Negative diffusion: jets, steps and layers

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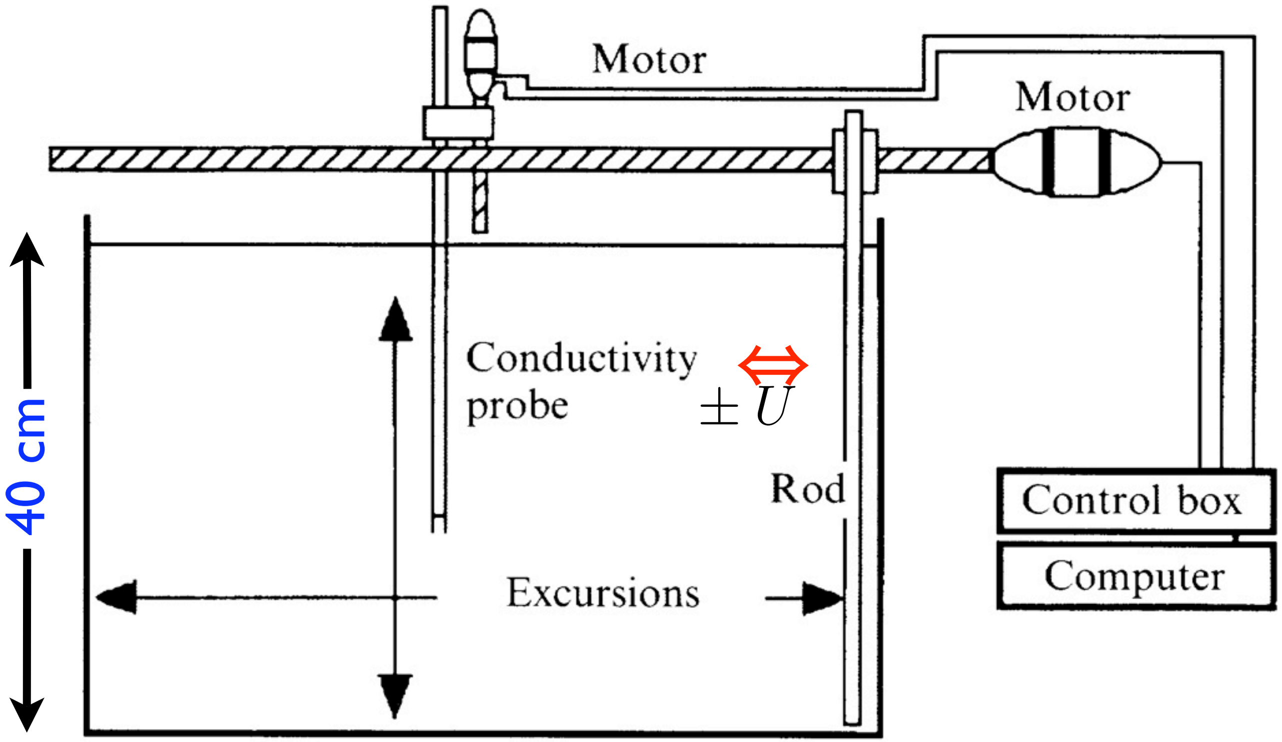
An experiment: mixing a stable salt gradient



Park, Whitehead & Gnanadeskian (1994)
Also Ruddick, McDougall & Turner (1989)
Rehmann & Koseff (2004a,b)
Linden (1978), Holford & Linden (1997)

Start with a strongly stable density gradient:

$$b = N^2 z$$

With molecular diffusion, the mixing time is:

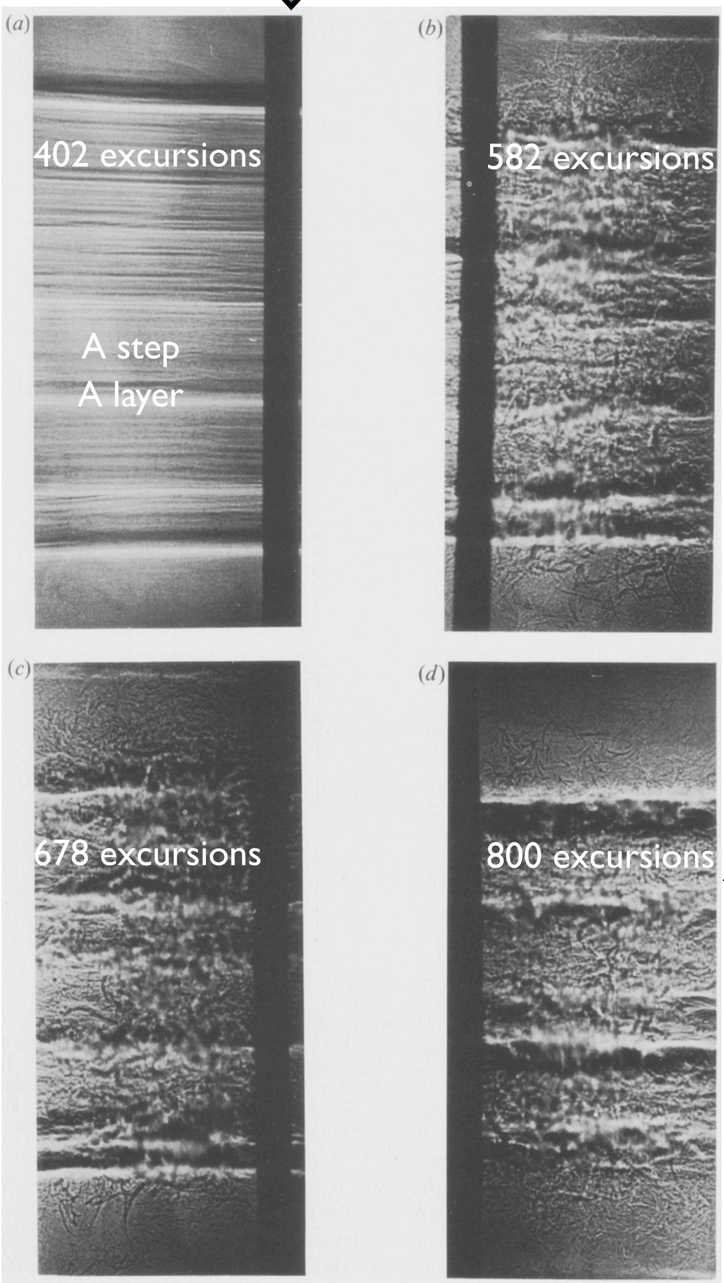
$$t_{\text{diff}} = \frac{H^2}{\kappa} \geq 1 \,\text{year}$$

With stirring, the experiment lasts a few hours.

$$\rho = \rho_0 \left[1 - g^{-1} b\right] \Leftarrow \text{"buoyancy"} \Rightarrow b = -g\beta(S - S_0)$$

Turbulence in the wake of the rod

Rod, D=2.26cm



$$\sqrt{Ri} = \frac{ND}{U} = 0.56$$

$$Re = \frac{UD}{\nu} = 547$$

(weakly stratified)

