Lecture 20:

Recap

This is the final session of this course and a chance for us to try to reflect on the content of the entire course.

Decibels and Powers of 10

Decibels ("db", not to be confused with decibars) quantify power or variance. We often focus on orders of magnitude. But power in decibels is on a $\log_{10}$ scale, with a factor of ten normalization.

$$P_{db} = 10\log_{10}(P/P_0), \tag{1}$$

where P_0 is a reference level of power: Variance is a squared quantity so

P/P_0	P relative to P_0 (db)
1000	30
10	10
2	3
0.5	-3
0.1	-10

$$P_{db} = 20\log_{10}(V/V_0). \tag{2}$$

By convention dB is used for sound pressure and db for everything else.

Rotary spectra

One final detail. We've discussed spectra for scalar quantities: temperature, wind speed, or atmospheric pCO_2. But what happens when we think about a vector quantity such as velocity? Of course you can treat the u and v components of velocity as separate scalars, but that might fail to capture the complexity of the overall motion. Rotary spectra provide a way to examine vector motions as a combined quantity.

To start thinking about rotary spectra, let's think about a single frequency. Imagine that you're sitting on the beach on a summer day. In the afternoon, as the land warms, wind starts blowing onshore from the ocean. This is the classic sea breeze. The wind direction reverses at night, when wind blows from the cold land to the warm ocean. Our cartoon schematics suggest that this is just an onshore/offshore circulation, but of course we live on a rotating planet, so the sea breeze, like everything else, rotates with the Earth's rotation. If we had a full anemometer, we would be able to identify this pattern of circulation.

For the diurnal cycle, as for any wind frequency, I can write the wind as:

$$u(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t) \tag{3}$$
$$v(t) = a_2 \cos(\omega t) + b_2 \sin(\omega t) \tag{4}$$

Or we can represent this as a complex number:

$$\begin{aligned} U(t) &= u(t) + iv(t) && (5)\\ &= a_1 \cos(\omega t) + b_1 \sin(\omega t) + a_2 \cos(\omega t) + b_2 \sin(\omega t) && (6)\\ &= (a_1 + ia_2)\cos(\omega t) + (b_1 + ib_2)\sin(\omega t) && (7) \end{aligned}$$

That gives us a vector motion in phase with the cosine and a vector motion in phase with the sine. But we're mixing complex amplitudes with real trigonometric functions in this form.

A different way to think of this is as a rotational components in the clockwise and counter-clockwise directions:

$$U(t) = U^+e^{i\omega t} + U^-e^{-i\omega t} \tag{8}$$
$$= U^+\left(\cos(\omega t) + i\sin(\omega t)\right)U^-\left(\cos(\omega t) - i\sin(\omega t)\right) \tag{9}$$
$$= (U^+ + U^-)\cos(\omega t) + (U^+ - U^-)i\sin(\omega t) \tag{10}$$

Here $e^{i\omega t}$ corresponds to counter-clockwise motion and $e^{-i\omega t}$ corresponds to clockwise motion. Since both expressions for U(t) have to be equivalent, this means that

$$U^+ = \frac{a_1 + b_2 + i(a_2 - b_1)}{2} \tag{11}$$
$$U^- = \frac{a_1 - b_2 + i(a_2 + b_1)}{2}. \tag{12}$$

The magnitude of these terms will give the rotary spectral components.

We can think of these terms as defining an ellipse with major axis $|U^+| + |U^-|$ and minor axis $||U^+| - |U^-||$. At each frequency, we can define three additional parameters. Two of these depend on angles, which we define as

$$\epsilon^+ = \tan^{-1}\left(\frac{a_2 - b_1}{a_1 + b_2}\right) \tag{13}$$
$$\epsilon^- = \tan^{-1}\left(\frac{a_2 + b_1}{a_1 - b_2}\right) \tag{14}$$

Then the orientation of the ellipse is:

$$\theta = \frac{\epsilon^+ + \epsilon^-}{2}, \tag{15}$$

and the phase of the ellipse (corresponding to the time when the velocity is at a maximum) is:

$$\phi = \frac{\epsilon^+ - \epsilon^-}{2}, \tag{16}$$

Finally, we can ask whether the motion is predominantly clockwise or counterclockwise by determining the sign of $|U^+| - |U^-|$.

