Coupled systems of two-dimensional turbulence

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Ordinary two-dimensional turbulence corresponds to a Hamiltonian dynamics that conserves energy and the vorticity on fluid particles. This paper considers *coupled systems* of two-dimensional turbulence with three distinct governing dynamics. One is a Hamiltonian dynamics that conserves the vorticity on fluid particles and a quantity analogous to the energy that causes the system members to develop a strong correlation in velocity. The other two dynamics considered are non-Hamiltonian. One conserves the vorticity on particles but has no conservation law analogous to energy conservation; the other conserves energy and enstrophy but it does not conserve the vorticity on fluid particles. The coupled Hamiltonian system behaves like two-dimensional turbulence, even to the extent of forming isolated coherent vortices. The other two dynamics behave very differently, but the behaviors of all four dynamics are accurately predicted by the methods of equilibrium statistical mechanics.

1. Introduction

Freely decaying two-dimensional Navier-Stokes turbulence is governed by

$$\zeta_t = J(\zeta, \psi) + \nu \nabla^2 \zeta, \tag{1.1}$$

where $\psi(x, y, t)$ the streamfunction at Cartesian location (x, y) and time t, $\zeta = \nabla^2 \psi$ is the vorticity, ν is the viscosity coefficient, $\mathbf{v} = (u, v) = (-\psi_y, \psi_x)$ is the velocity, and $J(A, B) \equiv A_x B_y - B_x A_y$. Subscripts denote partial derivatives. Our boundary conditions are spatial periodicity, $\psi(x + 2\pi, y) = \psi(x, y + 2\pi) = \psi(x, y)$. The theory of two-dimensional turbulence enjoys an extensive literature. The pioneering papers are by Fjortoft (1953), Kraichnan (1967), and Batchelor (1969). For a recent review see Boffetta and Ecke (2012). Much of the theory is based upon fundamental conservation laws. When $\nu = 0$, the dynamics (1.1) conserves the energy

$$E = \frac{1}{2} \iint d\mathbf{x}\, \nabla\psi \cdot \nabla\psi, \tag{1.2}$$

and every quantity of the form

$$\iint d\mathbf{x}\, G(\zeta), \tag{1.3}$$

where the integration is over the periodic domain and $G(\cdot)$ is an arbitrary function. The invariants (1.3) include the enstrophy

$$Z = \frac{1}{2} \iint d\mathbf{x}\, \zeta^2 \tag{1.4}$$

as a special case.

In this paper we study coupled systems of two-dimensional turbulence with conservation laws analogous to, but substantially different from, those of (1.1). Our focus is on the degree to which system behavior depends on conservation laws and can be predicted by the methods of equilibrium statistical mechanics. As an example of a coupled system, consider

$$\begin{aligned}\zeta_{1t} &= J(\zeta_1, \psi_2) + \nu\nabla^2\zeta_1, \\ \zeta_{2t} &= J(\zeta_2, \psi_1) + \nu\nabla^2\zeta_2,\end{aligned} \tag{1.5}$$

where the numerical subscripts denote the two system members, and here and throughout this paper $\nabla^2\psi_i = \zeta_i$. If the initial conditions are such that $\psi_1(x, y, 0) = \psi_2(x, y, 0)$, then the two system members evolve as identical copies. What happens if $\psi_1(x, y, 0) \neq \psi_2(x, y, 0)$? We shall show that, if the initial conditions are to any degree positively correlated (in a sense explained below), then the two systems eventually become perfectly correlated (in the same sense) and evolve together as a single, nearly identical, system. Moreover, this somewhat surprising behavior is accurately predicted by the methods of equilibrium statistical mechanics.

Our motivation for studying coupled systems like (1.5) is to better understand the dynamics (1.1) of ordinary two-dimensional turbulence. Suppose, as in the numerical solutions to be described, that the initial conditions on ψ_1 and ψ_2 correspond to the same wavenumber spectrum but with a phase difference in Fourier coefficients that controls the correlation between the systems. Then the two systems evolve in a statistically identical manner. That is, the statistics of $\psi_1(x, y, t)$ match the statistics of $\psi_2(x, y, t)$ at all later times. This means that, according to (1.5), the vorticity ζ_1 is advected by a velocity field that is *statistically identical* to that associated with ψ_1 but lacks the precise connection implied by $\nabla^2\psi_1 = \zeta_1$. At the beginning of this study it was thought that similarities between the solutions of the coupled system (1.5) and ordinary two-dimensional turbulence (1.1), including the tendency to form isolated coherent vortices, could be controlled by the extent to which the initial conditions on (1.5) were correlated, as, for example, measured by the velocity correlation coefficient