Shadowgraph visualization of the turbulent wake

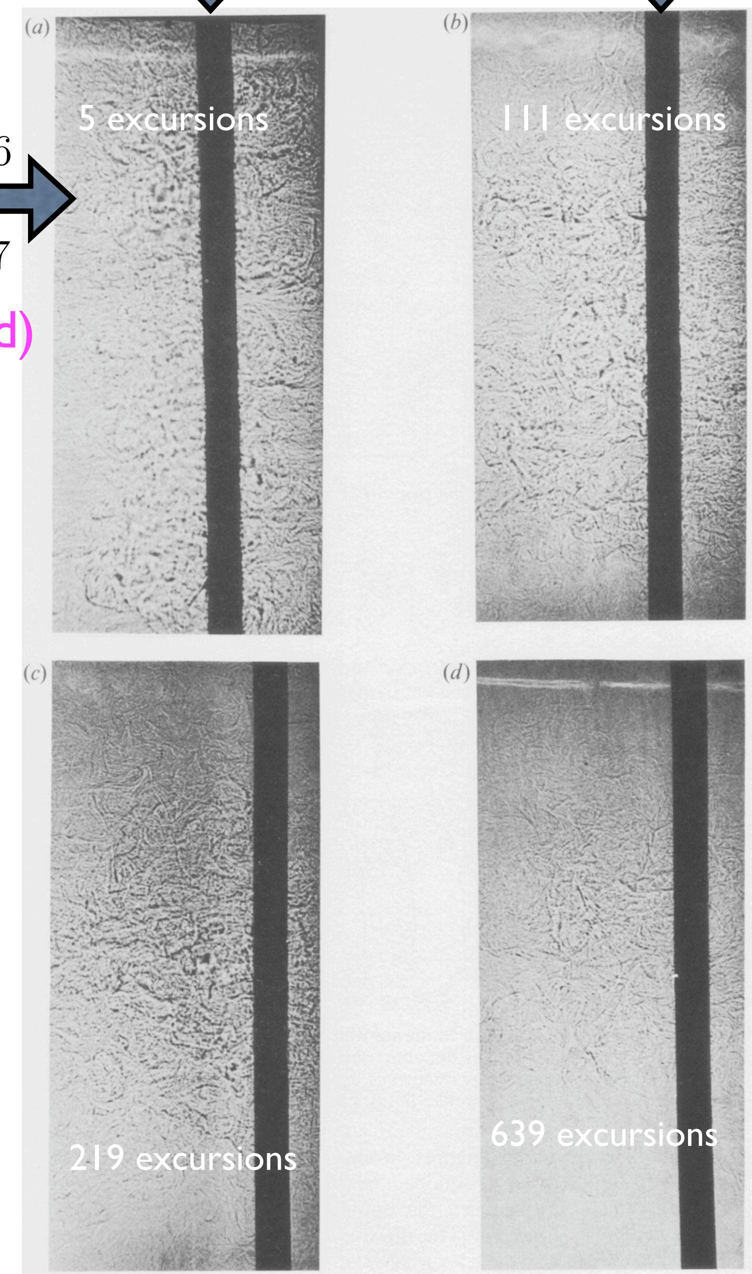
$$\sqrt{Ri} = \frac{ND}{U} = 1.46$$

$$Re = \frac{UD}{\nu} = 547$$

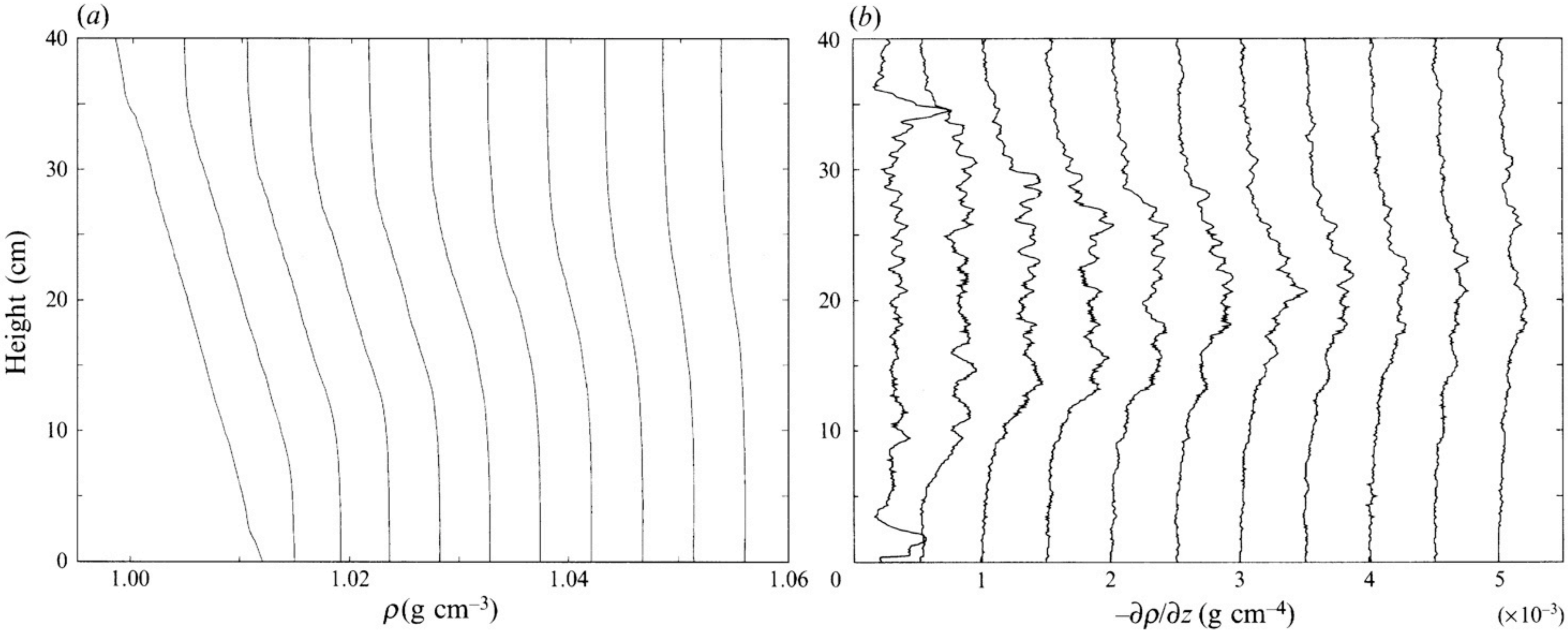
(strongly stratified)

