Lecture 16:

Reading: Bendat and Piersol, Ch. 9.1-9.2

Recap

We've looked at a couple examples of coherence calculations along with some (incomprehensible) figures from published cases. The key feature of coherence is that it allows you to decide if two records vary in a consistent way at any given frequency. This is a subtle point: if you view the world from the perspective of a Fourier transform, everything is sinusoidal, and naturally all data records vary sinusoidally, although the phasing of record x could differ from the phasing of record y. When we compute coherence, we ask whether the phasing (at frequency σ or wavenumber k) between x and y is consistent between different chunks of the data records. To compute coherence we need to segment our data; for the same reasons that we detrend and window when we compute spectra, we should detrend and window when we compute coherence.

Coherence calculations produce two results: a coherence that varies between 0 and 1, and a phase that varies between $-\pi$ and π (or equivalently from 0 to 360 degrees or -180 to 180 degrees.)

As a reminder, we defined coherence as:

$$\gamma_{xy}(f_k) = \sqrt{\frac{S_{xy}(f_k)}{S_{xx}(f_k)S_{yy}(f_k)}}, \tag{1}$$

where we needed S_{xx}, S_{yy} and S_{xy} to represent averages of multiple segments. This had an uncertainty threshold:

$$\sqrt{\beta} = \sqrt{1-\alpha^{1/(n_d-1)}}, \tag{2}$$

where n_d is the number of segments, α is the significance level and is tyically 0.05 for a 95% significance level (see Thomson and Emery).

The coherence phase is:

$$\phi(f_k) = \tan^{-1}(-Q(f_k)/C(f_k)), \tag{3}$$

where Q is the imaginary part of the co-spectrum S_{xy}, and C is the real part of the co-spectrum.

We left off with a couple of burning questions. First, we observed that in a test case with overlapping segments, we obtained error bars that were visibly too small if we simply counted the number of segments. Some Monte Carlo tests will show that with overlapping segments, you do need to adjust the number of degrees of freedom. Since you're working with the spectrum, use the same corrections that you used for overlapping segments with spectra (but scaled by a factor of 2):

Window type	Equivalent number of segments (n_d)
Boxcar	2/3
Triangle	8/9
Hanning	8/19 $\approx$ 0.95
Hamming	$\sim$0.90

Table 1: Effective number of independent segments relative to the total number of segments, using 50% overlap. (With no overlap, assume n_d segments.)

Finally, we need to tackle phase errors.....

Uncertainties of coherence and phase: What do we believe?

We talked about coherence uncertainties last time, but we didn't finish sorting out phase uncertainties, and I gave you three different formulations. Which one do we trust?

First a little terminology. Bendat and Piersol provide a good discussion of bias and uncertaintites in spectral estimators. As a starting point, the variance of the quantity that we want to estimate is

$$\text{var}[\tilde{A}] = E[\tilde{A}^2] - A^2, \tag{4}$$

where A is the true value, and $\tilde{A}$ is the unbiased estimate (so $E[\tilde{A}] = A$. For spectral estimators we tend to talk about the normalized error:

$$\epsilon^2 = \frac{\text{var}\tilde{A}}{A^2}. \tag{5}$$

Bendat and Piersol first derive relationships for the variance of the spectrum and cross-spectrum in the case of one segment and two degrees of freedom (see appendix). They then note that variance scales with $1/n$, where n is the number of degrees of freedom, so that variance can be inferred for spectra and cross-spectra with any number of degrees of freedom (by dividing by n_d the number of segments).

As we noted last time, one common formulation for phase uncertainty is:

$$\delta_\phi = \sin^{-1}\left[t_{\alpha,2n_d}\sqrt{\frac{1-\gamma_{xy}^2}{2n_d\gamma_{xy}^2}}\right] \tag{6}$$

where $t_{\alpha,2n_d}$ is identified as the "Student t distribution", and is actually the inverse of the Student t distribution ("tinv" in Matlab). Given an upper cut-off point of $\alpha/2 = 0.975$ for the cdf of the t-distribution, we're looking for the corresponding value of the function. In case you have doubts, check Table A9.3 of Koopmans, which shows, for example, that $t(0.975, 20) = 2.086$.