$$C(\mathbf{v}_1, \mathbf{v}_2) = \frac{\langle \mathbf{v}_1 \cdot \mathbf{v}_2 \rangle}{\langle \mathbf{v}_1 \cdot \mathbf{v}_1 \rangle^{1/2} \langle \mathbf{v}_2 \cdot \mathbf{v}_2 \rangle^{1/2}}. \tag{1.6}$$

Here $\langle \cdot \rangle$ denotes a statistical average. This hope was encouraged by the fact that when $\nu = 0$ the dynamics (1.5) conserves $\iint d\mathbf{x}\, \mathbf{v}_1 \cdot \mathbf{v}_2$, suggesting that the velocity correlation set by the initial conditions might tend to persist. (Throughout this paper we identify statistical averages with averages over the periodic domain.) As we shall see, the actual behavior is considerably more subtle.

When $\nu = 0$, both (1.1) and (1.5) take the general Hamiltonian form

$$\frac{dF}{dt} = \{F, H\}, \tag{1.7}$$

where $F[\zeta]$ is an arbitrary functional of the vorticity ζ, $\{\cdot, \cdot\}$ is the Poisson bracket, and H is the Hamiltonian. The dynamics (1.1) corresponds to the Poisson bracket

$$\{F, H\} = \iint d\mathbf{x}\, \zeta\, J\left(\frac{\delta F}{\delta \zeta}, \frac{\delta H}{\delta \zeta}\right) \tag{1.8}$$

and to the Hamiltonian (1.2). The choice $F = \zeta$ yields (1.1) with $\nu = 0$ and $\psi = -\delta H/\delta\zeta$. The Hamiltonian (1.2) is conserved by the antisymmetry property of (1.8). The *Casimirs* (1.3) are conserved for any choice of Hamiltonian.

The coupled dynamics (1.5) corresponds to the Poisson bracket

$$\{F,H\} = \iint d\mathbf{x}\,\zeta_1\, J\left(\frac{\delta F}{\delta\zeta_1},\frac{\delta H}{\delta\zeta_1}\right) + \iint d\mathbf{x}\,\zeta_2\, J\left(\frac{\delta F}{\delta\zeta_2},\frac{\delta H}{\delta\zeta_2}\right) \tag{1.9}$$

and to the Hamiltonian

$$H = \iint d\mathbf{x}\, \nabla\psi_1\cdot\nabla\psi_2. \tag{1.10}$$

Thus (1.5) conserves the velocity correlation (1.10) and Casimirs of the form $\iint d\mathbf{x}\,(G_1(\zeta_1)+G_2(\zeta_2))$. In particular, the enstrophies of the two coupled systems are separately conserved. Thus when $\nu = 0$ the coupled system (1.5) conserves the numerator of (1.6) but *not* the denominator. In contrast, two *uncoupled* systems of ordinary two-dimensional turbulence, which correspond to the bracket (1.9) and to the Hamiltonian

$$\frac{1}{2}\iint d\mathbf{x}\,\left(\nabla\psi_1\cdot\nabla\psi_1+\nabla\psi_2\cdot\nabla\psi_2\right), \tag{1.11}$$

conserve the denominator of (1.6) but not the numerator. These differences prove critical to our subsequent analysis.

It is possible to define a perfectly solvable dynamics of, for example, the form (1.1) that is *not* Hamiltonian. That is, it is possible to imagine a dynamics of the form (1.1) in which the streamfunction ψ depends on ζ in a well-defined manner, but in which ψ is *not* the functional derivative of any functional. (This is analogous to the statement that it is possible to define vector fields that are not the gradient of any scalar.) Such a non-Hamiltonian system typically has no conservation law analogous to (1.2), but it could still preserve all of the invariants (1.3) associated with the bracket (1.8). A somewhat trivial example of such a system would be (1.1) with an arbitrarily prescribed $\psi(x, y, t)$. In this case ζ is simply a passive scalar. We consider more interesting examples of coupled non-Hamiltonian systems below.