From PWG94

Rod, D=2.26cm



Strong stirring, PWG94

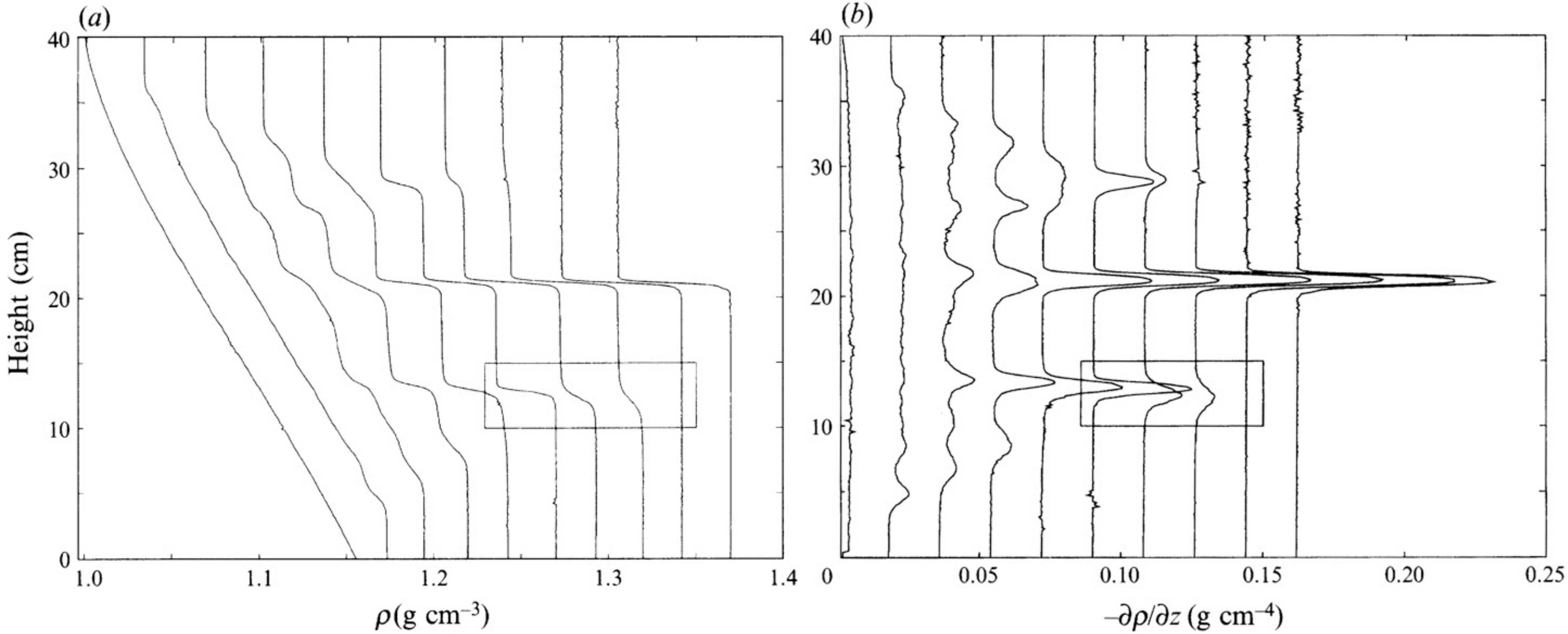


$$\sqrt{Ri} = \frac{ND}{U} = 0.56 \quad \text{and} \quad Re = \frac{UD}{\nu} = 547$$

Weakly stratified: turbulent mixing looks like accelerated molecular diffusion.

$\rho = \rho_0 \left[1 - g^{-1} b\right]$ ⇦ "buoyancy" ⇨ $b = -g\beta(S - S_0)$

Moderate stirring PWG94: layer formation

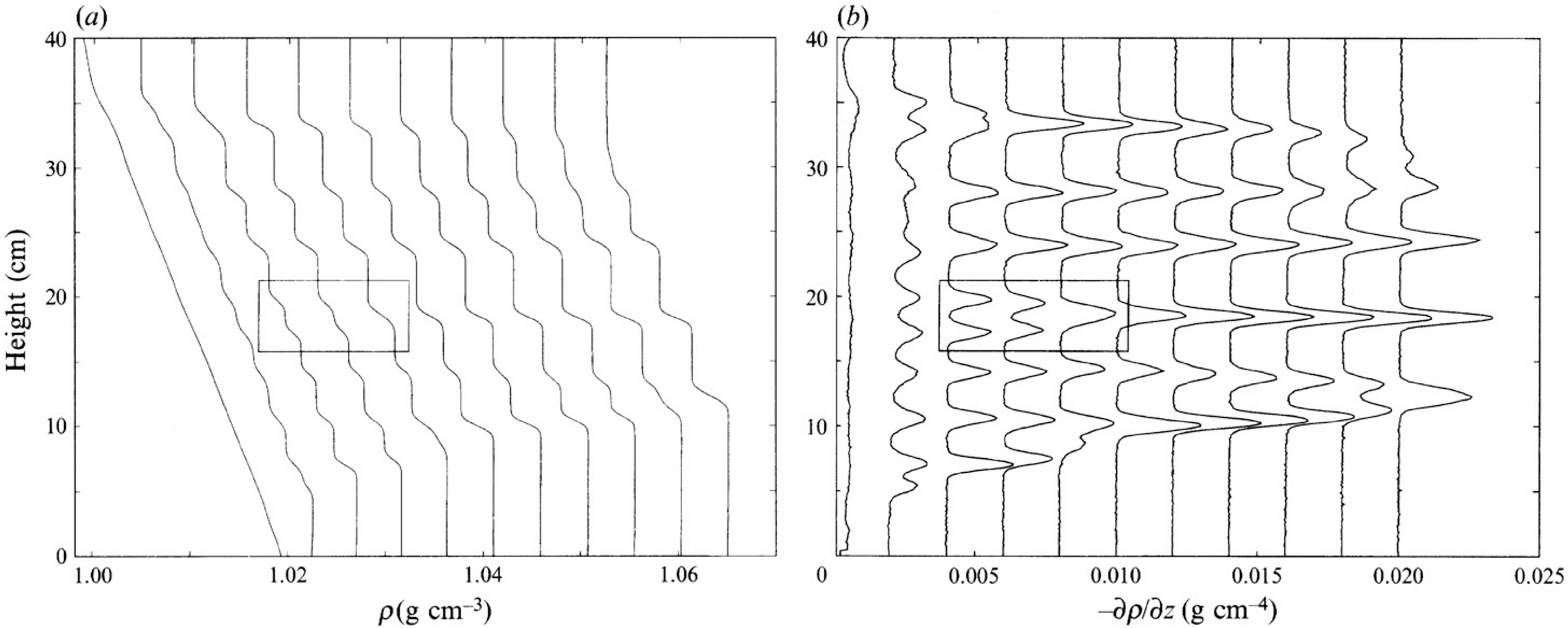


$$\sqrt{Ri} = \frac{ND}{U} = 1.83 \quad \text{and} \quad Re = \frac{UD}{\nu} = 547$$

Moderately stratified: turbulent mixing amplifies buoyancy gradients.

