Lecture 17:

Reading: Bendat and Piersol, Ch. 6.1

Recap

Last week and this week, we looked at coherence and coherence errors. Today we'll examine some examples, and then I want to take this one step further by looking at the transfer function.

Transfer function:
We discussed the fact that coherence is analogous to a correlation coefficient. It tells us if two things vary in tandem in a consistent way, but it doesn't tell us how big they are or how to use one variable to approximate a second variable. If we want to look at relative sizes, or if we want to approximate a variable y based on its relationship with z, in the time domain we think about finding a regression coefficient. We're used to solving a least-squares fitting equation of the form

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1}$$

with a solution of the form

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \tag{2}$$

For illustration we can simplify this to a case where the matrix $\mathbf{A}$ has only column, that is where we regress $\mathbf{y}'$ (with the prime telling us that the mean has been removed, since we don't want to complicate our least-squares fit) against just one variable, making $\mathbf{A}$ a column vector (e.g. $\mathbf{z}'$). In this case x becomes a scalar and the matrix inverse $(\mathbf{A}^T\mathbf{A})^{-1}$ is just the reciprocal of the variance of $\mathbf{z}'$.

$$x = \frac{\langle y'z'\rangle}{\langle z'^2\rangle}. \tag{3}$$

Compare this with the correlation coefficient of the demeaned variables

$$r = \frac{\langle y'z'\rangle}{\sqrt{\langle y'^2\rangle\langle z'^2\rangle}}. \tag{4}$$

Viewed in this way, the correlation coefficient r and the regression coefficient x (another term for the least-squares fit with only one variable) look nearly the same, aside from the normalization. Both the correlation coefficient and the regression coefficient convey useful information. And that might make you think that in the Fourier-transform domain there should be a form analogous to regression.

The term for the Fourier domain analog to regression is the transfer function:

$$\hat{H}_{zy}(f) = \frac{\hat{G}_{zy}(f)}{\hat{G}_{zz}(f)}, \tag{5}$$

which provides a (complex-numbered) recipe for mapping from z to y.

Formally, we talk about the transfer function when we think about constructing a linear system:

$$\mathcal{L}(y(t)) = z(t) \tag{6}$$

If $\mathcal{L}$ is a linear operator, then we could think of this relationship as a convolution:

$$y_t = \int_{-\infty}^{\infty} h(u)z(t-u)\,du \tag{7}$$

or if we Fourier transform, this would state:

$$Y(f) = H(f)Z(f). \tag{8}$$

Transfer functions (or gain functions): a proper example

If we want to look at relative sizes of two Fourier-transformed quantities, we can look at the transfer function (also known as the gain function):

$$\hat{H}_{zy}(f) = \frac{\hat{G}_{zy}(f)}{\hat{G}_{zz}(f)}, \tag{9}$$

which provides a (complex-numbered) recipe for mapping from z to y. Then, if

$$y_t = \int_{-\infty}^{\infty} h(u) z(t-u)\, du, \tag{10}$$

the Fourier transform is:

$$Y(f) = H(f)Z(f). \tag{11}$$

Consider it this way. Suppose

$$x(t) = \frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y \tag{12}$$

Then by Fourier transforming, we have:

$$\begin{aligned} Z(f) &= -f^2 Y(f) + i\alpha f Y(f) + \beta Y(f) && (13)\\ &= Y(f)\left[\beta - f^2 + i\alpha f\right] && (14) \end{aligned}$$

so

$$Y(f) = \frac{1}{[\beta - f^2 + i\alpha f]} Z(f) \tag{15}$$

and

$$H(f) = \frac{1}{[\beta - f^2 + i\alpha f]} \tag{16}$$

This is a nice framework for solving differential equations, but can we use it to gain insights into our data as well? We'll save that for next time.