2. Equilibrium statistical mechanics

Kraichnan (1967) applied the methods of equilibrium statistical mechanics to ordinary two-dimensional turbulence. (See also Kraichnan (1975), Kraichan and Montgomery (1980), Salmon (1998) and Bouchet and Venaille (2012).) The theory applies to a numerical model—that is, to a model of (1.1) with a finite number of gridpoints or modes—that conserves analogues of the energy (1.2) and enstrophy (1.4) when the viscosity $\nu = 0$. If the viscosity is switched off, then the motion in the phase space spanned by the modal or gridpoint values of ζ is nondivergent, and the probability distribution approaches the canonical ensemble based upon (1.2) and (1.4). (Other invariants of the form (1.3) typically do not survive the truncation in modes.) This equilibrium state, which represents the 'target state' towards which the nonlinear terms in (1.1) acting by themselves would drive the flow, corresponds to the energy spectrum

$$E(k) = \frac{k}{\alpha+\beta k^2}. \tag{2.1}$$

The 'inverse temperatures' α and β are determined by the requirements

$$\int_{k_0}^{k_{max}} E(k)\,dk = E_0, \qquad \int_{k_0}^{k_{max}} k^2E(k)\,dk = Z_0, \tag{2.2}$$

where k_0 corresponds to the periodic box size, k_{max} is the maximum wavenumber in the model, and E_0 and Z_0 are the prescribed constant values of the energy and enstrophy. Interest attaches to the limit $k_{max} \to \infty$ with k_0, E_0, Z_0 held fixed. In that limit, $\alpha/\beta \to -k_0^2$ and all of the energy 'condenses' into the lowest wavenumber k_0. The excess enstrophy $Z_0 - k_0^2 E_0$ is swept out to infinite wavenumber.

The inviscid equilibrium of the coupled system (1.5) is closely analogous to Kraichnan's result. First we define 'collective variables'

$$\psi = \frac{1}{2}(\psi_1 + \psi_2), \qquad \tau = \frac{1}{2}(\psi_1 - \psi_2), \tag{2.3}$$

for which the quadratic invariants take a diagonal form. In terms of ψ, τ, $\zeta_\psi \equiv \nabla^2\psi$, and $\zeta_\tau \equiv \nabla^2\tau$, the enstrophies take the form

$$\begin{aligned} Z_1 &= \iint d\mathbf{x}\, \zeta_1^2 = \iint d\mathbf{x}\left(\zeta_\psi^2 + 2\zeta_\psi\zeta_\tau + \zeta_\tau^2\right), \\ Z_2 &= \iint d\mathbf{x}\, \zeta_2^2 = \iint d\mathbf{x}\left(\zeta_\psi^2 - 2\zeta_\psi\zeta_\tau + \zeta_\tau^2\right), \end{aligned} \tag{2.4}$$

and the Hamiltonian (1.10) takes the form

$$H = \iint d\mathbf{x}\, \nabla\psi\cdot\nabla\psi - \iint d\mathbf{x}\, \nabla\tau\cdot\nabla\tau. \tag{2.5}$$

By the statistical equivalence of the systems $Z_1 = Z_2$, hence $\iint d\mathbf{x}\, \zeta_\psi\zeta_\tau = 0$, and we have the single enstrophy invariant

$$\bar{Z} = \iint d\mathbf{x}\left(\zeta_\psi^2 + \zeta_\tau^2\right). \tag{2.6}$$

Thus for quadratic invariants we need only consider (2.5) and (2.6), which are the analogues of the energy and enstrophy in the corresponding theory of ordinary two-dimensional turbulence. However, we note that (2.5) is proportional to the *difference* between the energy associated with the ψ-mode and the energy associated associated with the τ-mode. Again, neither of these energies is itself conserved.

