Lecture 9:

Reading: Bendat and Piersol, Ch. 11.5.1-11.5.2

Recap

We've now examined how to compute spectra and how to use the χ^2 distribution to put error bars on our spectra. Now, we'll need to think about background trends and problems associated with finite length records.

Detrending

Last time we noted that the Fourier transform of a linear trend puts energy into every possible frequency, meaning that if you don't detrend your data, any residual trend will give you a red spectrum. You can spend a lot of energy interpreting red spectra that result from linear trends, but they can really be viewed as an artifact of the data. Consider the linear trend in SST that we examined earlier in class. That linear trend produces a strongly red spectrum.

We can estimate the spectral slope by eye by looking at the drop in y per order of magnitude in x. If the spectrum drops one order of magnitue in y for a half an order of magnitude increase in x, then it has a slope of σ^{-2}. More formally, you could determine this slope through a least-squares fitting procedure.

Some further convolution examples

Let's return to convolution for one more example. We've pointed out that convolution in the time domain is equivalent to multiplication in the frequency domain.

What if I compute the convolution of a boxcar filter with itself?

$$\begin{aligned}
y(\tau) &= \int_{-\infty}^{\infty} h(t)h(\tau-t)dt && (1)\\
&= \int_0^1 h(\tau-t)dt \text{ for } 0<\tau-t<1 && (2)\\
&= \int_0^1 h(\tau-t)dt \text{ for } -\tau<-t<1-\tau && (3)\\
&= \int_0^1 dt \text{ for } \tau-1<t<\tau && (4)\\
&= \begin{cases} \int_0^\tau dt=\tau, & \text{for } 0<\tau\le 1\\ \int_{\tau-1}^1 dt = 1-(\tau-1)=2-\tau & \text{for } 1<\tau\le 2\end{cases} && (5)
\end{aligned}$$

This is definitely easier to consider graphically, and it will show that boxcar filter convolved with itself becomes a triangle filter.

What if we repeat this again and again? It's actually analogous to what we considered for the central limit theorem. Applying a filter to a filter to a filter will eventually give us a Gaussian.

The Fourier Transform of a Boxcar

When we deal with data records of finite length, we're always taking finite sized segments of data, sort of like Fourier transforming a boxcar filter. Usually we think of a Fourier transform of this form:

$$X = \int_{-\infty}^{\infty} x(t)e^{-i\sigma t}\, dt. \quad (6)$$

If $x(t)$ is a boxcar filter, this becomes:

$$\int_{-\infty}^{\infty} h(t)e^{-i\sigma t}\,dt = \int_{-1/2}^{1/2} e^{-i\sigma t}\,dt = \frac{e^{-i\sigma t}}{i\sigma}|_{-1/2}^{1/2} = \frac{e^{i\sigma/2} - e^{-i\sigma/2}}{i\sigma} = \frac{\sin(\sigma/2)}{\sigma/2} \tag{7}$$

This is the sinc function, more typically written as:

$$\text{sinc}\left(\frac{x}{\pi}\right) = \frac{\sin(x)}{x} \tag{8}$$

or

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{9}$$

Remember that convolution in the time domain corresponds to multiplication in the frequency domain. In this case, we're doing the opposite. We multiplied our data by a boxcar window in the time domain, and that's equivalent to convolving with a sinc function in the frequency domain:

$$\hat{X}(\sigma) = \int_{-\infty}^{\infty} X(\omega)W(\sigma - omega)\,d\omega \tag{10}$$

where

$$W(\sigma) = \frac{\sin(\sigma T)}{\sigma T} = \text{sinc}\left(\frac{\sigma T}{\pi}\right), \tag{11}$$

for a record length of 2T.

