Lecture 17:

Reading: Bendat and Piersol, Ch. 6.1

Recap

Last week and this week, we looked at coherence and coherence errors. Today I want to take this one step further by looking at the transfer function.

Transfer function:

We discussed the fact that coherence is analogous to a correlation coefficient. It tells us if two things vary in tandem in a consistent way, but it doesn't tell us how big they are or how to use one variable to approximate a second variable. If we want to look at relative sizes, or if we want to approximate a variable y based on its relationship with z, in the time domain we think about finding a regression coefficient. We're used to solving a least-squares fitting equation of the form

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1}$$

with a solution of the form

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \tag{2}$$

For illustration we can simplify this to a case where the matrix $\mathbf{A}$ has only column, that is where we regress $\mathbf{y}'$ (with the prime telling us that the mean has been removed, since we don't want to complicate our least-squares fit) against just one variable, making $\mathbf{A}$ a column vector (e.g. $\mathbf{z}'$). In this case x becomes a scalar and the matrix inverse $(\mathbf{A}^T\mathbf{A})^{-1}$ is just the reciprocal of the variance of $\mathbf{z}'$.

$$x = \frac{\langle y'z'\rangle}{\langle z'^2\rangle}. \tag{3}$$

Compare this with the correlation coefficient of the demeaned variables

$$r = \frac{\langle y'z'\rangle}{\sqrt{\langle y'^2\rangle\langle z'^2\rangle}}. \tag{4}$$

Viewed in this way, the correlation coefficient r and the regression coefficient x (another term for the least-squares fit with only one variable) look nearly the same, aside from the normalization. Both the correlation coefficient and the regression coefficient convey useful information. And that might make you think that in the Fourier-transform domain there should be a form analogous to regression.

The term for the Fourier domain analog to regression is the transfer function:

$$\hat{H}_{zy}(f) = \frac{\hat{G}_{zy}(f)}{\hat{G}_{zz}(f)}, \tag{5}$$

which provides a (complex-numbered) recipe for mapping from z to y.

Formally, we talk about the transfer function when we think about constructing a linear system:

$$\mathcal{L}(y(t)) = z(t) \tag{6}$$

If $\mathcal{L}$ is a linear operator, then we could think of this relationship as a convolution:

$$y_t = \int_{-\infty}^{\infty} h(u)z(t-u)\,du \tag{7}$$

or if we Fourier transform, this would state:

$$Y(f) = H(f)Z(f). \tag{8}$$

Transfer functions (or gain functions): a proper example

If we want to look at relative sizes of two Fourier-transformed quantities, we can look at the transfer function (also known as the gain function):

$$\hat{H}_{zy}(f) = \frac{\hat{G}_{zy}(f)}{\hat{G}_{zz}(f)}, \tag{9}$$

which provides a (complex-numbered) recipe for mapping from z to y. Then, if

$$y_t = \int_{-\infty}^{\infty} h(u)z(t-u)\,du, \tag{10}$$

the Fourier transform is:

$$Y(f) = H(f)Z(f). \tag{11}$$

Consider it this way. Suppose

$$x(t) = \frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y \tag{12}$$

Then by Fourier transforming, we have:

$$\begin{aligned} Z(f) &= -f^2 Y(f) + i\alpha f Y(f) + \beta Y(f) \tag{13} \\ &= Y(f)\left[\beta - f^2 + i\alpha f\right] \tag{14} \end{aligned}$$

so

$$Y(f) = \frac{1}{[\beta - f^2 + i\alpha f]} Z(f) \tag{15}$$

and

$$H(f) = \frac{1}{[\beta - f^2 + i\alpha f]} \tag{16}$$

This is a nice framework for solving differential equations, but can we use it to gain insights into our data as well? First some rules:

1. Linearity: If a given linear system has an input $z_1(t)$ which corresponds to an output $y_1(t)$, and input $z_2(t)$ corresponds to output $y_2(t)$m then a summed input $z(t) = \alpha z_1(t) + \beta z_2(t)$, will produce an output $y(t) = \alpha y_1(t) + \beta y_2(t)$.

2. Time invariance: If an input is delayed in time by τ, then the output is as well: If $z(t) \to z(t+\tau)$, then $y(t) \to y(t+\tau)$.

3. Causality: If h(t) represents an impulse, then it should be zero for $t < 0$. A response cannot occur before the forcing.