The minus sign that appears in (2.5) is the only difference between the inviscid equilibrium statistical mechanics of (1.1) and that of (1.5). The canonical ensemble corresponding to (2.5) and (2.6) leads to the energy spectra

$$E_\psi(k) = \frac{k}{\alpha + \beta k^2}, \qquad E_\tau(k) = \frac{k}{-\alpha + \beta k^2}, \tag{2.7}$$

where α is the 'inverse temperature' that corresponds to H, and β corresponds to $\bar{Z}$. The sign difference in the denominators corresponds to the minus sign in (2.5). The constants α and β are determined by

$$H_0 = \int_{k_0}^{k_{max}} \left(E_\psi(k) - E_\tau(k)\right)\, dk \equiv E_\psi - E_\tau \tag{2.8}$$

and

$$\bar{Z}_0 = \int_{k_0}^{k_{max}} \left(k^2 E_\psi(k) + k^2 E_\tau(k)\right)\, dk. \tag{2.9}$$

First suppose that H_0, the prescribed value of (2.5), vanishes, corresponding to initially uncorrelated systems. Then $\alpha = 0$ and the enstrophy $\bar{Z}$ is equipartitioned among Fourier modes. This corresponds to an energy spectrum of the form $E(k) \equiv E_\psi(k) + E_\tau(k) = C_0 k^{-1}$ with the constant C_0 determined by (2.9) to be $C_0 = 2\bar{Z}_0/(k_{max}^2 - k_0^2)$. Let $k_{max} \to \infty$ with k_0 and $\bar{Z}_0$ held fixed. Then the total energy $E = E_\psi + E_\tau$ vanishes like $k_{max}^{-2} \ln k_{max}$. In summary, as more high wavenumbers become available to the system, the low wavenumbers hold a smaller and smaller fraction of the total enstrophy, and the total energy, which is not conserved, simply vanishes.

Now suppose that $H_0 > 0$, corresponding to systems with positive initial correlation. Again we let $k_{max} \to \infty$. The analysis closely parallels the corresponding analysis of (2.1) and (2.2). We find that (2.7)-(2.9) imply

$$\beta \sim \frac{k_{max}^2}{(\bar{Z}_0 - k_0^2 H_0)}, \qquad \frac{\alpha}{\beta} \sim -k_0^2 + 2k_0^2 e^{-2k_{max}^2 H_0/(\bar{Z}_0 - k_0^2 H_0)}. \tag{2.10}$$

Let k_1 be any wavenumber between k_0 and k_{max}. From (2.10) it follows that as $k_{max} \to \infty$,

$$E_\psi = \int_{k_0}^{k_1} E_\psi(k)dk \to H_0, \qquad E_\tau = \int_{k_0}^{k_1} E_\tau(k)dk \to 0. \tag{2.11}$$

For the correlation coefficient (1.6), we obtain

$$C(\mathbf{v}_1, \mathbf{v}_2) = \frac{E_\psi - E_\tau}{E_\psi + E_\tau} \to 1. \tag{2.12}$$

In summary, the coupled dynamics conserves $H = E_\psi - E_\tau$. If $H = 0$ the system can attain the maximum entropy state of enstrophy equipartition that requires the energy to vanish, $E_\psi = E_\tau = 0$. If $H > 0$, this zero-energy state is unattainable, but enstrophy equipartition proceeds farthest if all of H is concentrated in E_ψ and in the lowest wavenumber k_0. If $H < 0$, then the analysis proceeds as above, but with the symbols ψ and τ everywhere interchanged. One concludes that all of $E_\tau - E_\psi$ ends up in E_τ with $E_\psi = 0$ and $C(\mathbf{v}_1, \mathbf{v}_2) = -1$.

Inviscid numerical experiments confirm the predictions of equilibrium statistical mechanics: If the velocity fields are even slightly positively (negatively) correlated, then the coupled system approaches the state of perfectly correlated (anticorrelated) velocity fields, losing whatever energy is required. Viscous numerical experiments evolve similarly to the inviscid cases. However, viscous experiments with $H > 0$ resemble ordinary two-dimensional turbulence in the formation of isolated coherent structures, whereas in experiments with $H \leq 0$ the vorticity behaves like a passive scalar. We shall not present these results here; instead we proceed to more general cases of greater interest.

Consider a 'ring' of N coupled systems governed by the equations

$$\zeta_{\alpha t} = \frac{1}{2} J(\zeta_\alpha, \psi_{\alpha-1} + \psi_{\alpha+1}) + \nu \nabla^2 \zeta_\alpha, \qquad \alpha = 1, \ldots, N \tag{2.13}$$

with $\psi_0 = \psi_N$, $\psi_{N+1} = \psi_1$, and N an even integer. The system (1.5) corresponds to $N = 2$. When $\nu = 0$ (2.13) is a Hamiltonian system with Hamiltonian

$$H = \frac{1}{2} \sum_{\alpha=1}^{N} \iint d\mathbf{x}\; \nabla\psi_\alpha \cdot \nabla\psi_{\alpha+1} \tag{2.14}$$

and a bracket equal to the obvious generalization of (1.9). As in the case $N = 2$, the equilibrium statistical mechanics is most easily worked out using variables that diagonalize the invariants. Let

$$\psi_\alpha = \sum_{n=-N/2}^{N/2-1} \hat{\psi}_n \; e^{i2\pi n\alpha/N}. \tag{2.15}$$