While these parameters can be used to assess motions at a single frequency, more broadly they can be extracted from the Fourier transform to tell us about all frequencies. To do this, we just have to remember that $a_1 + ib_1$ are the Fourier coefficients for u and $a_2 + ib_2$ represent the Fourier coefficients for v, so we can extract the values that we need.

If we separately Fourier transform u and v, then the clockwise spectrum is the positive frequencies for the Fourier transform of u plus i times the negative frequencies for the Fourier transform of v. the counterclockwise spectrum is the opposite: negative frequencies for the Fourier transform of u plus i times the positive frequencies for the Fourier tranform of u.

It's useful to consider some limiting cases. Suppose that v is zero and u us proportional to $\cos(\omega t)$. Then only a_1 is non-zero, so U^+ and U^- are both $a_1/2$, the major axis is a_1. The angles ϵ^+ and ϵ^- are both 0, so the orientation angle $\theta = 0$ and the time of maximum $\phi = 0$, consistent with cosine being a maximum when $t = 0$. You can work through other cases to see how they express themselves in rotary form.

(For a good discussion of this, check our course notes by Miles Sundermeyer (U. Mass Dartmouth).) 1

Putting it all together

In this course, we've looked at a broad range of strategies for analyzing time series. Can we make some decisions about how we might plan an experiment and analyze our data?

Let's consider a central problem of physical oceanography. Can we evaluate the ocean response to wind:

$$\frac{\partial u}{\partial t} + u \cdot \nabla \mathbf{u} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau^x}{\partial z} \tag{17}$$

$$\frac{\partial v}{\partial t} + v \cdot \nabla \mathbf{u} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau^y}{\partial z} \tag{18}$$

Suppose we cross out a few terms. What do we need to measure to evaluate whether we've crossed out the right terms? Can we show that one of these is a reasonable approximation?

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial \tau^x}{\partial z} \tag{19}$$

$$-fv = \frac{1}{\rho}\frac{\partial \tau^x}{\partial z} \tag{20}$$

$$-fv + -\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial \tau^x}{\partial z} \tag{21}$$

$$\tag{22}$$

Methods we've explored in this class include:

1. Basic statistics: means, standard deviation, variance, standard error
2. Probability density function
3. Least-squares fitting
4. One-dimensional spectra (with windowing and uncertainties)
5. Two-dimensional spectra
6. Monte Carlo methods for evaluating confidence limits
7. Coherence

Intangibles

Besides the formal topics for which you've done problem sets, this class has aimed to start you thinking more like a data analyst. This has thrown you into the thorny world of real data problems, and I'm immensely grateful to you for your persistence. Some life lessons from this class:

1. In science, we favor evidence-based decision making over shoot-from-the-hip opinons. Data analysis gives you a set of tools for this.
2. Your good judgement matters in deciding how to approach a data analysis problem. You should always ask yourself how your understanding of the physics can inform your approach.

3. Even the rigors of the peer review process cannot guarantee the fidelity of published sources of information. Be skeptical and inquisitive.

4. You have the tools at your disposal to address your skepticism. Fake data and Monte Carlo methods are always an option.

5. Many questions have not been answered carefully, and there is room for you to make significant contributions. (The pier data and some of the other records that we used in class have barely been scratched.)

6. Just because something appears to be significant at the 95% level doesn't guarantee that it is a robust signal.

7. If your results are wildly dependent on the details of your methodology, that might mean that paying attention to methodology matters, but it also could be a warning sign that you're trying to identify a signal that is more wishful thinking than real signal.

8. The methods that we apply to real data can be exactly analogous to problems that we've done in this class, or they can be surprisingly divergent.