4. Sequential application: If the output of one linear system is an input to a second system, then the frequency response is

$$H_{12}(f) = H_1(f) \cdot H_2(f) \tag{17}$$

So suppose we measure $y(t)$ and $z(t)$. Can we determine h or H? We know that

$$Y(f) = H(f)Z(f) \qquad (18)$$

Let's multiply both sides of the equation by the complex conjugate of Z to form the cross-spectrum:

$$\frac{Y(f)Z^*(f)}{T} = H(f)\frac{Z(f)Z^*(f)}{T} \qquad (19)$$

This becomes

$$G_{zy}(f) = H(f)G_{zz}(f) \qquad (20)$$

so

$$H(f) = \frac{G_{zy}(f)}{G_{zz}(f)} \qquad (21)$$

Salinity spiking examples

A classic example comes from the 'salinity spiking' problem, which arises due to the different response times of temperature and conductivity sensors:

$$S = S(T(t), C(t - \Delta t)). \qquad (22)$$

Our challenge now is to use the transfer (or gain) function to assess the salinity spiking, and maybe even to correct it. Conductivity is strongly dependent on temperature, and we have to remove the temperature effect to determine salinity. But the faster response times of conductivity sensors relative to temperature sensors are a source of confusion. Here's a basic procedure.

1. Identify a segment of the water column in which temperature and conductivity should be well behaved, with fluctuations due to temperature only. (This isn't essential, but it will give us a good shot at unraveling the sensor response time issues that lead to salinity or density spiking.) We'll identify the true values as T and C, and the measured values as $\hat{T}$ and $\hat{C}$.

2. Collect a lot of profiles of data.

3. Now treat this as a linear system:

$$
\begin{align}
\hat{T}(k) &= H_T(k)T(k) \qquad &(23) \\
\hat{C}(k) &= H_C(k)T(k) \qquad &(24) \\
& &(25)
\end{align}
$$

   Here $H_T(k)$ is the spatial/frequency response of the temperature sensor, and $H_C(k)$ is the spatial/frequency response of the conductivity sensor. The use of $T(k)$ in the conductivity equation might seem a little crazy, but it's really important, since we're asserting that salinity is unimportant in our (hypothetical) study region.

4. Compute cross spectra:

$$
\begin{align}
\hat{G}_{TT}(k) &= \frac{\langle \hat{T}^*(k)\hat{T}(k) \rangle}{N\Delta t} \qquad &(26) \\
\hat{G}_{CC}(k) &= \frac{\langle \hat{C}^*(k)\hat{C}(k) \rangle}{N\Delta t} \qquad &(27) \\
\hat{G}_{TC}(k) &= \frac{\langle \hat{T}^*(k)\hat{C}(k) \rangle}{N\Delta t} \qquad &(28) \\
& &(29)
\end{align}
$$

Then if we substitute in the expessions linking the observed values to the true values we obtain:

$$\frac{\langle \hat{T}^*(k)\hat{C}(k)\rangle}{N\Delta t} = \frac{\langle (H_T(k)T(k))^* H_C(k)T(k)\rangle}{N\Delta t} \tag{30}$$

$$= [H_T^*(k)H_C(k)]\frac{\langle T^*(k)T(k)\rangle}{N\Delta t} \tag{31}$$

$$\hat{G}_{TC}(k) = [H_T^*(k)H_C(k)]G_{TT}(k) \tag{32}$$

The same applies for the temperature spectrum:

$$\hat{G}_{TT}(k) = \frac{\langle \hat{T}^*(k)\hat{T}(k)\rangle}{N\Delta t} \tag{33}$$

$$= [H_T^*(k)H_T(k)]G_{TT}(k) \tag{34}$$

So the ratio of these becomes:

$$\frac{\hat{G}_{TC}(k)}{\hat{G}_{TT}(k)} = \frac{H_T^*(k)H_C(k)}{|H_T|^2} = \frac{H_C(k)}{H_T(k)} \tag{35}$$

This means that even without knowing the response function H, we can compute the ratio of the response functions from the transfer function, the ratio of the cross-spectrum to the spectrum.