Thus ψ_α and $\hat{\psi}_n$ form a discrete Fourier transform pair. The Hamiltonian (2.14) takes the form

$$H = \sum_n \hat{E}_n \cos(2\pi n/N), \tag{2.16}$$

where

$$\hat{E}_n = \iint d\mathbf{x}\, \nabla\hat{\psi}_n \cdot \nabla\hat{\psi}_{-n} \tag{2.17}$$

is the energy in Fourier mode n. The Hamiltonian (2.16) is conserved but the $\hat{E}_n$ are not. If the N systems are statistically identical then their N enstrophies are equal and we have the single enstrophy invariant

$$\bar{Z} = \sum_n \hat{Z}_n, \tag{2.18}$$

where

$$\hat{Z}_n = \iint d\mathbf{x}\, \nabla^2\hat{\psi}_n \cdot \nabla^2\hat{\psi}_{-n}. \tag{2.19}$$

Equlibrium statistical mechanics based upon (2.16) and (2.18) predicts the modal energy spectra

$$\hat{E}_n(k) = \frac{k}{\alpha\cos(2\pi n/N) + \beta k^2}. \tag{2.20}$$

When $N = 1$ (2.20) reduces to (2.1), and when $N = 2$ (2.20) reduces to (2.7). If H vanishes then the equilibrium state is enstrophy equipartition with vanishing energy and vanishing correlation between the N coupled systems. However, if $H > 0$ then $\alpha/\beta \to -k_0^2$ as $k_{max} \to \infty$. In this limit $\hat{E}_0 = \int dk \hat{E}_0(k) \to H$ and all other $\hat{E}_n$ vanish. The velocity fields of the N coupled systems become perfectly correlated.

In the remainder of this paper, we consider coupled systems composed of $N = 3$ members, for which the dynamics (2.13) takes the form

$$\begin{aligned}
\zeta_{1t} &= \frac{1}{2} J(\zeta_1, \psi_2 + \psi_3) + \nu\nabla^2\zeta_1, \\
\zeta_{2t} &= \frac{1}{2} J(\zeta_2, \psi_1 + \psi_3) + \nu\nabla^2\zeta_2, \qquad \text{HCM} \\
\zeta_{3t} &= \frac{1}{2} J(\zeta_3, \psi_1 + \psi_2) + \nu\nabla^2\zeta_3.
\end{aligned} \tag{2.21}$$

We refer to (2.21) as the *Hamiltonian coupled model* (HCM). The case of three coupled systems is interesting because it offers other interesting possibilities. Consider, for example, the alternative dynamics

$$\begin{aligned}
\zeta_{1t} &= J(\zeta_1, \psi_3) + \nu\nabla^2\zeta_1, \\
\zeta_{2t} &= J(\zeta_2, \psi_1) + \nu\nabla^2\zeta_2, \qquad \text{NHCM} \\
\zeta_{3t} &= J(\zeta_3, \psi_2) + \nu\nabla^2\zeta_3,
\end{aligned} \tag{2.22}$$

in which each system's vorticity is a advected by the streamfunction of the *preceding* system only. When $\nu = 0$ both HCM and (2.22) conserve

$$\iint d\mathbf{x}\, G_\alpha(\zeta_\alpha), \alpha = 1, 2, 3, \tag{2.23}$$

where the $G_\alpha(\cdot)$ are arbitrary functions. However, when $\nu = 0$ HCM is a Hamiltonian system that also conserves

$$H = \frac{1}{2} \iint d\mathbf{x}\, \left(\nabla\psi_1 \cdot \nabla\psi_2 + \nabla\psi_2 \cdot \nabla\psi_3 + \nabla\psi_3 \cdot \nabla\psi_1\right), \tag{2.24}$$

