Lecture 18:

Recap

Last time we worked through a coherence example in detail and looked at links between coherence and co-variance. Now we just have a few more details to wrap up.

Multi-tapers and spectral peaks

Spectra can come in two flavors. Some have distinct single peaks, associated with tides. Some have large-scale structure associated with the general red structure of the ocean. If we want to find exactly the right peaks, then we can try different strategies to what we use when we want to find the general structure.

When we have narrow peaks, they aren't always easy to differentiate, particularly if our sampling is a bit coarse compared with the signals we'd like to detect. Consider the following case of a sinusoidal cycle that might or might not be well sampled, depending how long our instruments survive:

```
time=1:.5:120;
A=2*cos(2*pi*time/30)+cos(2*pi*time/60);
B=A(1:200);
C=[A(1:200) zeros(1,40)];

plot(time,A,time(1:200),B,'LineWidth',3)
set(gca,'FontSize',16)
xlabel('time','FontSize',16)
ylabel('amplitude','FontSize',16)

fA=fft(A);
fB=fft(B);
fC=fft(C);

frequency1=(0:120)/120;
frequency2=(0:100)/100;

loglog(frequency1,abs(fA(1:121)).^2,frequency2,abs(fB(1:101)).^2,...
  frequency1,abs(fC(1:121)).^2,'LineWidth',3)
set(gca,'FontSize',16)
xlabel('frequency','FontSize',16)
ylabel('spectral density','FontSize',16)
legend('full record','truncated record','zero padding')
```

When you look at this example, you might conclude that without perfect sampling of full sinusoidal cycles, we'll never find the correct spectral peaks. In essence this is a windowing problem.

If we don't have adequate resolution what are our options?

1. Possibility 1. Pad the short record with zeros to make it as long as we want. Since resolution is $f = 1/(N\Delta t)$. In this case, we'll see the impact of a sinc function bleeding into the frequencies that we'd like to resolve. Clearly this doesn't fully solve our problem.

2. Possibility 2. Obtain a longer record. This will be critical if we really want to resolve our signal.

Even if our record is norminally long enough, we also need to figure out how to optimize our detection of spectral peaks. Earlier this quarter we looked at the impact of windows, and examples from the Harris (1978) study showed how much impact a good windowing strategy can have in identifying spectral peaks. (For continuous spectra, windowing approaches work well.)

Formally, you'll recall that we can represent our record length problems using a convolution of our data with a finite width filter:

$$\hat{X}(\sigma) = \int_{-\infty}^{\infty} X(\omega) W(\sigma - \omega)\, d\omega, \tag{1}$$

where

$$W(\sigma) = \frac{\sin(\sigma T)}{\sigma T} = \text{sinc}\left(\frac{\sigma T}{\pi}\right) \tag{2}$$

This means that the spectrum is essentially convolved with $W(\sigma)^2$. But as we noted earlier, we can switch from a boxcar window to a triangle window or something a bit more Gaussian than that and cut down on the sidelobes in our window to obtain a cleaner spectrum, although we have to widen the central peak of the window in the frequency domain, which means de-emphaizing the beginning and end of the data series.

What if we want to improve our resolution. Consider Rob Pinkel's example of a record equivalent to

$$x = 100\cos(2\pi 20.5/10001) + 80\cos(2\pi 30.4/100001) + 100\cos(2\pi 40.8/100001) + 10\cos(2\pi 50.3/100001) \tag{3}$$

The quality of our spectral estimate will depend on the length of our record. (Why is that? The resolution is the lowest resolved frequency.) So what can we do to improve resolution. One strategy would be to pad our record with zeros to make it as long we want. That buys us something, but it gives us plenty of spectral ringing.

If we really want to optimize resolution, we can try a multitaper approach. (See for example Ghil et al, Reviews of Geophysics, 2001). In a multitaper approach, we replace our single window with a set of tapers. The tapers are designed to minimize spectral leakage, and they are referred to as "discrete prolate spheroidal sequences" or "Slepian" tapers (after Slepian, who studied them). Tapers are used like windows—they pre-multiply the data, Fourier transforms are coputed, and then the spectrum is computed as a weighted sum of all of the squared Fourier transforms. This effectively averages over an ensemble of windows to minimize variance. This is very effective for extracting narrow peaks that would otherwise be undetectable. Matlab has a multi-taper method package ('pmtm'), but if you really want this to work, you probably want to dig into the guts of the algorithm a bit further. Here's the Matlab example, modified slightly to make a longer record:

```
fs = 1000;                   % Sampling frequency
t = (0:3*fs)/fs;               % One second worth of samples
A = [1 2];                   % Sinusoid amplitudes
f = [150;140];               % Sinusoid frequencies

xn = A*sin(2*pi*f*t) + 0.1*randn(size(t));
pmtm(xn,4,[],fs)
```