$\rho = \rho_0 \left[1 - g^{-1} b\right]$ ⇦ "buoyancy" ⇨ $b = -g\beta(S - S_0)$

Weak stirring PWG94: the interior regime

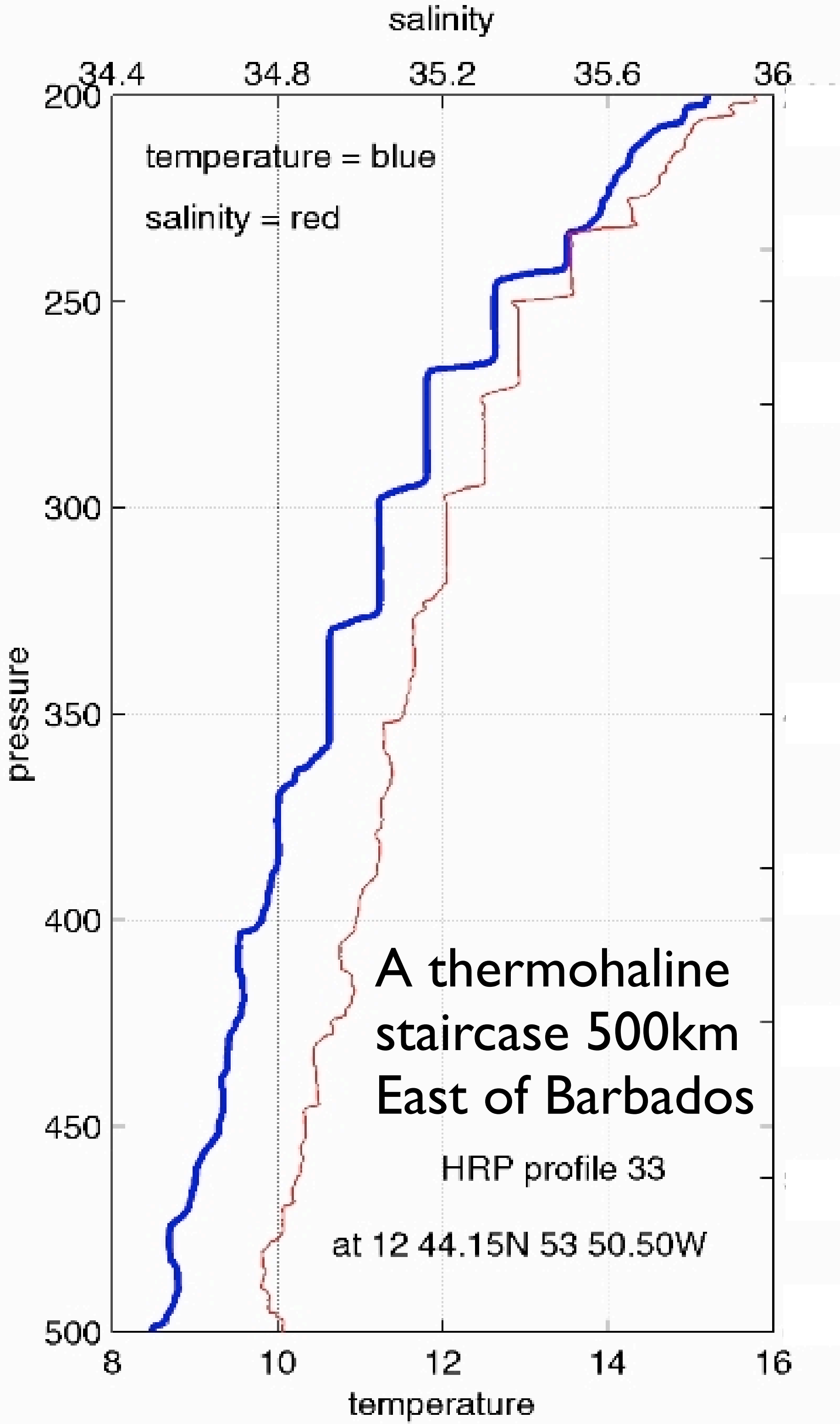


$$\sqrt{Ri} = \frac{ND}{U} = 1.64 \quad \text{and} \quad Re = \frac{UD}{\nu} = 226$$

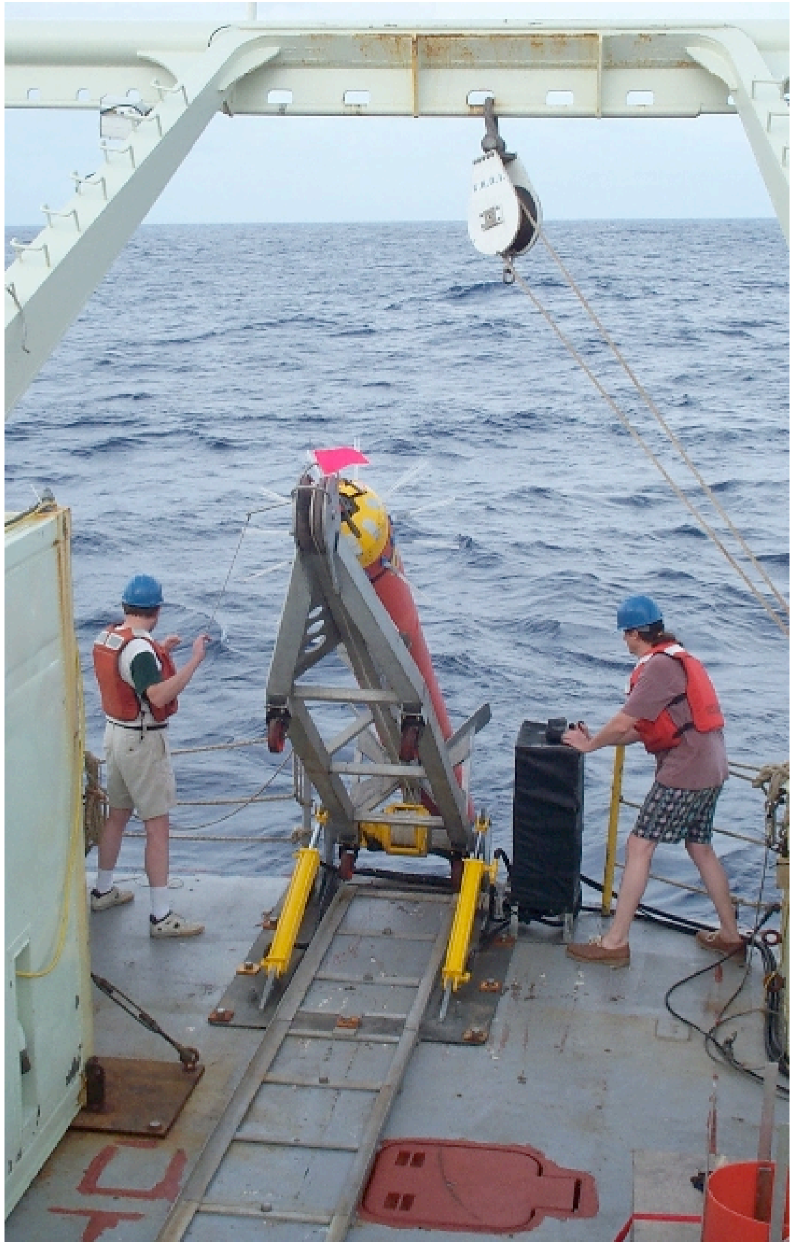
Strongly stratified: steps and layers everywhere....

$$\rho = \rho_0 \left[1 - g^{-1} b\right] \Leftarrow \text{"buoyancy"} \Rightarrow b = -g\beta(S - S_0)$$

Steps & layers in the ocean



A thermohaline staircase 500km East of Barbados

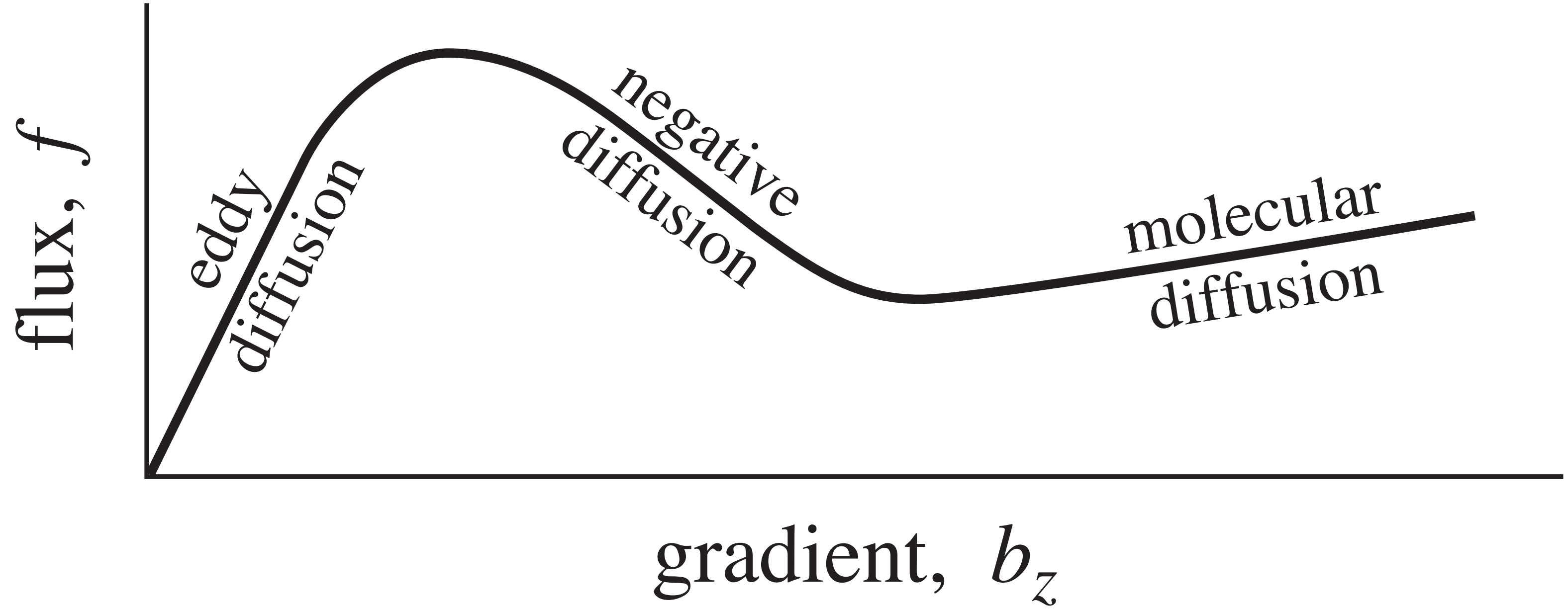


WHOI Ocean Mixing Group

“Is strongly stratified turbulence stable?”

