Problem Set 1: Solutions

1. A Gaussian distribution with zero mean has the following form:

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-x^2}{2\sigma^2}\right] \tag{1}$$

Show that the first four moments of the pdf are as follows:

$$\begin{aligned}
\mu_1 &= 0. \\
\mu_2 &= \sigma^2 \\
\mu_3 &= 0. \\
\mu_4 &= 3\sigma^4
\end{aligned}$$

solution:

$$\begin{aligned}
\mu_1 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= -\frac{\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(-\frac{x}{\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= -\frac{\sigma}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2\sigma^2}\right) |_{-\infty}^{\infty} = 0.
\end{aligned} \tag{2}$$

This can also be solved by inspection, since we are integrating an odd function and an even function.

$$\begin{aligned}
\mu_2 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \left[-\sigma^2 x \exp\left(-\frac{x^2}{2\sigma^2}\right)\right] |_{-\infty}^{\infty} + \frac{\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= \sigma^2
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mu_3 &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \left[-\sigma^2 x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right] |_{-\infty}^{\infty} + \frac{\sigma^2}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2x \exp\left(-\frac{x^2}{2\sigma^2}\right)\, dx \\
&= 0.
\end{aligned} \tag{4}$$

$$\begin{aligned}\mu_4 &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right)\,dx \\ &= \frac{1}{\sigma\sqrt{2\pi}}\left[-\sigma^2 x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} 3x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\,dx \\ &= 3\sigma^2\mu_2 = 3\sigma^4\end{aligned}\tag{5}$$

2. Repeat the same procedures for a symmetrical exponential distribution of the form:

$$F(x) = \frac{1}{\sigma\sqrt{2}}\exp\left[\frac{-|x|\sqrt{2}}{\sigma}\right]\tag{6}$$

Is the 4th moment of the pdf bounded?

solution:

$$\begin{aligned}\mu_1 &= \frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} x \exp\left(-\frac{|x|\sqrt{2}}{\sigma}\right)\,dx \\ &= \frac{1}{\sigma\sqrt{2}}\int_{0}^{\infty} x \exp\left(-\frac{x\sqrt{2}}{\sigma}\right) - x \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\,dx \\ &= 0.\end{aligned}\tag{7}$$

$$\begin{aligned}\mu_2 &= \frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{|x|\sqrt{2}}{\sigma}\right)\,dx \\ &= \frac{2}{\sigma\sqrt{2}}\int_{0}^{\infty} x^2 \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\,dx \\ &= \int_{0}^{\infty} 2x \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\,dx \\ &= -\frac{2\sigma}{\sqrt{2}}\int_{0}^{\infty} \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\,dx \\ &= \sigma^2\end{aligned}\tag{8}$$

$$\begin{aligned}\mu_3 &= \frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} x^3 \exp\left(-\frac{|x|\sqrt{2}}{\sigma}\right)\,dx \\ &= \frac{2}{\sigma\sqrt{2}}\int_{0}^{\infty} (x^3 - x^3) \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\,dx = 0.\end{aligned}\tag{9}$$

$$\begin{aligned}
\mu_4 &= \frac{1}{\sigma\sqrt{2}}\int_{-\infty}^{\infty} x^4 \exp\left(-\frac{|x|\sqrt{2}}{\sigma}\right)\, dx \\
&= \frac{2}{\sigma\sqrt{2}}\frac{\sigma}{\sqrt{2}}\int_{0}^{\infty} 4x^3 \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\, dx \\
&= \frac{4\sigma}{\sqrt{2}}\int_{0}^{\infty} 3x^2 \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\, dx \\
&= -\frac{12\sigma^2}{2}\int_{0}^{\infty} 2x \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\, dx \\
&= 6\sqrt{2}\sigma^3\int_{0}^{\infty} \exp\left(-\frac{x\sqrt{2}}{\sigma}\right)\, dx \\
&= 6\sigma^4
\end{aligned} \tag{10}$$

3. On the course website (under homework), you'll find a current meter record identified as 'rcm00001.str', containing a variety of measured quantities. Download the record and compute the following quantities. The header lists the parameters contained in each column of the record.

