Lecture 11: Degrees of freedom for overlapping segments, and other approaches to computing spectra

Reading: Bendat and Piersol, Ch. 5.2.1

Recap

We've looked at a basic strategy for computing spectra, the Welch's overlapping segment method, and we've talked about the value of segmenting, with overlaps, detrending, and applying a taper or window. Now let's wrap up our consdieration of the number of degrees of freedom. Then we'll look at some alternate approaches to computing spectra.

Analytic approach to degrees of freedom for overlapping segments

Following Percival and Walden, we define the following terms:

$$\begin{aligned} N &= \text{total length of record} \\ N_B &= \text{number of blocks} \\ N_S &= \text{segment length or block size} \\ n &= \text{shift factor or number of points of overlap between segments} \\ h &= \text{window, normalized so that } h^2 \text{ sums to } 1. \end{aligned}$$

Percival and Walden point out that the covariance between adjacent segments determines the adjustment to the degrees of freedom, and this depends on h.

Last time, I wrote out the expression for the variance of the spectral estimate:

$$\text{var}\left\{\hat{S}^{(WOSA)}(f)\right\} = \frac{1}{N_B}\sum_{j=0}^{N_B-1}\left(\text{var}\left\{\hat{S}_{jn+1}(f)\right\} + \frac{2}{N_B}\sum_{j<k}\text{cov}\left\{\hat{S}_{jn+1}(f), \hat{S}_{kn+1}(f)\right\}\right), \quad (1)$$

where j and k are indices for separate but overlapping segments. The variance of the jth spectrum converges to the canonical spectrum:

$$\text{var}\left\{\hat{S}_{jn+1}(f)\right\} \approx S^2(f). \quad (2)$$

The covariance depends on the overlap of the tapers or windows:

$$\text{cov}\left\{\hat{S}_{jn+1}(f), \hat{S}_{kn+1}(f)\right\} \approx S^2(f)\left|\sum_{t=1}^{N_S} h_t h_{t+|k-j|n}\right|^2, \quad (3)$$

with h_t defined to be zero when t is out of range (i.e. $t > N_S$). This means that:

$$\begin{aligned} \text{var}\left\{\hat{S}^{(WOSA)}(f)\right\} &\approx S^2(f)\left(1 + \frac{2}{N_B}\sum_{j<k}\left|\sum_{t=1}^{N_S} h_t h_{t+|k-j|n}\right|^2\right) \quad (4) \\ &= S^2(f)\left(1 + 2\sum_{m=1}^{N_B-1}\left(1-\frac{m}{N_B}\right)\left|\sum_{t=1}^{N_S} h_t h_{t+mn}\right|^2\right). \quad (5) \end{aligned}$$

Thus for a full record with arbitrary overlap:

$$\nu \approx \frac{2N_B}{1 + 2\sum_{m=1}^{N_b-1}\left(1-\frac{m}{N_B}\right)\left|\sum_{t=1}^{N_S} h_t h_{t+mn}\right|^2}. \quad (6)$$

This formulation allows for arbitrary levels of overlap, so you could imagine starting a new segment every data point and having to contend with with lots of complicated covariances between adjacent segments.

For practical purposes, we typically work with 50% overlap, so $n = N_S/2$. In this case, Percival and Walden show that the equation for the effective degrees of freedom simplifies to

$$\nu \approx \frac{2N_B}{1 + 2\left(1 - \frac{1}{N_B}\right)\left|\sum_{t=1}^{N_S/2} h_t h_{t+N_S/2}\right|^2}. \tag{7}$$

In the limit of large N_B and many samples, it's relatively straightforward to find an analytic solution:

$$\nu \approx \frac{2N_B}{1 + 2\left|\int_0^{L/2} h(t)h(t+L/2)\,dt\right|^2}. \tag{8}$$

subject to the requirement that the window normalization is:

$$\int_0^L h(t)^2\,dt = 1. \tag{9}$$

Thus for a boxcar filter, $h(t) = 1/\sqrt{L}$, and

$$\int_0^{L/2} h(t)h(t+L/2)\,dt = \int_0^{L/2} \frac{1}{L}\,dt = \left.\frac{t}{L}\right|_0^{L/2} = \frac{1}{2}. \tag{10}$$

Thus

$$\nu \approx \frac{2N_B}{1 + 2\left|\frac{1}{2}\right|^2} = \frac{2N_B}{1 + \frac{1}{2}} = \frac{4N_B}{3}. \tag{11}$$

Either analytically, or by plugging in normalized discrete window values h_t, we can compute the adjustments to our effective degrees of freedom shown in Table 1. You'll see that these values provide a fairly effective match to the values that you obtained from Monte Carlo simulation.

Window type	Equivalent degrees of freedom (ν)
Boxcar	4/3
Triangle	16/9
Hanning	36/19 $\approx$ 1.90
Hamming	$\sim$1.80

Table 1: Effective number of degrees of freedom relative to the total number of segments, using 50% overlap. (With no overlap, the equivalent degrees of freedom would be double the number of segments.)