This produces an impressive two spectral peaks. Of course this example isn't too tricky. Here's what we get if we take the same data and split them into 6 non-overlapping segments, even with no windowing or detrending:

```
fa=fft(reshape(xn(1:3000),500,6));
semilogy(mean(abs(fa(1:250,:)').^2))
```

These are reassuringly similar results.

At this point, we should sort out in our heads the definition of a decibel.

Decibels ("db", not to be confused with decibars) quantify power or variance. We often focus on orders of magnitude. But power in decibels is on a $\log_{10}$ scale, with a factor of ten normalization.

$$P_{db} = 10\log_{10}(P/P_0), \tag{4}$$

where P_0 is a reference level of power: Variance is a squared quantity so

P/P_0	P relative to P_0 (db)
1000	30
10	10
2	3
0.5	-3
0.1	-10

$$P_{db} = 20\log_{10}(V/V_0). \tag{5}$$

By convention dB is used for sound pressure and db for everything else.

Transfer function: If we want to look at relative sizes, we can look at the transfer function:

$$\hat{H}_{xy}(f) = \hat{G}_{xy}(f)\hat{G}_{xx}(f), \tag{6}$$

which provides a (complex-numbered) recipe for mapping from x to y.

Formally, we talk about the transfer function when we think about constructing a linear system:

$$\mathcal{L}(y(t)) = x(t) \tag{7}$$

If $\mathcal{L}$ is a linear operator, then we could think of this relationship as a convolution:

$$y_t = \int_{-\infty}^{\infty} h(u)x(t-u)\,du \tag{8}$$

or if we Fourier transform, this would state:

$$Y(\sigma) = H(\sigma)X(\sigma). \tag{9}$$

Consider it this way. Suppose

$$x(t) = \frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y \tag{10}$$

Then by Fourier transforming, we have:

$$\begin{aligned} X(\sigma) &= -\sigma^2 Y(\sigma) + i\alpha\sigma Y(\sigma) + \beta Y(\sigma) && (11)\\ &= Y(\sigma)\left[\beta - \sigma^2 + i\alpha\sigma\right] && (12)\end{aligned}$$

so

$$Y(\sigma) = \frac{1}{[\beta - \sigma^2 + i\alpha\sigma]} X(\sigma) \tag{13}$$

and

$$H(\sigma) = \frac{1}{[\beta - \sigma^2 + i\alpha\sigma]} \tag{14}$$

This is a nice framework for solving differential equations, but can we use it to gain insights into our data as well? First some rules:

1. Linearity: If a given linear system has an input $x_1(t)$ which corresponds to an output $y_1(t)$, and input $x_2(t)$ corresponds to output $y_2(t)$m then a summed input $x(t) = \alpha x_1(t) + \beta x_2(t)$, will produce an output $y(t) = \alpha y_1(t) + \beta y_2(t)$.

2. Time invariance: If an input is delayed in time by τ, then the output is as well: If $x(t) rightarrow x(t+\tau)$, then $y(t) \rightarrow y(t+\tau)$.

3. Causality: If h(t) represents an impulse, then it should be zero for $t < 0$. A response cannot occur before the forcing.

4. Sequential application: If the output of one linear system is an input to a second system, then the frequency response is
$$H_{12}(\sigma) = H_1(\sigma) \cdot H_2(\sigma) \tag{15}$$

So suppose we measure $y(t)$ and $x(t)$. Can we determine h or H? We know that

$$Y(\sigma) = H(\sigma) X(\sigma) \tag{16}$$

Let's multiply both sides of the equation by the complex conjugate of X to form the cross-spectrum:

$$\frac{Y(\sigma)X^*(\sigma)}{\Delta\sigma} = H(\sigma)\frac{X(\sigma)X^*(\sigma)}{\Delta\sigma} \tag{17}$$

This becomes

$$C_{xy}(\sigma) = H(\sigma) S_{xx}(\sigma) \tag{18}$$

so

$$H(\sigma) = C_{xy}(\sigma)/S_{xx}(\sigma) \tag{19}$$