a. Plot the empirical pdfs of u, v, T, speed, and direction. (Hint: You can use the Matlab "hist" function to do this, but be sure to normalize it correctly so that the area under the curve is 1.)

b. From the data, compute the first four moments of the pdfs. Based on the kurtosis, do the data have normal distributions? Which quantities are most nearly Gaussian?

solution:

Moments of the pdf are as follows:

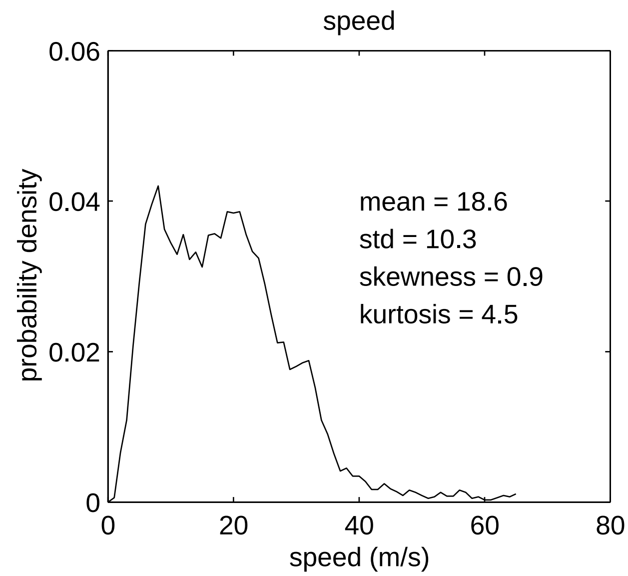
moment	speed	direction	u	v	temperature
1	18.5625	262.9646	-15.4836	3.2741	1.8338
2	106.6	4599.6	162.6	38.1	0.03
3	1031	-7.5×10^5	-141	-35.5	-0.002
4	5.13×10^4	1.75×10^8	1.09×10^5	7.3×10^3	0.006

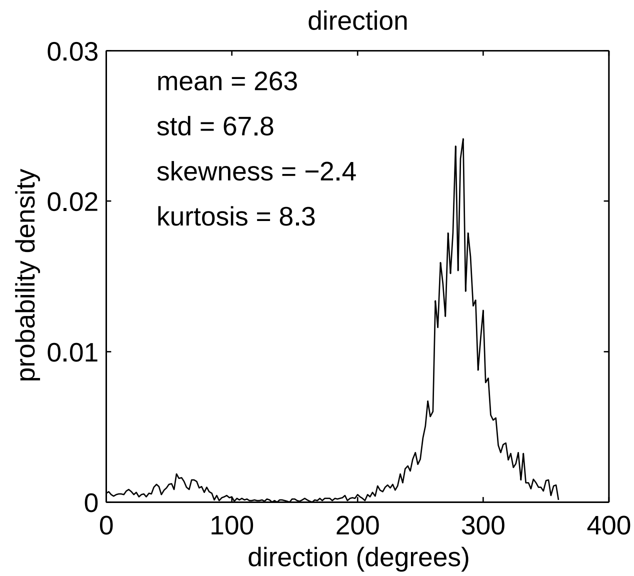
4. (bonus) Find a time series data set of your own choosing. Plot the time series and pdf. Compute the first 4 moments of the pdf. Is it Gaussian? Preferably send me an electronic version of the figure so that we can post them for class inspection.

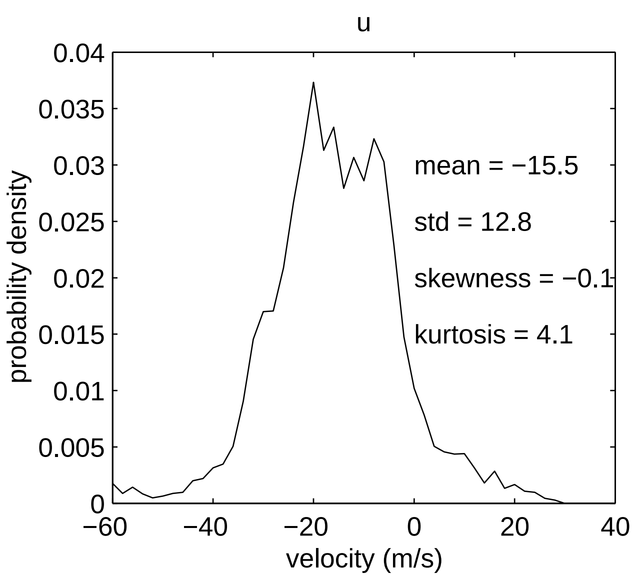
evaluation criteria:

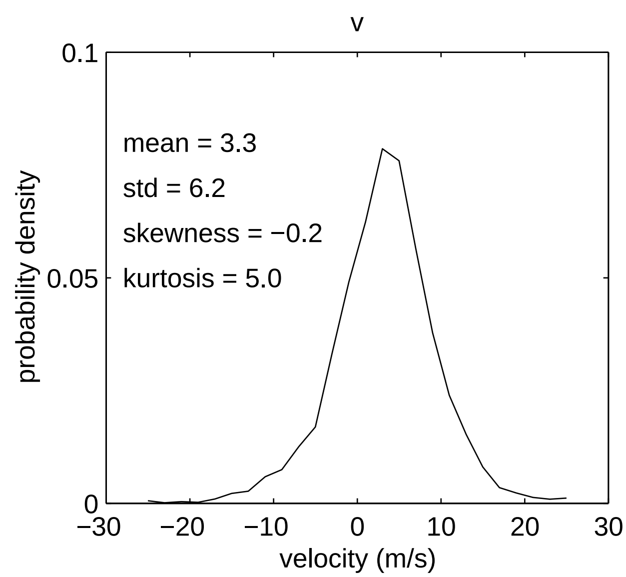
Is the pdf normalized so that its integral is 1?

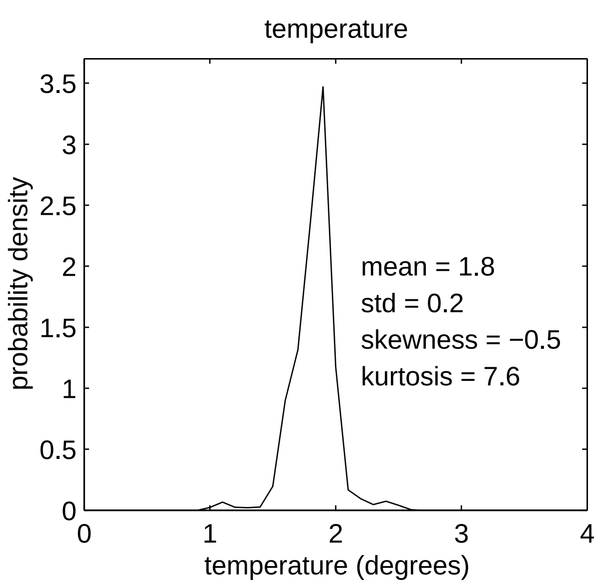
Have the moments (or the mean, standard deviation, skewness, and kurtosis) been computed?


speed
probability density
0.06
0.04
0.02
0
mean = 18.6
std = 10.3
skewness = 0.9
kurtosis = 4.5
0 20 40 60 80
speed (m/s)


direction
probability density
0.03
0.02
0.01
0
mean = 263
std = 67.8
skewness = −2.4
kurtosis = 8.3
0 100 200 300 400
direction (degrees)


u
probability density
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
mean = −15.5
std = 12.8
skewness = −0.1
kurtosis = 4.1
−60 −40 −20 0 20 40
velocity (m/s)


v
probability density
0.1
0.05
0
mean = 3.3
std = 6.2
skewness = −0.2
kurtosis = 5.0
−30 −20 −10 0 10 20 30
velocity (m/s)


temperature
probability density
3.5
3
2.5
2
1.5
1
0.5
0
mean = 1.8
std = 0.2
skewness = −0.5
kurtosis = 7.6
0 1 2 3 4
temperature (degrees)