$
\frac{\partial v}{\partial t} = c^2 \left[ -\mathbf{v} \cdot \nabla v + \frac{1}{2} \frac{\partial}{\partial y} \left( u^2 + v^2 + w^2 / \mu^2 \right) \right] - \frac{\partial \phi}{\partial y} \tag{7.12b}
$$

$$
\frac{\partial w}{\partial t} = c^2 \left[ -\mathbf{v} \cdot \nabla w + \frac{\mu^2}{2} \frac{\partial}{\partial z} \left( u^2 + v^2 + w^2 / \mu^2 \right) \right] - \mu^2 \frac{\partial \phi}{\partial z} + \mu^2 \theta \tag{7.12c}
$$

$$
\frac{\partial \phi}{\partial \tau} = -c^2 \nabla \cdot \mathbf{v} \tag{7.12d}
$$

$$
\frac{\partial}{\partial t} \left( \frac{\phi}{c^2} \theta \right) = -\nabla \cdot (\mathbf{v} \theta) \tag{7.12e}
$$

limits on the standard Boussinesq equations (2.17) except that the vertical momentum equation is replaced by (7.1). In particular, a scale analysis shows that (7.12e) approximates (2.17c) when the scale depth $c/N$ is much greater than the fluid depth $H$, where $N$ is the scale for the Vaisala frequency. Thus, if the horizontal domain size $L$ is much greater than the internal deformation radius $NH/\mu$, the most stringent requirement on sound speed is $c/f \gg L$.

8. Ocean section experiments

To test the buoyancy split and the modifications proposed in the previous section, we consider rotating, $y$-independent flow in the $xz$-plane. As $c \to \infty$ the dynamics (7.12) (with viscous and diffusive terms added) limits on

$$
\left( \frac{\partial}{\partial \tau} - A_h \left( \partial_{xx} + \partial_{yy} \right) - A_v \partial_{zz} \right) u = -\frac{\partial \psi}{\partial \left( x, z \right)} + f v - \frac{\partial \phi}{\partial x} \tag{8.1a}
$$

$$
\left( \frac{\partial}{\partial \tau} - A_h \left( \partial_{xx} + \partial_{yy} \right) - A_v \partial_{zz} \right) v = -\frac{\partial \psi}{\partial \left( x, z \right)} - f u \tag{8.1b}
$$

$$
\left( \frac{\partial}{\partial \tau} - A_h \left( \partial_{xx} + \partial_{yy} \right) - A_v \partial_{zz} \right) w = -\frac{\partial \psi}{\partial \left( x, z \right)} + \mu^2 \left( -\frac{\partial \phi}{\partial z} + \theta \right) \tag{8.1c}
$$

$$
\left( \frac{\partial}{\partial \tau} - K_h \left( \partial_{xx} + \partial_{yy} \right) - K_v \partial_{zz} \right) \theta = -\frac{\partial \psi}{\partial \left( x, z \right)} \tag{8.1d}
$$

where $\psi$, defined by $u = -\psi_x$ and $w = \psi_z$, is the streamfunction for the flow in the $xz$-plane; and $A_h, A_v, K_h, K_v$ are horizontal and vertical eddy viscosity and diffusion coefficients. We solve (8.1) by the algorithm

$$
D_{xz} R_x R_y R_z S_x S_y S_z T_x T_y T_z S_x S_y S_z T_x T_y T_z R_x R_y R_z D_{xz} \tag{8.2}
$$

where $D_{xz}$ represents both the viscosity and diffusion splits. All the splits in (8.2) act for the time $\Delta x/c = \Delta z/\mu c$. The viscosity split incorporates the prescribed wind stress and the no-slip boundary conditions at solid boundaries. The diffusion split incorporates surface heat flux and the no-flux boundary conditions at solid boundaries. All the other
splits retain the conservation properties described in previous sections. The computational domain (e.g. Figure 6) is the east-west section with rugged bathymetry and maximum depth $H=1.08$ km between the southern tip of San Clemente Island and the site of our computations at La Jolla, California, a distance $L=101.4$ km to the east. Besides the specified geometry and the forcing/initial conditions, the only parameters of the model are the horizontal grid spacing $\Delta x$, the vertical grid spacing $\Delta z$, the sound speed $c$, and the eddy coefficients $A_h, A_v, K_h, K_v$.