Corresponding, the spectrum is convolved with $W(\sigma)^2 = \text{sinc}(\sigma T/\pi)^2$. This has a central peak of $\pm\pi/T$. The window will have a maximum value

$$|W(\sigma)|^2 < (\sigma T)^{-2} \tag{12}$$

The side lobes of the window are definitely problematic. And it turns out that we're not stuck with them. If we can widen the central peak of W (the Fourier transform of our window), we can lower the impact of the side lobes. To do this, we'll want to forego the rectangular window in the time domain and choose something that lets us attenuate the beginnings and ends of each segment of our data. What if we chose a triangle window? That will already give us fewer side lobes.

But we can keep going to find a window that looks more like an exponential. Leading possibilities:

1. Cosine taper:

$$w(t) = \cos^{\alpha}\left(\frac{\pi t}{2T}\right) \tag{13}$$

   with $\alpha = [1, 4]$.

2. Hanning window or "raised cosine" window (developed by von Hann):

$$w(t) = \cos^2\left(\frac{\pi t}{2T}\right) = \frac{1 + \cos(\pi t/T)}{2} = 0.5 + 0.5\cos(\pi t/T) \tag{14}$$

3. Hamming window. This variant of the Hanning window was developed by Hamming.

$$w(t) = 0.54 + 0.46\cos(\pi t/T) \tag{15}$$

The Hamming window has less energy in the first side lobe but more in the distant side lobes.

Some other options include a Blackman-Harris window or a Kasier-Bessel window, and Harris (1978, Use of windows for harmonic analysis, *Proc. IEEE*) provides detailed discussion of options.

So how do you use a window?

1. First you must demean your data—otherwise, the window will shift energy from the mean into other frequencies. If you're working in segments, you should demean (and detrend) each segment before you do anything further.

2. Second, for a segment with N points, multiply by a window that is N points wide.

3. Since the window attentuates the impact of the edge of each segment, you can use segments that overlap (typically by 50%). This will give you (almost) twice as many segments, so instead of ν degrees some larger number.

4. Now Fourier transform, scale appropriately (e.g. by $\sqrt{8/3}$ for a Hanning window, to account for energy attenuation) and compute amplitudes.

Will the window preserve energy in your system? Not necessarily. You can normalize it appropriately, but windowing can shift the background energy level of your spectrum relative to the spectral peaks, and you'll want to keep track of this.

How many degrees of freedom do you have for overlapping windows. Not 2ν but close to that. Bendat and Piersol usefully say that overlapping by 50% will recover about 90% of the stability lost due to tapering.

So a quick recap. When we filter, we convolve the filter with our data in the time domain, which is equivalent to multiplying in the frequency domain. When we window, we multiply by a tapered window in the time domain, which is equivalent to convolving in the frequency domain.

Once you've created overlapping, windowed segments, then you'll need to figure out how many independent segments you really have. Clearly at a minimum you should have the equivalent of the number of segments that you would have if you did no overlapping. If you have N data points divided into segments that are $2M$ wide, then the minimum number of segments is $N/(2M)$. But with windowing, the end points of each segment are used less than the middle, making the overlapping segments more independent, so perhaps you have N/M segments.

Thomson and Emery's book provides the following table (Table 5.5) identifying equivalent degrees of freedom. (They lifted the table from Priestley (*Spectral Analysis and Time Series*, 1981, Table 6.2.)

Window type	Equivalent degrees of freedom (ν)	multiplier $\times$ double number segments
Truncated peridogram (boxcar)	N/M	m/2
Bartlett (triangle)	$3N/M$	1.5 m
Daniell (sinc)	$2N/M$	m
Parzen	$3.708614N/M$	1.354 m
Hanning	$8/3N/M$	4/3 m
Hamming	$2.5164N/M$	1.25 m

Priestley includes another table (Table 6.1) that contains a different set of numbers.

Window type	$N/M\,\mathrm{var}\hat{h}/h^2$
Truncated peridogram (boxcar)	2
Bartlett (triangle)	2/3
Daniell (sinc)	1
Parzen	0.539285
Hanning	0.75
Hamming	0.7948