An analogous relationship also holds:

$$\frac{\hat{G}_{TC}(k)}{\hat{G}_{CC}(k)} = \frac{H_T^*(k)H_C(k)}{|H_C|^2} = \frac{H_T(k)}{H_C(k)} \tag{36}$$

5. Now use this information to correct the conductivity sensor to have the same response as the temperature sensor. Here we'll define our corrected conductivity as $\hat{\hat{C}}(k)$, and we want to understand its relationship with the observed temperature $\hat{T}(k)$ and the true temperature $T(k)$.

$$\hat{\hat{C}}(k) = \alpha \hat{T}(k) = \alpha H_T(k)T(k) \tag{37}$$

This means we need a correction of the form:

$$\hat{\hat{C}}(k) = \hat{C}(k) \cdot P(k) = \alpha\hat{T}(k), \tag{38}$$

and our task is to figure out $P(k)$. We can also write:

$$\hat{C}(k) \cdot P(k) = H_C(k)T(k)P(k) \tag{39}$$

so putting this together:

$$\alpha H_T(k)T(k) = H_C(k)T(k)P(k) \tag{40}$$

Thus

$$P(k) = \frac{\alpha H_T(k)}{H_C(k)} = \alpha\frac{\hat{G}_{TC}(k)}{\hat{G}_{CC}(k)} \tag{41}$$

where α is real. So we do a bit of curve fitting to optimize our correction.

A typical correction might allow for errors both in the response time and a direct time lag:

$$\tau\frac{d\hat{T}(t)}{dt} + \hat{T}(t) = T(t - L) \tag{42}$$

(from Giles and McDougall, Deep-Sea Research, 1986) and this suggests corrections both in the frequency and time domain, either by minimizing phase differences or by maximizing correlation. We can Fourier transform this to find:

$$-i2\pi f\tau\mathcal{F}(\hat{T}) + \mathcal{F}(\hat{T}) = \mathcal{F}(T)e^{-i2\pi fL}, \tag{43}$$

implying that

$$(1 - i2\pi f\tau)e^{i2\pi fL}\mathcal{F}(\hat{T}) = \mathcal{F}(T). \tag{44}$$

Since conductivity has the fast response, one strategy is to treat $\hat{C}$ as behaving like the true temperature T. So hypothetically:

$$\hat{T}(k) = H_T(k)T(k) = \frac{1}{(1 - i2\pi f\tau)e^{i2\pi fL}}T(k) \tag{45}$$

$$\hat{C}(k) = H_C(k)T(k) = T(k) \tag{46}$$

$$\tag{47}$$

implying a correction P(k) of the form:

$$P(k) = \frac{\alpha H_T(k)}{H_C(k)} = \alpha\frac{1}{(1 - i2\pi f\tau)e^{i2\pi fL}}. \tag{48}$$

Transfer function: Practical implication

You might be wondering exactly how to compute a transfer function, so here's a practical example.

```
lambda=10; % 10 m wavelength
V=0.3; % 0.3 m/s propagation
n2s=0.2; % noise-to-signal ratio
time=(1:5000)';
x=n2s*randn(5000,1)+cos(2*pi/lambda*V*time);
y=n2s*randn(5000,1)+cos(2*pi/lambda*V*(time)+pi/2);


% start with a segmented data approach:
xx=[reshape(x,500,10) reshape(x(251:4750),500,9)];
yy=[reshape(y,500,10) reshape(y(251:4750),500,9)];
fxx=fft(xx.*(hanning(500)*ones(1,19)));
fyy=fft(yy.*(hanning(500)*ones(1,19)));
sxx=abs(fxx(1:251,:)).^2;
syy=abs(fyy(1:251,:)).^2;
sxy=conj(fxx(1:251,:)).*fyy(1:251,:);
coher1=abs(mean(sxy,2)).^2./(mean(sxx,2).*mean(syy,2));
phase1=atan2(-imag(mean(sxy,2)),real(mean(sxy,2)));
```

```
semilogx(0:250,coher1)
transfer=mean(sxy,2) ./mean(sxx,2);
semilogx(0:250,coher1,0:250,abs(transfer))
set(gca,'FontSize',14)
xlabel('frequency','FontSize',14)
ylabel('coherence or transfer function','FontSize',14)
legend('coherence','transfer function amplitude')
```