But when we plot this up, for our white noise case, it seems to be a complex number, since we've ended up with some out of range values for the arcsine—perhaps this isn't surprising since the phase is ill-defined for white noise. Bendat and Piersol provide a different formulation, which has the virtue of producing a real number:

$$\text{std}\left[\phi_{xy}(f)\right] \approx \frac{\left[1-\gamma_{xy}^2(f)\right]^{1/2}}{\left|\gamma_{xy}(f)\right|\sqrt{2n_d}} \tag{7}$$

Zwiers and Von Storch quote Hannan (1970)[1] and provide:

$$\delta_\phi = \sin^{-1}\left[t_{(1+p)/2,2n_d-2}\frac{\gamma_{xy}^{-2}-1}{2n_d-2}\right], \tag{8}$$

where p is the confidence interval (e.g. 0.95), so $(1+p)/2$ and $(1-p)/2$ provide the limits for $p\%$ significance levels. However, Zwiers and Von Storch have misquoted Hannan (1970), who actually have a form equivalent to this:

$$\delta_\phi = \sin^{-1}\left[t_{(1+p)/2,2n_d-2}\left\{\frac{\gamma_{xy}^{-2}-1}{2n_d-2}\right\}^{1/2}\right], \tag{9}$$

which is exactly equivalent to Koopmans (1974). In Matlab, these become:

[1] Hannan, 1970, *Multiple Time Series*, John Wiley & Sons, 536 pp. (See p. 257, equation 2.11)

```
% cab is squared coherence between a and b
% for example:
%cab=abs(mean(fab,2)) ./sqrt(abs(mean(faa,2)) .* abs(mean(fbb,2)));

alpha = .05;
nd=10; % # of segments
p=1-alpha;
delta_phase = asin(tinv(.95,2*nd)*...
  sqrt((1-abs(cab).^2)./(abs(cab).^2*(2*nd))));
delta_phase2 = sqrt((1-cab.^2)./(abs(cab).^2*2*nd));
delta_phase3 = asin(tinv(.975,2*nd-2)*(1 ./cab.^2-1)/(2*nd-2));
```

The expressions are similar, though not identical. Which is most appropriate? We can test this out by creating a fake data set with a known phase relationship:

```
a=randn(100,1000)+ cos(2*pi/10*(1:100)')*ones(1,1000);
b=randn(100,1000) + sin(2*pi/10*(1:100)')*ones(1,1000);
fa=fft(a);
fb=fft(b);
fab=conj(fa).*fb;
faa=conj(fa).*fa;
fbb=conj(fb).*fb;

cab=abs(mean(fab,2)) ./sqrt(abs(mean(faa,2)) .* abs(mean(fbb,2)));

m=10;
clear phase_c
for i=1:1000/m
 phase_c(:,i)=atan2(-imag(mean(fab(:,(i-1)*m+1:i*m),2)),...
      real(mean(fab(:,(i-1)*m+1:i*m),2)));
end

nd=m;
delta_phase = asin(tinv(.95,2*nd)*...
  sqrt((1-abs(cab).^2)./(abs(cab).^2*sqrt(2*nd))));
delta_phase2 = sqrt((1-cab.^2)./(abs(cab).^2*2*nd));
delta_phase3 = asin(tinv(.975,2*nd-2)*(1 ./cab.^2-1)/(2*nd-2));
% compare results
[delta_phase(11) delta_phase2(11) delta_phase3(11)]
std(phase_c(11,:))
```

It's clear from these tests that (a) the distribution of the phases should be roughly Gaussian, (b) Bendat and Piersol's representation for the standard deviation of the phase (delta_phase2) is relatively reliable, (c) the inverse sine formulations should produce phase errors representing the 95th percentile.

Variance preserving spectra

One of the virtues of the properly normalized spectrum is that the area under the curve should represent the signal variance within a specific frequency band:

$$\text{variance in a band} = \int_{f-\Delta f/2}^{f+\Delta f/2} |X(f)|^2 \, df \tag{10}$$

This sounds good as a concept, but in log-log space, it's challenging to figure out what the area under the curve really represents. Variance-preserving spectra provide a visual representation for this.