Idealized, flat-bottom, homogeneous-fluid solutions were found to agree closely with linear boundary-layer theory provided that the acoustic deformation radius $c/\nu$ was sufficiently large and boundary layers were well-resolved. In particular, the interior northward velocity agrees with the theoretical prediction $v_c = \sqrt{2} \tau^\nu / \sqrt{\nu A_y}$ to within 0.01\% when $c/\nu=10L$; when the sidewall (i.e. Stewartson) boundary-layer thickness $\delta_{St} \equiv (A_h L / \nu)^{1/3}$ is at least $4\Delta x$; and when the Ekman boundary-layer thickness $\delta_{Ek} \equiv (A_v / \nu)^{1/2}$ is at least $4\Delta z$. When $c/\nu=2L$, this error increased to 5.3\%. When $\delta_{Ek} = 2\Delta z$, this error increased to 3.0\%. In all the calculations to be presented, we take $c/\nu=10L$. Since our coarsest-resolution experiments correspond to $\Delta x = 510$ m and $\Delta z = 5.43$ m, we choose $A_h = 6.69$ m\,s$^{-1}$ to make $\delta_{St} = 4 \times 510$ m, and we choose $A_v = 3.715 \times 10^{-2}$ m\,s$^{-1}$ to make $\delta_{Ek} = 4 \times 5.43$ m. For the diffusion coefficients we take $K_h = A_h$ and $K_v = 5.0 \times 10^4$ m\,s$^{-1}$. We use these same values in all the experiments discussed.

All of these eddy coefficients are somewhat larger than the values typically used for the interior regions of ocean circulation models. Here, as is normally done there, it would be possible to allow these coefficients to vary between relatively large values in the boundary layers and much smaller values in the ocean interior. However, in this initial study, it was thought best to keep the number of parameters to an absolute minimum by allowing only constant values for the eddy coefficients.

We discuss only a few of many dozens of experiments performed. Because nothing varies in the $y$-direction, our solutions cannot be realistic; they serve only to demonstrate model behavior. Figure 6 and 7 show the streamfunction $\psi$, the northward flow $v$, and the buoyancy $\theta$ at time $t=50$ days in solutions that begin from a state of rest and uniform stratification $d\theta/dz$. The initially uniform stratification corresponds to a change of 1 unit of $\sigma_\theta$ (hereafter one sigma unit) from deepest ocean bottom to the ocean surface. In both solutions $\Delta x = 510$ m and $\Delta z = 5.43$ m, corresponding to a maximum of 200 grid points in each direction. In the solution of Figure 6, the buoyancy force has been turned off, thus $\theta$ is a passive scalar. In the solution of Figure 7, the buoyancy force is on. Both solutions are driven by a uniform southward wind stress $\tau^\nu = -1$ cm\,s$^{-2}$. In both solutions, the southward wind stress drives a strong westward flow in the surface Ekman layer. In the homogeneous-fluid solution of Figure 6, downwelling near San Clemente Island feeds a bottom Ekman layer in which the flow returns to the mainland. In the solution of Figure 7, the stratification resists downwelling, and a stronger flow closes at intermediate depth. A density inversion caused by the strong westward flow in the upper Ekman layer produces a weak convective cell in Figure 7c. The oscillations in Figures 6a and 7a are sound waves generated by the flow impinging on San Clemente Island. They disappear as the resolution or the horizontal viscosity is increased.
Figures 8 and 9 show solutions designed to test the performance of the model when buoyancy forces are strong. The initial condition is a statically unstable state of rest with $\phi \equiv c^2$ and

$$\theta = -\theta_0 \exp(-z/100 \, \text{m})$$

(8.3)

where the amplitude $\theta_0$ corresponds to 0.10 sigma units. In this case the model rapidly adjusts to a state of hydrostatic balance by generating sound waves. It is possible to avoid this brief adjustment phase by choosing the initial pressure to be in hydrostatic balance with the initial buoyancy, but it seems simpler to let the sound waves do their work. In any case, the sound waves are quickly overwhelmed by the much larger, convective motions of primary interest.

The solution of Figure 8 has the same spatial resolution as the experiments shown in Figures 6 and 7, corresponding to a maximum of 200 grid points in both directions. In the solution of Figure 9, the horizontal grid spacing has been reduced from $\Delta x = 510$ m to $\Delta x = 204$ m, corresponding to a maximum of 500 grid points in the horizontal direction. Because of the aspect-ratio trick, convective instability is well resolved in both Figures 8 and 9. However, there is a real, physical difference between these two solutions, because they correspond to different values of $\mu$ (0.0106 and 0.0267, respectively). Once again, exact Boussinesq dynamics corresponds to $\mu = 1$, and is achieved only by allowing $\Delta x = \Delta z$. The solution of Figure 9 is taking advantage of the increased horizontal resolution over Figure 8, but, because of the aspect-ratio trick, its convection cells remain well resolved. Because of the increased resolution, the convection is stronger and proceeds more rapidly in Figure 9 than in Figure 8; the times in the two figures have been chosen to represent the flows at similar stages of development. At the time corresponding to Figure 8d, the vertical rms velocity is 88.9 m day$^{-1}$; in Figure 9d it is 224 m day$^{-1}$. In both figures, convection occurs fastest near the two coastlines. Our method easily accommodates the rapid descent of dense water along the rough, stair-step topography.