whereas (2.22) is a non-Hamiltonian system with no conservation law analogous to (2.24). That is, there exists no $H[\zeta_1, \zeta_2, \zeta_3]$ such that $\psi_3 = -\delta H/\delta\zeta_1$, etc. Hence we refer to (2.22) as the *non-Hamiltonian coupled model* (NHCM). As a third alternative to HCM and NHCM, we consider

$$\begin{aligned}
\zeta_{1t} &= \frac{1}{2}J(\zeta_2, \psi_3) + \frac{1}{2}J(\zeta_3, \psi_2) + \nu\nabla^2\zeta_1, \\
\zeta_{2t} &= \frac{1}{2}J(\zeta_1, \psi_3) + \frac{1}{2}J(\zeta_3, \psi_1) + \nu\nabla^2\zeta_2, \qquad \text{KM} \\
\zeta_{3t} &= \frac{1}{2}J(\zeta_1, \psi_2) + \frac{1}{2}J(\zeta_2, \psi_1) + \nu\nabla^2\zeta_3,
\end{aligned} \tag{2.25}$$

which, like NHCM, is non-Hamiltonian. We call (2.25) the *Kraichnan model* (KM) because it resembles a model proposed by Kraichnan (1961, 1991) in connection with his direct interaction approximation. When $\nu = 0$ KM conserves the average energy,

$$E = \frac{1}{2}\iint d\mathbf{x}\ \left(\nabla\psi_1 \cdot \nabla\psi_1 + \nabla\psi_2 \cdot \nabla\psi_2 + \nabla\psi_3 \cdot \nabla\psi_3\right), \tag{2.26}$$

and the average enstrophy,

$$Z = \frac{1}{2}\iint d\mathbf{x}\ \left((\nabla^2\psi_1)^2 + (\nabla^2\psi_2)^2 + (\nabla^2\psi_3)^2\right), \tag{2.27}$$

but has no other conservation laws of the form (2.23). Thus KM has the same equilibrium statistical mechanics as ordinary two-dimensional turbulence. The equilibrium statistical mechanics of NHCM is enstrophy equipartition.

3. Numerical solutions

In this section we compare numerical solutions of two-dimensional turbulence governed by (1.1) (hereafter called TDT) to solutions of HCM, NHCM, and KM. The 2π-periodic domain is considered to be covered by n^2 gridpoints corresponding to wavenumbers that range between $k_0 = 1$ and $k_{max} = n/\sqrt{2}$. All the solutions have the same initial energy spectrum $E(k)$, sharply peaked at wavenumber $k = 4$, with initial rms fluid velocity $u_{rms} = 1$. Thus $t = 2\pi$ corresponds to the time required for fluid particles to traverse the periodic domain. We set the initial correlation between the 3 members of each of the coupled systems HCM, NHCM and KM by random phase variations in the Fourier coefficients.

First we consider inviscid experiments, to which the predictions of equilibrium statistical mechanics should apply quantitatively. Figure 1 shows the wavenumber-dependent analogue,

$$C(k) = \frac{\langle \mathbf{v}_1(\mathbf{k}) \cdot \mathbf{v}_2(\mathbf{k})\rangle}{\langle \mathbf{v}_1(\mathbf{k}) \cdot \mathbf{v}_1(\mathbf{k})\rangle^{1/2}\langle \mathbf{v}_2(\mathbf{k}) \cdot \mathbf{v}_2(\mathbf{k})\rangle^{1/2}}, \tag{3.1}$$

of (1.6) for HCM averaged over times near $t = 100$ in an inviscid calculation in which $n = 32$ and the initial correlation $C(k) = 0.5$. (Note that $C(k)$, in contrast to C defined by (1.6), would have the same value if $\mathbf{v}$ were replaced by ζ.) The relatively small value of n permits the long time integrations required to verify the statistical mechanical theory. Also shown in Figure 1 are the analytically computed

$$C(k) = \frac{k_0^2}{2k^2 - k_0^2} \tag{3.2}$$

corresponding to the asymptotic limit $k_{max} \to \infty$, and the $C(k)$ obtained by solving for

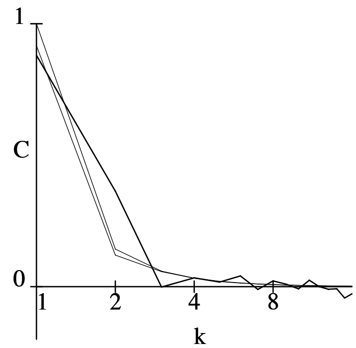

FIGURE 1. Upper curve: The correlation $C(k)$ between systems in an inviscid solution of HCM. The two nearly identical, lower curves are the predictions of equilibrium statistical mechanics as computed analytically for the case $k_{max} \to \infty$, and more accurately by numerical means.