(Phillips 1972, Posmentier 1977)

No, not if stable buoyancy gradients suppress mixing: the turbulent flow spontaneously becomes spatially inhomogeneous.



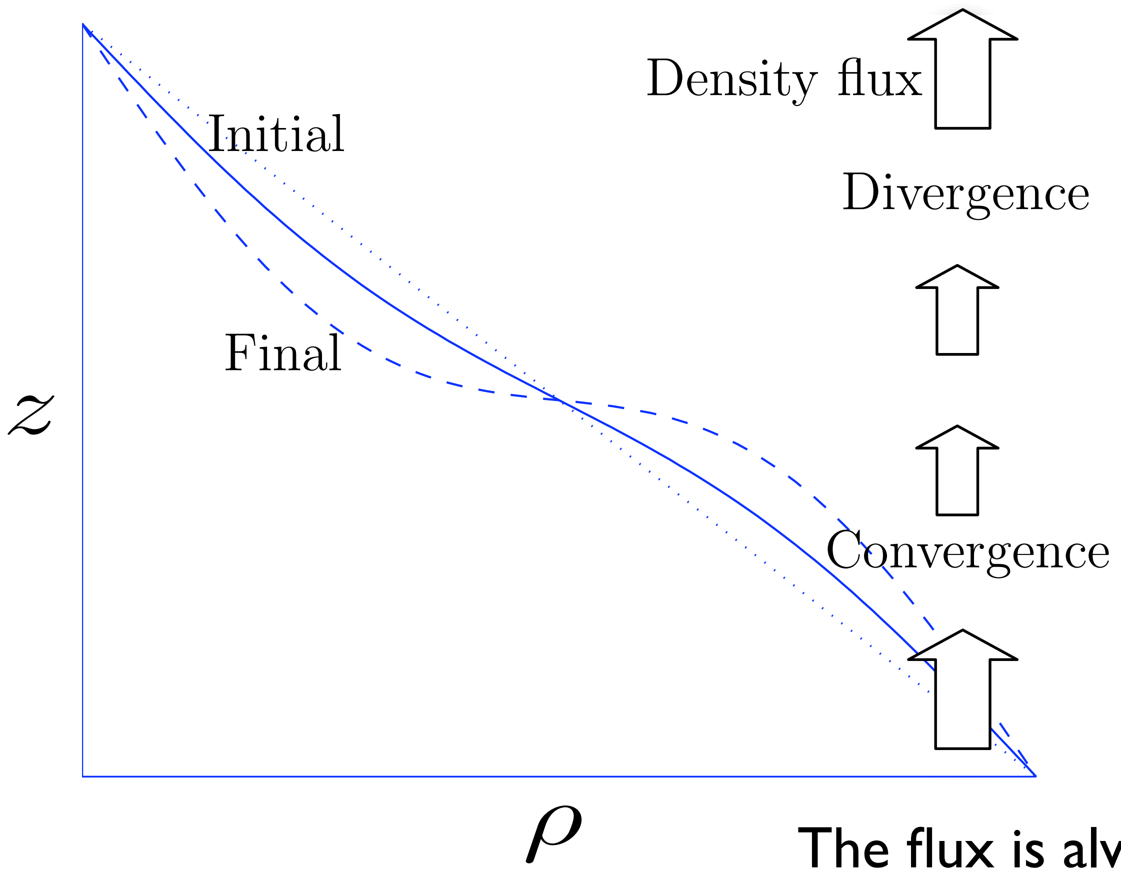
Conservation of mean buoyancy:

$$b_t = \frac{\partial f}{\partial z} = f'(b_z) b_{zz} \,, \qquad f = -\overline{w'b'} + \kappa b_z$$

The linear instability of Phillips & Posmentier

P&P explain the onset of layering as a negative-diffusion instability:

$$b_t = f_z \quad \text{and} \quad b = N^2 z + \hat{b}_z \quad \Rightarrow \quad \hat{b}_t = f'\left(N^2\right)\hat{b}_{zz}$$



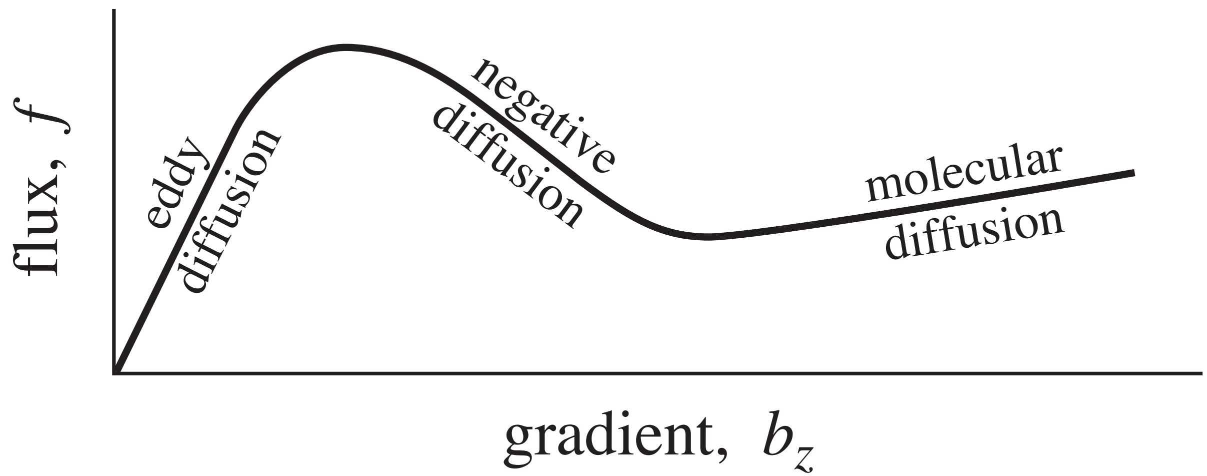
The flux is always down-gradient.

The diffusivity is:

$$D = \frac{\mathrm{d}f}{\mathrm{d}g}$$

(not f/g!?)

Problems with the P & P model



$$b_t = f(b_z)_z$$

(1) There is no mechanism to arrest interface steepening.

(2) The problem is ill-posed: there is no high wavenumber cut-off.

$$\hat{b}_t = f'\left(N^2\right)\hat{b}_{zz}$$

Beyond the linear instability?

To explain the experiments we need a more elaborate model (BLSY1998):

$$e(z,t) = \text{TKE} \qquad \text{and} \qquad b(z,t) = \text{mean buoyancy}$$

Physical arguments suggest the model:

$$b_t = (\ell e^{1/2} b_z)_z \qquad e_t = \beta(\ell e^{1/2} e_z)_z - \ell e^{1/2} b_z - \alpha \ell^{-1} e^{3/2} + \mathcal{P}$$

conversion TKE→PE

dissipation of TKE, ε

The mixing length:

$$\frac{1}{\ell^2} = \frac{1}{D^2} + \gamma \frac{b_z}{e}$$

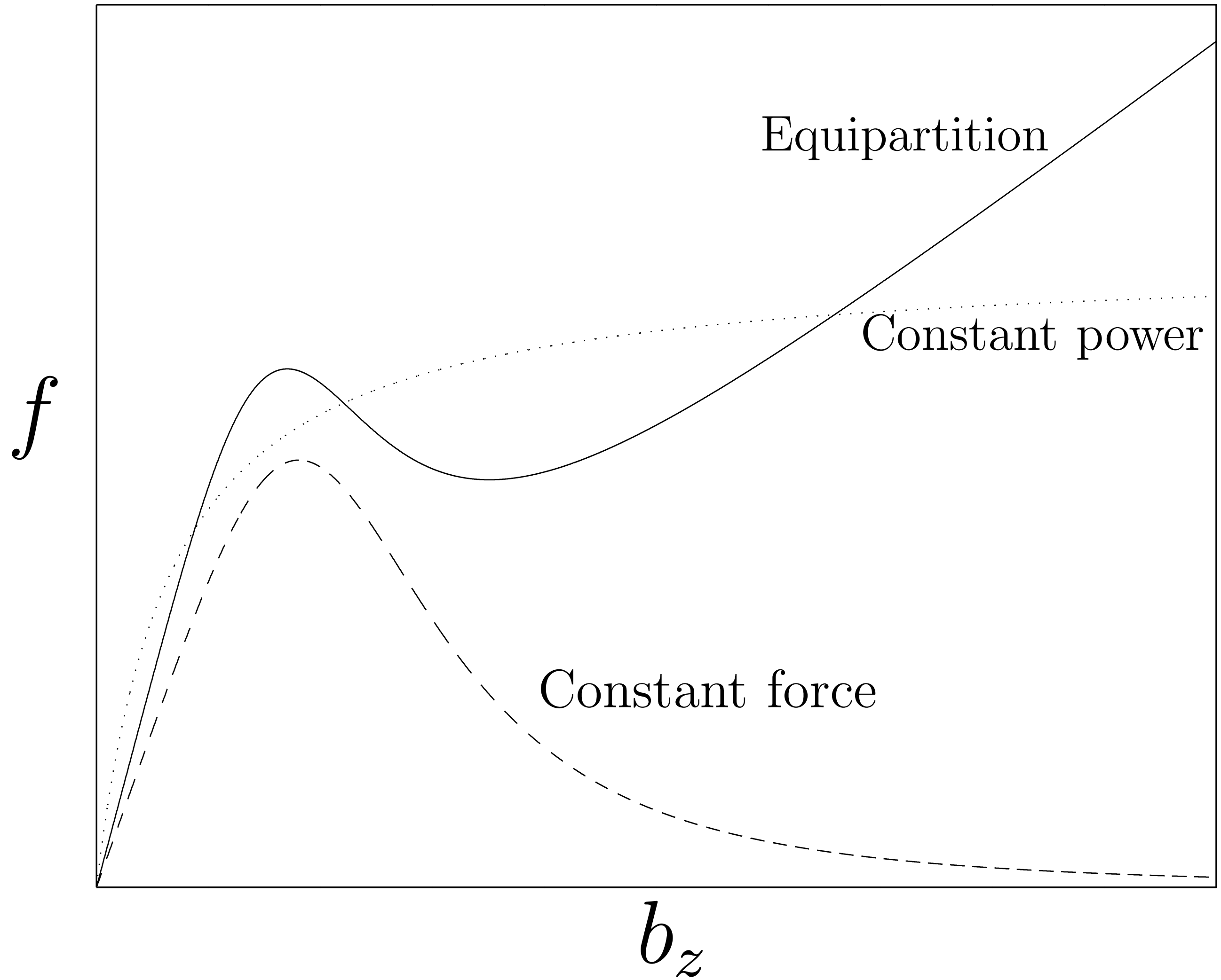
Conservation of energy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^H e - zb \, \mathrm{d}z = \int_0^H \mathcal{P} - \alpha \ell^{-1} e^{3/2} \, \mathrm{d}z$$

We must specify the energy production $\mathcal{P}$......

Completing the model

$$b_t = (\ell e^{1/2} b_z)_z \qquad e_t = \beta(\ell e^{1/2} e_z)_z - \ell e^{1/2} b_z - \alpha \ell^{-1} e^{3/2} + \mathcal{P}$$



The most satisfactory assumption is:

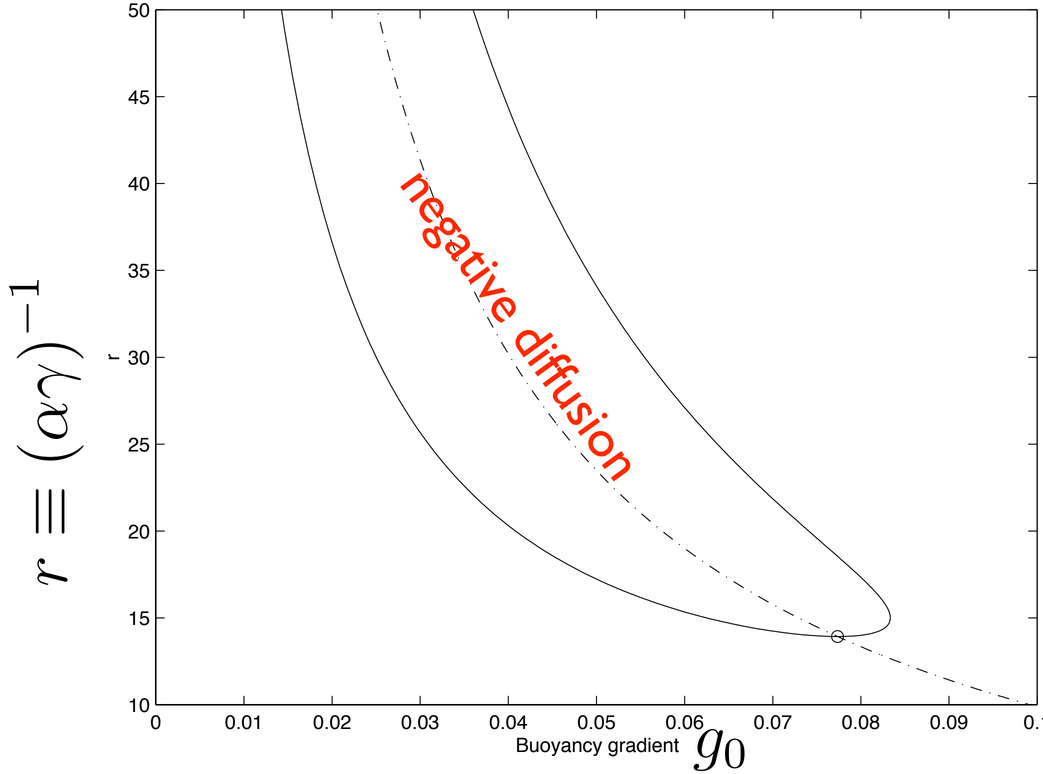
$$\mathcal{P} = \alpha \frac{e^{1/2}}{\ell} U^2$$