9. Discussion

The key features of our model are its extreme simplicity and its massively parallel form. Each of the propagators in (8.1) represents a simple operation. For example, each of the sound-wave propagators $S_x, S_y, S_z$, which merely shift Riemann invariants left and right, corresponds to a computer subroutine with only 26 lines of fortran. The other propagators are comparably simple. Both the sound wave propagators and the buoyancy propagators $T_x, T_y, T_z$ operate along propagation lines that require no elaborate treatment of the boundaries. At the beginning of the calculation, one must use the bathymetric data to determine the beginning and ending location of each propagation line, but, once these locations are stored, very complicated boundary shapes require no more operations than very simple ones. The rotation propagators $R_x, R_y, R_z$ operate on interior fluid points independently. Thus every propagation line in the case of the sound and buoyancy propagators, and every interior point in the case of the rotation propagators, could be sent to a different processor. The forcing and dissipation propagators could easily be handled in the same way.
The tremendous advantage of massively parallel form should easily compensate for the primary disadvantage of our model: the very short time cycle required to satisfy the condition that the acoustic deformation radius $c/f$ be larger than the domain size. For a domain size of, say, 4000 km and a horizontal grid spacing of, say, $\Delta x = 5$ km, the sound speed must exceed 25,000 km per day, and the time cycle $2 \Delta x/c$ must be less than half a minute. While this sound speed is still about 5 times smaller than the actual sound speed, such time steps are indeed very short. On the other hand, these short time steps should be compared not to the time steps in conventional models, but to the iteration steps required to solve elliptic equations at fixed times. The iteration steps are not so easy to program in parallel form.

Although our complete model is somewhat novel, none of its key ingredients is completely new. For example, operator splitting (sometimes called fractional steps) has been tried by many authors; see, for example, Skamarock (2006). However, our splitting method violates a frequently cited rule of conventional operator splitting that makes our method much more akin to the lattice Boltzmann method. In conventional splitting, one normally avoids separating fundamental physical balances into different splits. In our model, geostrophic balance is spread between $S_x$ and $S_y$, which contain the horizontal pressure gradient, and $R_z$, which contains the Coriolis force. Similarly, hydrostatic balance is spread between $S_z$, which contains the Coriolis force, and $T_z$, which contains the buoyancy force. If one analyzes the sub-sequence $S_zT_zT_zS_z$ of (8.2) in the manner of Section 3, one finds an exact static eigenvector corresponding to hydrostatic balance. However, the existence of such a static state is somewhat remarkable when one considers that $S_zT_zT_zS_z$ achieves hydrostatic balance by alternately accelerating fluid particles in $T_z$ and then decelerating them in $S_z$. In fact, the individual components $S_zT_z$ and $T_zS_z$ each have static eigenvectors corresponding to physical states with persistent vertical velocities of opposite sign. Such is the magic of Strang splitting!

Although the analytical dynamics (2.11-13) conserve all the invariants (2.18), the finite-difference equations conserve or semi-conserve only that subset of invariants noted in the preceding sections. (Since, as explained in Section 7, the aspect-ratio trick is equivalent to a formal rescaling of the vertical coordinate, it alters but does not destroy conservation laws.) In particular, the finite-difference equations do not conserve potential vorticity. However, the retained conservation laws—mass, energy, buoyancy, and buoyancy-squared—lend our method great stability. When breakdown occurs, it is always because the local Mach number becomes too high, and the breakdown consists of a spurious local generation of sound waves. In fact, relatively vigorous sound waves are always present, but if the sound speed is sufficiently large—that is, if the time step is sufficiently small—then the sound waves are negligible in comparison to the slow hydrodynamic motions of fundamental interest.

All ocean circulation models face fundamental difficulties associated with the scale separation between the horizontal and vertical directions. Conventional approaches to these problems include sub-grid-scale closures that explicitly mix buoyancy to neutral stability, and greatly enhanced vertical mixing coefficients in regions of static instability. In this paper we choose the aspect-ratio trick because it allows us to apply each of our propagators exactly twice in a time cycle. However, this is an arbitrary choice, and our algorithm is flexible enough to entertain alternatives. Suppose, for example, that $\Delta z$
were 50 times smaller than $\Delta \chi$. One could keep the sound speed the same in all directions but then apply the subsequence $S_zT_zS_z$ 50 times as often as the other propagators. Alternatively, one could replace $S_zT_zS_z$ by a completely different physical operation that relaxes the system toward hydrostatic balance in each vertical column. Likewise, the need for very short time steps could likely be overcome by a modification to our method that singles out the barotropic mode for special treatment. In fact, the general approach of this paper leaves as much room for strategy and experimentation as do the more traditional methods. However, in this initial study, it was thought best to test the method in its simplest and purest form.