In this case, you'll see that the transfer function peak happens to be about the same size as the coherence, while the background noise for the transfer function is significantly larger than the background noise for the coherence. These results are entirely a function of the data that we use. The high background noise level stems from the fact that the coherence is scaled by x, which can be very small.
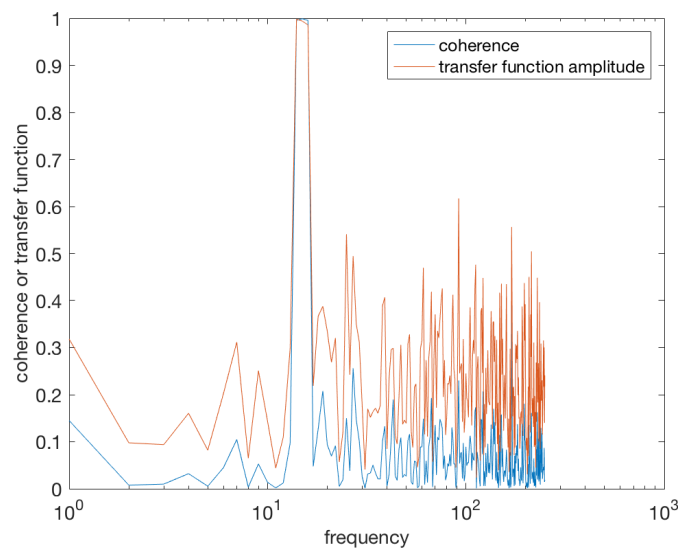

Figure 1: Transfer function example (red) compared with coherence (blue) for an artificial data sets consisting of white noise plus either a sine or cosine at one frequency.

Noise and coherence

Finally, let's take a quick look at what noise does to coherence and to the transfer function. Suppose we start with two related signals:

$$y(t) = c_1 x(t) + n(t), \tag{49}$$

where $n(t)$ is noise. Can we determine c_1 and $n(t)$ using the cross-spectrum? Formally, the cross spectrum is:

$$S_{XY}(f) = \frac{\langle X^*(f)Y(f) \rangle}{T} \tag{50}$$

$$= \frac{c_1 \langle X^*(f)X(f) \rangle + \langle X^*(f)N(f) \rangle}{T} \tag{51}$$

$$= c_1 S_{XX} \tag{52}$$

The squared coherence is of course:

$$\text{Coh}(f)^2 = \gamma_{XY}^2 = \frac{|S_{XY}|^2}{S_{XX}S_{YY}} \tag{53}$$

So what is S_{YY}? We can compute the spectrum of y:

$$S_{YY} = \frac{[c_1^2 \langle X^* X \rangle + c_1(\langle X^* N \rangle + \langle X N^* \rangle + \langle N^* N \rangle)]}{T} \tag{54}$$

$$= c_1^2 S_{XX} + S_{NN}, \tag{55}$$

where X and N are assumed to be uncorrelated. Then the squared coherence is:

$$\gamma_{XY}^2(f) = \frac{c_1^2 S_{XX}(f)^2}{S_{XX}(c_1^2 S_{XX} + S_{NN})} \tag{56}$$

$$= \frac{1}{\left(1 + \frac{S_{NN}(f)}{c_1^2 S_{XX}(f)}\right)} \tag{57}$$

The final term in the denominator is a measure of the noise-to-signal ratio. (In our example, we imposed it from the beginning.) So if we knew a lot about the causal relations between our records, we could use the coherence to extract a measure of the noise-to-signal ratio.

Noise and the transfer function

Now let's introduce noise in the Fourier transform domain, which perhaps implies a slightly more complicated relationship between x and y:

$$Y(f) = X(f)H(f) + N(f) \tag{58}$$

If we multiply through by X^*:

$$\langle X^* Y \rangle = \langle X^* X \rangle H(f) + \langle X^* N \rangle \tag{59}$$

Since the signal is uncorrelated with noise, this still gives us

$$H(f) = \frac{G_{xy}(f)}{G_{xx}(f)}, \tag{60}$$

so noise appears to have no impact on the results, but is it all so rosy?

Alternatively, we might imagine that our forcing x is noisy, so that

$$Y(f) = [X(f) + N(f)]H(f) \tag{61}$$

In doing this, we assume that the noise associated with X is uncorrelated with the signal Y—in other words, we assume that the response $y(t)$ should be responding to $x(t)$, but we've mismeasured the forcing as $x(t) + n(t)$. Then:

$$\langle (X + N)^* Y \rangle = \langle (X + N)^* (X + N) \rangle H(f) \tag{62}$$

And since y and n are uncorrelated in this formulation,

$$\hat{H}(f) = \frac{G_{xy}(f)}{G_{xx}(f) + G_{nn}(f)}, \tag{63}$$

which is biased low relative to the true response. (But the phase is unbiased.) This is analogous to the noise-related formulation that we looked at for coherence.