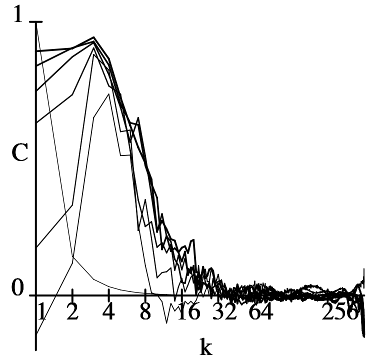

FIGURE 2. The correlation $C(k)$ between systems in a viscous solution of HCM averaged over the successive intervals $t = 0-1, 1-2, \dots, 5-6$. Darker lines correspond to later intervals. The monotonically decreasing curve is the asymptotic prediction (3.2) of equilibrium statistical mechanics.

α and β in the HCM equations analogous to (2.7), namely

$$E_\psi(k) = \frac{k}{\alpha + \beta k^2}, \qquad E_{\tau_1}(k) = E_{\tau_2}(k) = \frac{k}{-\alpha/2 + \beta k^2}, \tag{3.3}$$

by a method that does not assume $k_{max} \to \infty$ and accurately accounts for the discrete distribution of the wavenumbers. Here, τ_1 and τ_2 are the two (degenerate) eigenmodes besides $\psi = (\psi_1+\psi_2+\psi_3)/3$ that are needed to diagonalize the quadratic form associated with (2.24). The three curves in Figure 1 are almost indistinguishable. As predicted by equilibrium statistical mechanics, the corresponding $C(k)$ in inviscid solutions of NHCM and KM (which are not shown) decay rapidly to zero at all wavenumbers. The energy spectra (not shown) of all four dynamics also agree very closely with predictions. Thus, in the large-t, purely inviscid limit to which the theory should apply, equilibrium statistical mechanics offers a *quantitatively* accurate prediction of the numerical solutions.

Now we turn to viscous solutions, in which the viscous coefficient ν corresponds to a sub-grid-scale viscosity. In all the solutions described, ν is taken to be a slowly varying function of time, $\nu(t) = \zeta_{rms}(t)\Delta^2$, where ζ_{rms} is the rms vorticity and Δ is the distance between gridpoints. Figure 2 shows the correlation $C(k)$ averaged over successive time intervals up to $t = 6$ in a viscous solution of HCM with $n = 1024$. Darker lines correspond

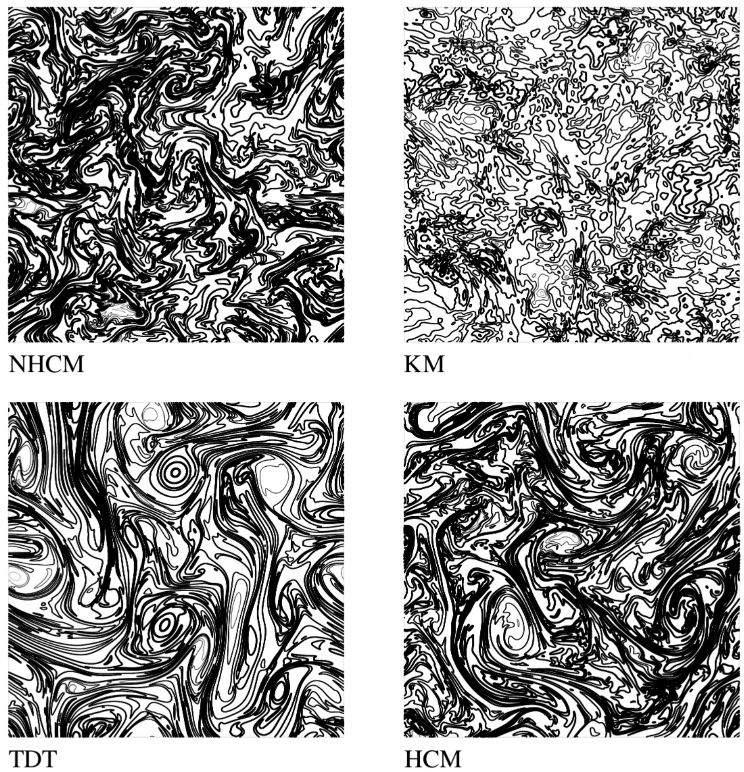

FIGURE 3. The vorticity ζ at $t = 6$ in viscous solutions of the four dynamics. In each of the dynamics HCM, NHCM and KM, only one of the three coupled systems is shown.