Although this paper reports only two-dimensional solutions, a three-dimensional version of the model has been tested at the relatively low resolution of $100^3$. Much more testing is required. However, it appears that the generalization from two to three space dimensions requires no new ideas or methods beyond those presented here.

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**REFERENCES**


Figure 1. Propagation lines for sound waves in a bounded domain. The waves propagate along the dashed lines through the interior of the fluid. Both components of the fluid velocity vanish at the boundary points.
Figure 2. Two measures of the error caused by Strang splitting of the propagators $S_x$ and $S_y$ in the “worst case” situation of a wave vector pointing at a 45-degree angle to the grid. (A) The ratio of the sound speed to its exact value $c$ as a function of the nondimensional wavenumber $\gamma$. The largest value $\gamma = \pi$ corresponds to the most poorly resolved wave. (B) The fraction of the energy in steady, nondivergent flow that is spuriously converted to sound waves in two time steps. No errors occur when the wave vector points in the $x$- or $y$-direction.
Figure 3. (a) The enstrophy spectrum corresponding to random initial conditions, and the spectra corresponding to inviscid solutions of the Arakawa and splitting algorithms at a time equal to the time required for fluid particles to traverse one tenth the periodic domain size. (b) The enstrophy spectra in viscous solutions of the Arakawa and splitting algorithms at a time equal to the time required for fluid particles to traverse one sixth the periodic domain size. The Reynolds number is 5000.
Figure 4. The merging of two initially overlapping Gaussian vortices as computed by the Arakawa algorithm (a, c) and by the splitting algorithm (b, d) at the two times $t=100, 300$. (e) The splitting algorithm solution at $t=100$ with a constant Coriolis parameter chosen to make the acoustic deformation radius equal to one tenth the periodic domain size. Darker contours correspond to larger values.
Figure 5. Splitting-algorithm solution of counter-rotating vortices in an arbitrarily shaped domain with no-slip boundaries. By time $t=10$, the initial vorticity (a) has evolved to the state (b) in which viscous boundary layers dominate. The windowed vorticity, obtained by multiplying the field in (b) by the window in (d), reveals the interior pattern (c). The windowed vorticity at $t=30$ (e) and a passive scalar (f) initially equal to the vorticity at the same time. An initially random passive scalar at (g) $t=0$ and (h) $t=30$ in the same flow. Darker contours correspond to large values.
Figure 6. The streamfunction (a), northward velocity (b), and “buoyancy” (c) in a east-west section between San Clemente Island on the left, and La Jolla, California on the right. In this experiment the buoyancy force is switched off, thus (c) depicts a passive scalar $\theta$ for which $d\theta/dz$ is initially a constant. The flow, which is driven by a uniform southward wind stress of magnitude $1 \text{ cm}^2 \text{ sec}^{-2}$, is westward in the surface Ekman layer and eastward in the bottom Ekman layer. The maximum $u$, $v$ and $w$ are respectively $3.47 \text{ km day}^{-1}$, $20.9 \text{ km day}^{-1}$, and $57.6 \text{ m day}^{-1}$, respectively. At the time shown (50 days after a state of rest) the velocity field is steady. Darker contours correspond to larger values.
Figure 7. The same as Figure 6 but with the buoyancy force turned on. The uniform initial stratification corresponds to a difference of one sigma unit between the ocean surface and deepest ocean bottom. The stratification concentrates the flow in the upper ocean, hence the maximum $u$, $v$ and $w$ (respectively $4.81 \text{ km day}^{-1}$, $59.8 \text{ km day}^{-1}$ and $131 \text{ m day}^{-1}$) are larger than in Figure 6. Static instability in the upper Ekman layer generates a small convective cell at mid-basin. Darker contours correspond to larger values.
Figure 8. The buoyancy at times (a) 14 days; (b) 18 days; (c) 20 days; and (d) the streamfunction at 20 days in an experiment beginning at a state of rest and the unstable straification (8.3). The grid spacings are $\Delta x = 510$ m and $\Delta z = 5.43$ m, corresponding to a maximum of 200 grid points in each direction. At the final time, the maximum eastward, northward, and vertical speeds are, respectively, 5.65 km day$^{-1}$, 21.4 km day$^{-1}$, and 268 m day$^{-1}$. 
Figure 9. The buoyancy at times (a) 5 days; (b) 6 days; (c) 8 days; and (d) the streamfunction at 8 days in an experiment beginning at the same, statically unstable, initial state as that shown in Figure 8. The only difference between this experiment and that shown in Figure 8 is that the horizontal grid spacing has been reduced to $\Delta x = 204$ m, corresponding to 500 grid points in the east-west direction. At the final time, the maximum eastward, northward, and vertical speeds are, respectively, 16.5 km day$^{-1}$, 30.9 km day$^{-1}$, and 837 m day$^{-1}$. 