We could test all of this with an example:

```
x2=x+randn(5000,1);
xx2=[reshape(x2,500,10) reshape(x2(251:4750),500,9)];
fxx2=fft(xx2.*(hanning(500)*ones(1,19)));
sxx2=abs(fxx2(1:251,:)).^2;
sxy2=conj(fxx2(1:251,:)).*fyy(1:251,:);
transfer2=mean(sxy2,2) ./mean(sxx2,2);

transfer3=mean(sxy2,2) ./mean(syy,2);

semilogx(0:250,abs(transfer),0:250,abs(transfer2),0:250,abs(transfer3))
set(gca,'FontSize',14)
xlabel('frequency','FontSize',14)
ylabel('transfer function','FontSize',14)
legend('H_{xy}','H_{x+n,y}','H_{y,x+n}')
```
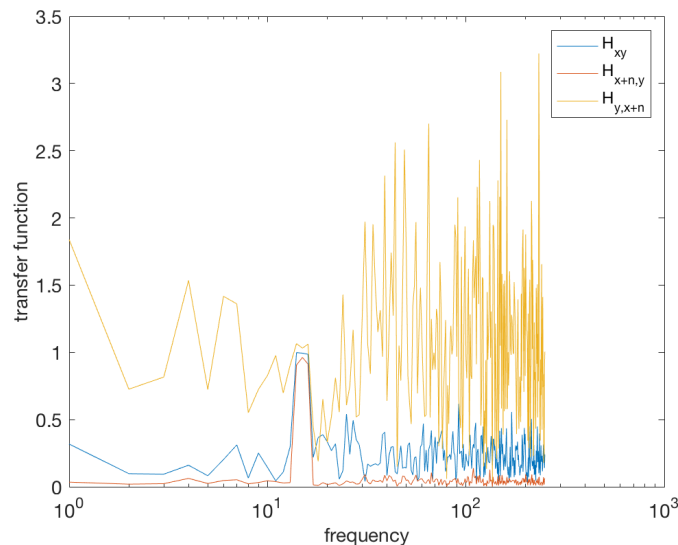

Figure 2: Transfer function for baseline case, vs examples with noise added. Noise substantially modifies the transfer function, and its impact depends on whether it appears in the denominator.

Uncertainties for transfer function

Finally we should comment briefly on the uncertainty of the transfer function or gain. Bendat and Piersol go through a detailed discussion of this (see their section 9.2.4) and ultimately write that the relative uncertainty of the transfer function is:

$$\epsilon\left[|\hat{H}_{xy}|\right] = \frac{s.d.\left[\left|\hat{H}_{xy}\right|\right]}{|H_{xy}|} \approx \frac{(1-\gamma_{xy}^2)^{1/2}}{|\gamma_{xy}|\sqrt{2n_d}}, \tag{64}$$

where $s.d.$ indicates the standard deviation. This expression is in fact the same as Bendat and Piersol's expression for the standard deviation of the phase.
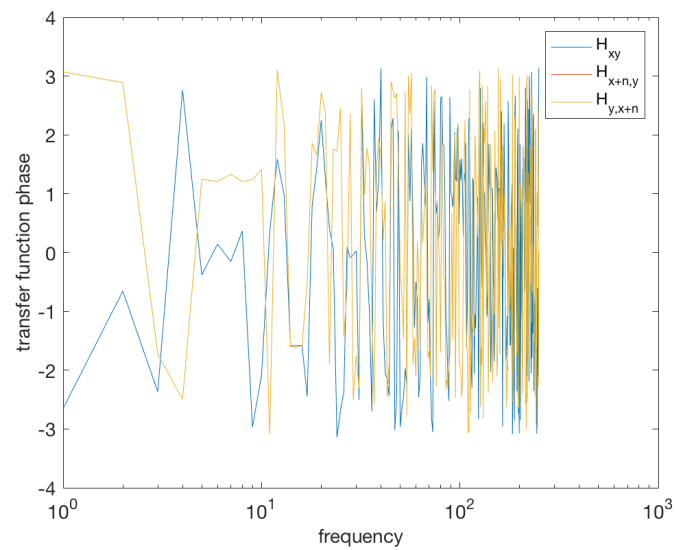
Figure 3: Phase for the transfer function examples shown in Figure 2. Phase depends on the signal, but does not depend on the spectral normalization that appears in the denominator, so the red and yellow lines are the same.