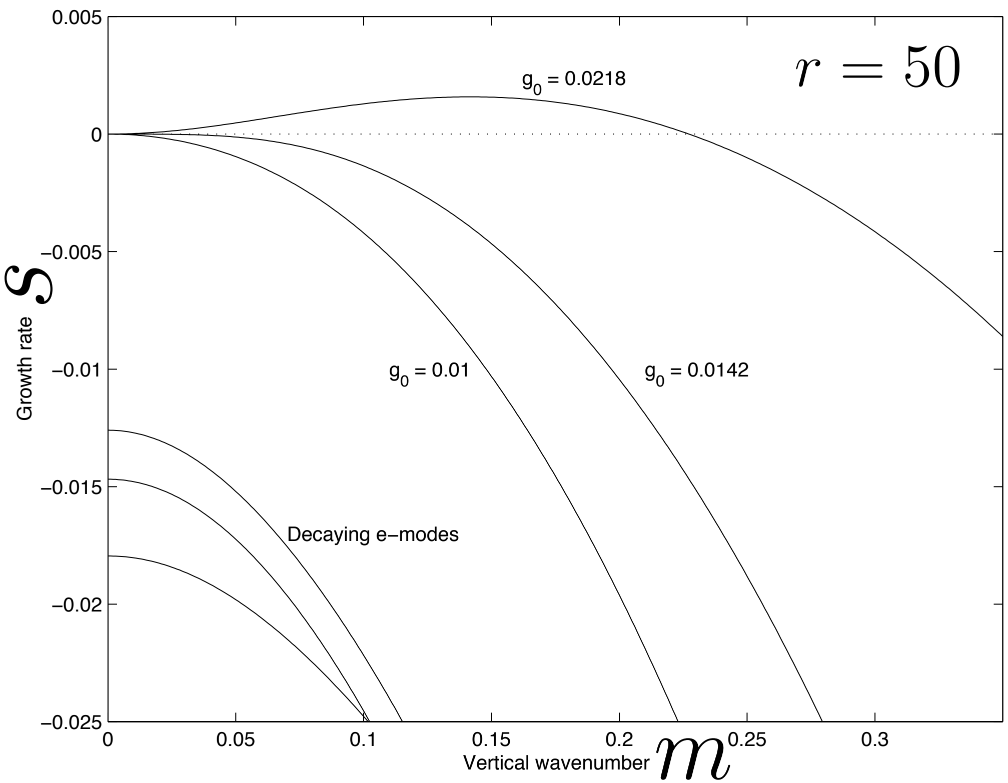
TKE density relaxes to the rod speed:

$$e_t = \cdots + \frac{e^{1/2}}{\ell} \left(U^2 - e\right)$$

eddy turnover rate

Linear stability of the spatially uniform solution





The non-dimensional system is:

$$
\begin{aligned}
g_t &= \left(\ell e^{1/2} g\right)_{zz}, \\
e_t &= \beta\left(\ell e^{1/2} e_z\right)_z - \ell e^{1/2} g + r^{-1}\ell^{-1}(1-e)e^{1/2}, \\
\ell &= \left(\frac{e}{e+g}\right)^{1/2}, \quad \text{and} \quad g \equiv b_z .
\end{aligned}
$$

Linearize around the homogeneous state:

$$g = g_0 + g_1 \mathrm{e}^{st+\mathrm{i}mz}$$

$$e = e_0 + e_1 \mathrm{e}^{st+\mathrm{i}mz}$$

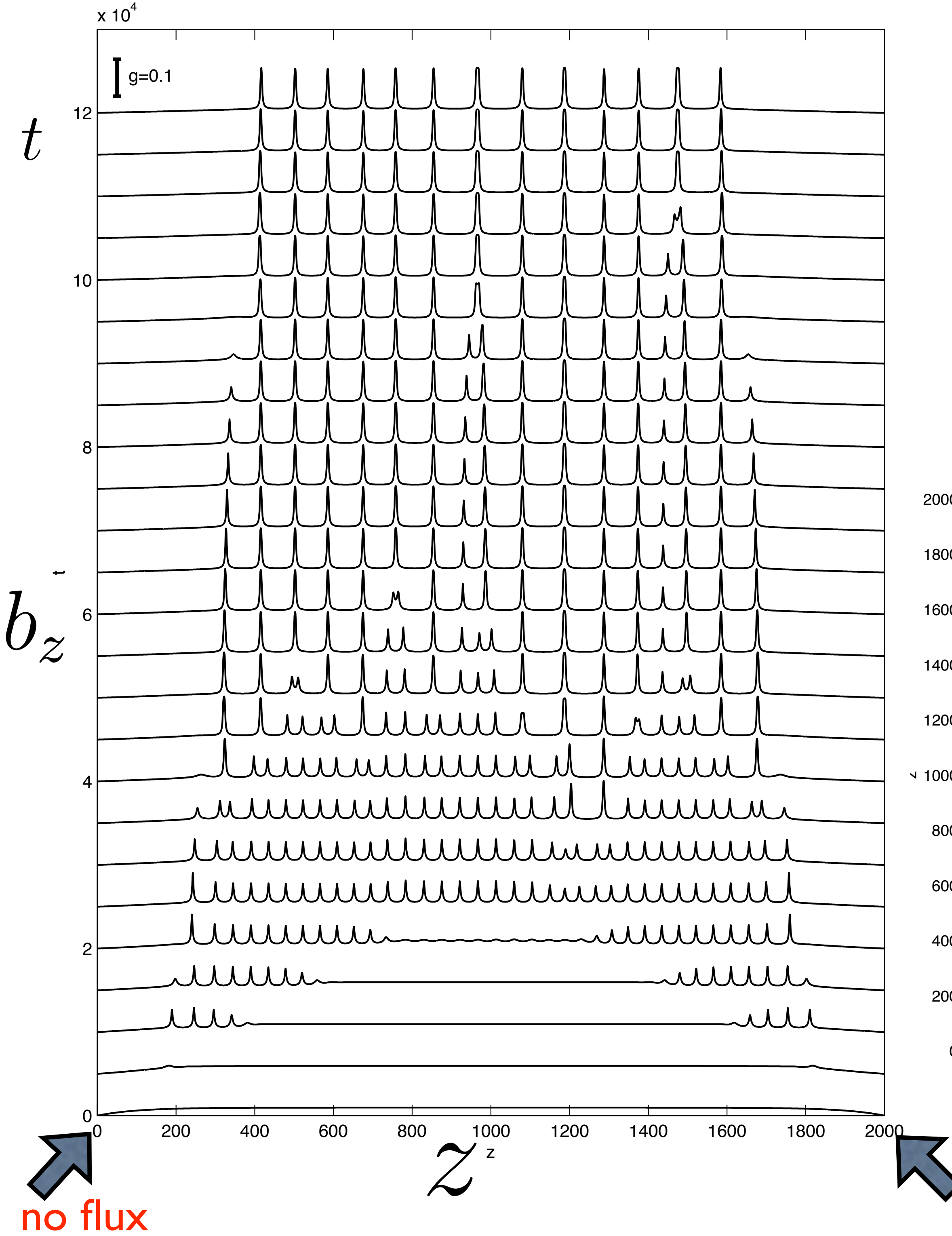
Note β is necessary for the high wavenumber cut-off.

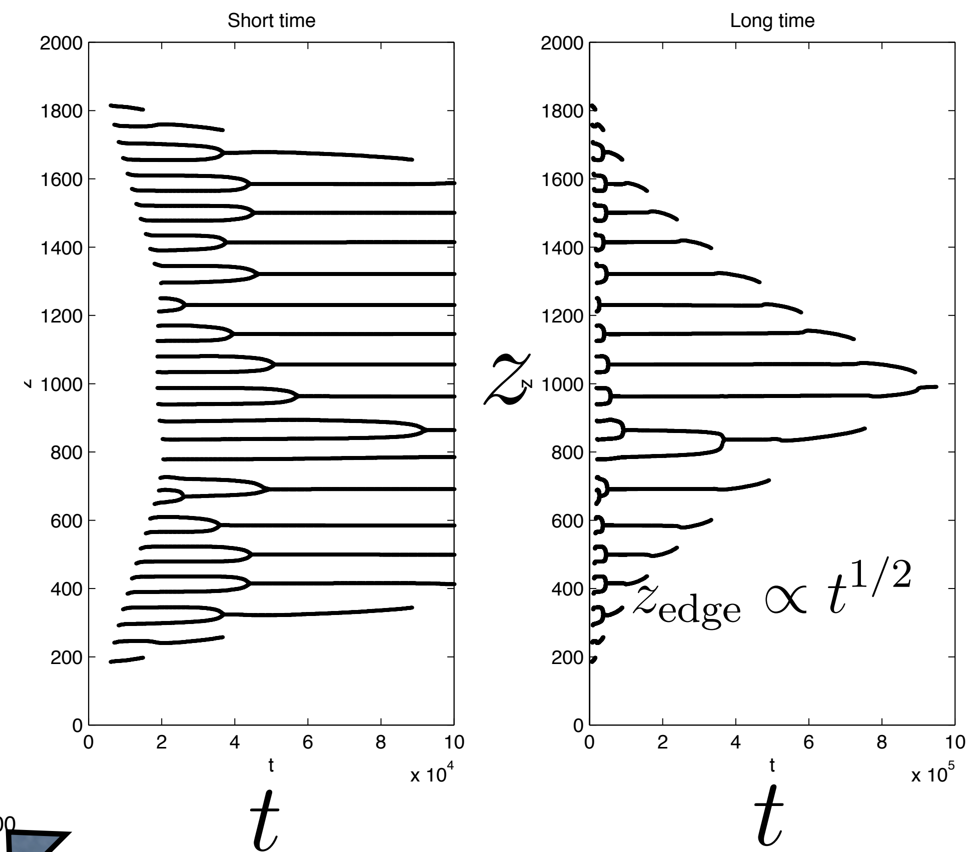
A solution of the model

(1) linear instability (P&P).