So what of the other texts? The 2014 edition of Thomson and Emery is as misleading as the earlier editions. Von Storch and Zwiers, who are usually fairly lucid on data analysis, strongly favor filtering in the frequency domain so don't consider the impact of windowing or tapering in the time domain. Priestley also focuses largely on spectra computed from the autocovariance and spectra computed by filtering the periodogram. Their published tables are intended to provide guidance on the "lag window" (e.g. $\lambda(t)$) for spectra computed from the autocovariance, and the "spectral window", $W(f)$, which is Fourier transform of the lag window. When $\lambda(t)$ and $W(f)$ are used as a Fourier transform pair, they should have equivalent impacts on the degrees of freedom.

Finally, Percival and Walden note that we can also consider overlaps other than 50%, by adjusting m in their original equation:

$$\nu \approx \frac{2N_B}{1 + 2\sum_{m=1}^{N_b-1}\left(1 - \frac{m}{N_B}\right)\left|\sum_{t=1}^{N_S-mn} h_t h_{t+mn}\right|^2}. \tag{12}$$

Their Figure 293 shows degrees of freedom as a function of overlap for the Hanning window. We can code this in Matlab to consider other windows as well, as illustrated in Figure 1:

```
Ns=512;
n=256;
N=Ns*100;
Nb_theory=N/Ns;

h=ones(Ns,1)/sqrt(Ns);
for n=1:Ns-1
Nb=round((N-Ns)/n+1);
sumh=[];
for m=1:Nb-1
 if(Ns-m*n>=1)
  sumh(m)=(1-m/Nb)*abs(sum(h(1:Ns-m*n).*h(1+m*n:Ns)))^2;
 end
end
denom=1+2*sum(sumh);
nu_boxcar(n)=2*Nb/denom;
end

h=sqrt(2/3/Ns)*(1-cos(2*pi*(1:Ns)/Ns));

for n=1:511
Nb=floor((N-Ns)/n+1);
sumh=[];
for m=1:Nb-1
 if(Ns-m*n>=1)
  sumh(m)=(1-m/Nb)*abs(sum(h(1:Ns-m*n).*h(1+m*n:Ns)))^2;
 end
end
denom=1+2*sum(sumh);
nu_hanning(n)=2*Nb/denom;
end


hold off
plot(1-(1:2:Ns-1)/Ns,nu_boxcar(1:2:end)/Nb_theory,'LineWidth',3);
hold on
plot(1-(1:2:Ns-1)/Ns,nu_hanning(1:2:end)/Nb_theory,'LineWidth',3);
```

```
set(gca,'FontSize',16)
xlabel('Fractional overlap between segments','FontSize',16)
ylabel('Effective dof relative to # non-overlapping segments',...
   'FontSize',16)
legend('Boxcar','Hanning')
```

Filtering in the frequency domain

When we talked about windowing, we noted that windowing in the time domain is equivalent to convolution in the frequency domain (and filtering in the time domain is equivalent to multiplication in the frequency domain.) This could lead you to an interesting conclusion. What if you skipped all the windowing and just did convolutions (i.e. filtering) in the frequency domain? In the limit in which you choose the same filter, these options should be the same.

This approach was originally developed by Daniell and is nicely discussed by von Storch and Zwiers (see their section 12.3.11). Daniell's original idea was to run a moving average over the Fourier transform of the full record. In this case the confidence intervals are determined by:

$$P\left(\chi^2_{\nu,1-\alpha/2} < \nu\frac{\hat{E}(f)}{E(f)} < \chi^2_{\nu,\alpha/2}\right) \tag{13}$$

where ν in this case is 2 $\times$ the number of frequencies averaged together.

The advantages of this approach are that it provides an unbiased estimate of the true spectrum. The width of our averaging forces us to tradeoff bias (minimized if we do less averaging) vs variance (minimized with more averaging). One virtue of averaging in the frequency domain is that we can apply different levels of averaging (with different error bars) depending on the frequency.

Using the auto-covariance to think about spectra.

Now let's look at spectra from a different perspective. First, let's remind ourselves of the definition of a convolution:

$$z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau. \tag{14}$$

When we talked about Parseval's theorem, my notes mentioned that the autocovariance is the convolution of $x(t)$ with its time reversal, $x(-t)$.

$$y(\tau) = \int_{-\infty}^{\infty} x(t)x(\tau+t)dt. \tag{15}$$

More formally, we might write this autocovariance as $R_{xx}(\tau)$.

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(\tau+t)dt. \tag{16}$$

Now, what if we Fourier transform R?

$$S_{xx}(f) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i2\pi f\tau}\, d\tau. \tag{17}$$

Formally, this and its inverse transform are the Wiener-Khinchine relations.