to later time intervals. The initially uniform correlation $C(k) = 0.5$ is rapidly wiped out in all but the most energetic wavenumbers. Subsequently $C(k) \to 1$ in all lower k, only gradually approaching the asymptotic prediction (3.2) of equilibrium statistical mechanics, in which the energy and positive correlation are concentrated in lowest wavenumber $k_0 = 1$. By $t = 6$ the overall velocity correlation (1.6) of HCM has increased from 0.5 to 0.88. In contrast, the velocity correlations of NHCM and KM have decreased from 0.5 to time-averaged values around 0.02. In TDT and KM, which conserve energy in the inviscid limit, the rms velocity u_{rms} at $t = 6$ is 0.988 and 0.982 respectively. In HCM, for which statistical mechanics predicts the asymptotic value $u_{rms} = 0.707$, the rms velocity has decayed from 1 to 0.752. In NHCM, for which the predicted energy vanishes asymptotically in the inviscid case, $u_{rms} = 0.585$ at $t = 6$. However, it is in the physical-space distribution of vorticity $\zeta(x, y, t)$ that the differences in behavior among the four distinct dynamics are most apparent. Figure 3 shows ζ at $t = 6$ in one representative member of each dynamics. At $t = 6$ the energy-containing scales of motion in HCM have become almost perfectly correlated, and the vorticity field resembles that of TDT even to the extent of exhibiting isolated coherent vortices. In contrast, the vorticity of NHCM has the 'fractal' appearance of a passive scalar, and KM resembles a Gaussian random field. In other solutions (not shown) in which $C(k) = 0$ initially, the correlation between systems in HCM remains zero as predicted by equilibrium statistical mechanics, and the vorticity fields of both HCM and NHCM resemble passive scalars at all times.

The models considered in this paper are artificially constructed models intended to illuminate the role of fundamental conservation laws in determining system behavior. However, some of our models have realistic antecedents. In particular, the two-system dynamics (1.5) is related to the dynamics of two-layer quasigeostrophic turbulence, which corresponds to the Hamiltonian

$$H = \frac{1}{2}\iint d\mathbf{x}\ \left(\nabla\psi_1 \cdot \nabla\psi_1 + \nabla\psi_2 \cdot \nabla\psi_2 + \frac{1}{2}k_R^2(\psi_1 - \psi_2)^2\right), \tag{3.4}$$

and to the bracket (1.9) with the vorticities ζ_1 and ζ_2 replaced by the potential vorticities $q_1 = \nabla^2\psi_1 + k_R^2(\psi_2 - \psi_1)/2$ and $q_2 = \nabla^2\psi_2 + k_R^2(\psi_1 - \psi_2)/2$. Here k_R is the wavenumber corresponding to the deformation radius. In contrast to (1.10), the coupling between quasigeostrophic layers arises from the available potential energy—the last term in (3.4). The case in which the two layers have the same statistics corresponds to the case of 'equivalent layers' considered by Salmon (1978). The two-layer analogues of (2.7) are

$$E_\psi(k) = \frac{k}{\alpha + \beta k^2}, \qquad E_\tau(k) = \frac{k}{\alpha + \beta(k^2 + k_R^2)}, \tag{3.5}$$

where, as defined by (2.3), ψ is the 'barotropic streamfunction' and τ is the 'baroclinic streamfunction.' In the limit $k_{max} \to \infty$, equilibrium statistical mechanics predicts that $\alpha/\beta \to -k_0^2$, and the correlation between layers takes the asymptotic form

$$C(k) = \frac{E_\psi(k) - E_\tau(k)}{E_\psi(k) + E_\tau(k)} = \frac{k_R^2}{2k^2 - 2k_0^2 + k_R^2}. \tag{3.6}$$

Thus $C(k) \approx 1$, corresponding to barotropic equilibrium flow, at all spatial scales larger than the deformation radius k_R^{-1}. For a more complete discussion see Salmon (1998).

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