Now let's think about starting with two functions, $x(t)$ and $y(t)$. We can write their Fourier transforms:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt \tag{18}$$

$$Y(f) = \int_{-\infty}^{\infty} y(t)e^{-i2\pi ft}\,dt. \tag{19}$$

So now let's define X times the complex conjugate of Y. (Why do we consider the complex conjugate? Because it's how we always multiply vectors.) We find the Fourier transform of the complex conjugate by substituting $-i$ for i everywhere it appears:

$$\begin{aligned} Y^*(f) &= \int_{t=-\infty}^{\infty} y(t)e^{i2\pi ft}\,dt & (20)\\ &= \int_{-t=-\infty}^{\infty} y(-t)e^{-i2\pi ft}\,d(-t) & (21)\\ &= \int_{t=\infty}^{-\infty} -y(-t)e^{-i2\pi ft}\,dt & (22)\\ &= \int_{t=-\infty}^{\infty} y(-t)e^{-i2\pi ft}\,dt. & (23) \end{aligned}$$

So

$$X(f)Y^*(f) = \int_{-\infty}^{\infty} k(t)e^{-i2\pi ft}\,dt. \tag{24}$$

For the moment, we have no idea what $k(t)$ should be, but we should be able to figure it out. If $X(f)Y^*(f)$ is a product in the frequency domain, then $k(t)$ should be a convolution in the time domain:

$$k(t) = \int_{-\infty}^{\infty} x(u)y(u-t)\,du. \tag{25}$$

(Remember that we'd normally use $t-u$ for a convolution; but here we reverse the sign to be consistent with using the complex conjugate $Y^*(f)$.) We used a derivation very similar to this in about Lecture 7, when we wanted to persuade ourselves that Parseval's theorem would work. But now we revisit this with a goal of looking closely at this convolved quantity, which represents the autocovariance. We can plug $k(t)$ into our equation to check this.

$$\begin{aligned} \int_{-\infty}^{\infty} k(t)e^{-i2\pi ft}\,dt &= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} x(u)y(u-t)\,du\right\} e^{-i2\pi ft}\,dt & (26)\\ &= \int_{-\infty}^{\infty}\int_{u-t=\infty}^{-\infty} -x(u)y(u-t)e^{i2\pi f(u-t)}e^{-i2\pi fu}\,d(u-t)\,du & (27)\\ &= \int_{-\infty}^{\infty} x(u)e^{-i2\pi fu}\left\{\int_{-\infty}^{\infty} y(u-t)e^{i2\pi f(u-t)}\,d(u-t)\right\}du & (28)\\ &= \int_{-\infty}^{\infty} x(u)e^{-i2\pi fu}Y^*(f)\,du & (29)\\ &= Y^*(f)\int_{-\infty}^{\infty} x(u)e^{-i2\pi fu}\,du & (30)\\ &= X(f)Y^*(f). & (31) \end{aligned}$$

Here we've taken advantage of the fact that the integral runs from $-\infty$ to $+\infty$ which lets us treat $u - t$ as a variable that depends only on t.

So we can think about what happens when $x(t) = y(t)$, so that

$$k(t) = \int_{-\infty}^{\infty} x(u)x(u-t)\,du = R_{xx}(-t). \tag{32}$$

This means that $k(t)$ is the autocovariance of x. The autocovariance is symmetric, so we could also write this as

$$k(t) = \int_{-\infty}^{\infty} x(u+t)x(u)\,du = R_{xx}(t). \tag{33}$$

Regardless

$$|X(f)|^2 = \int_{-\infty}^{\infty} k(t)e^{-i2\pi ft}\,dt. \tag{34}$$

This says that the Fourier transform coefficients squared (what we use when we compute spectra) are equivalent to the Fourier transform of the autocovariance.

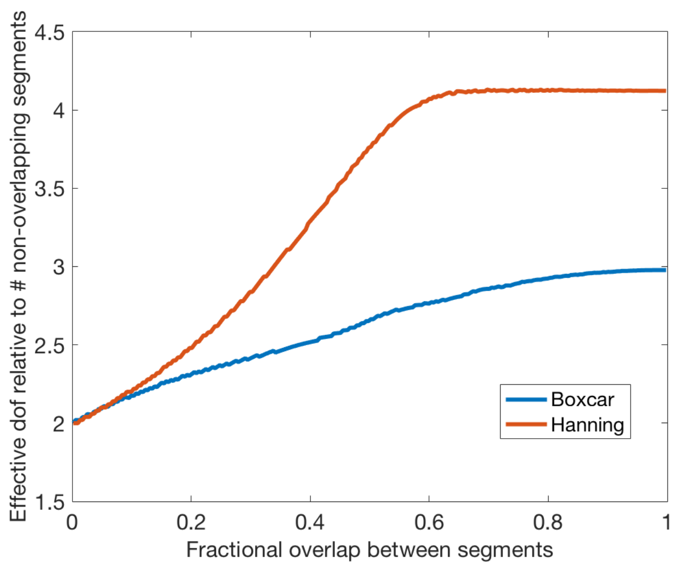

Figure 1: Ratio of degrees of freedom ν relative to nominal number of segments available if no overlapping is used for Hanning window and boxcar window.