(2) pattern development via merger.

(3) invasion of the edge layers and ultimate homogenization.

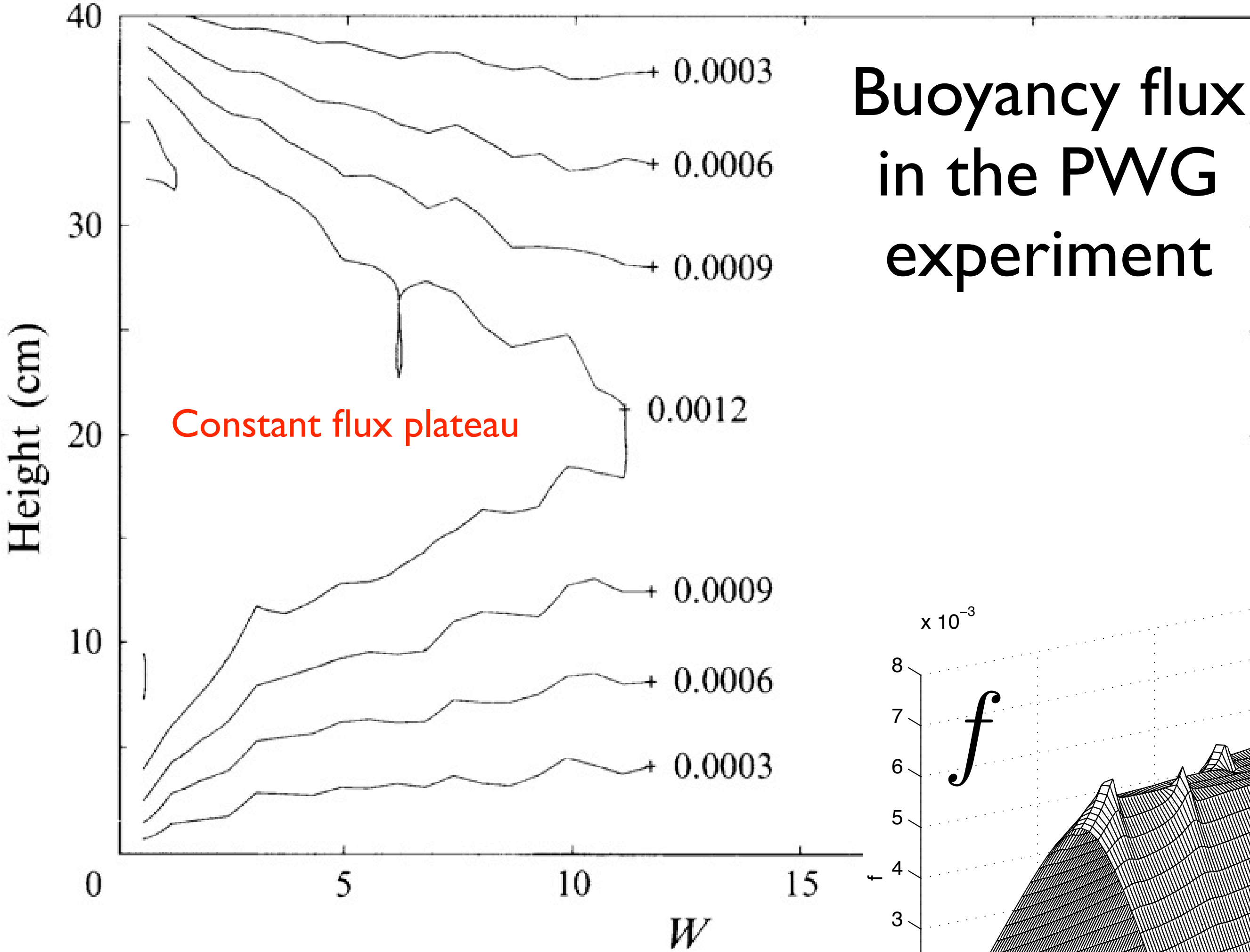




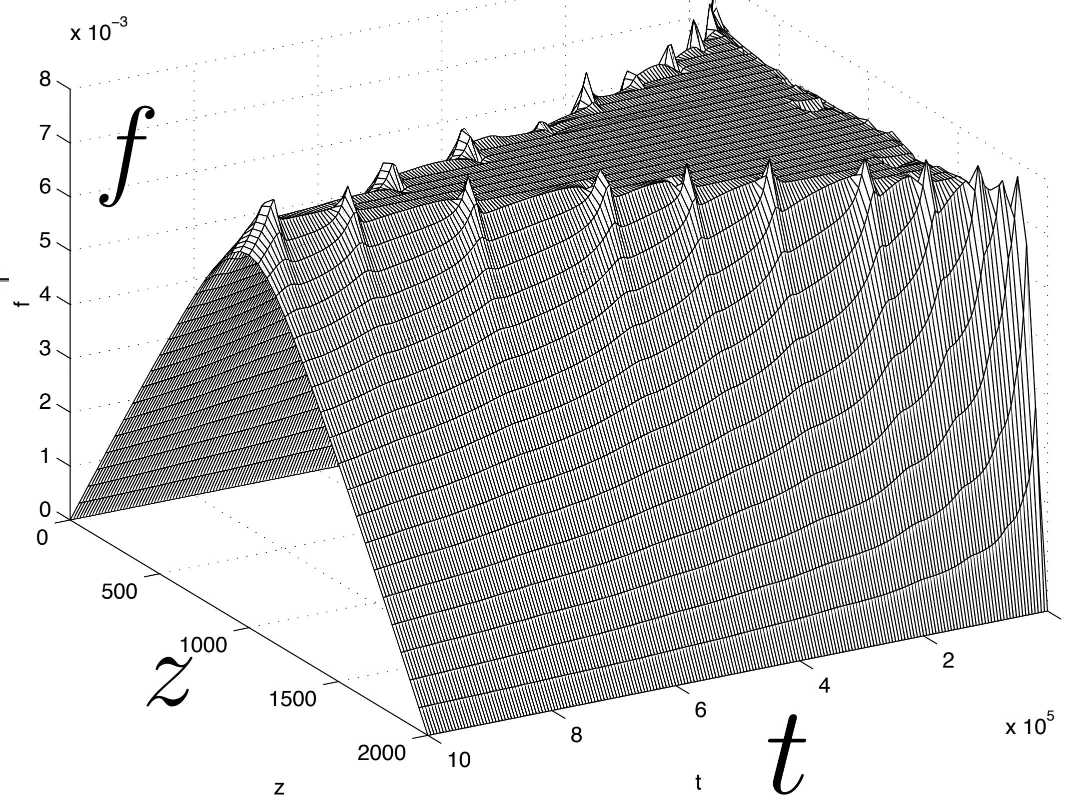
$r = 50$

The spiky interior has constant flux

Buoyancy flux in the PWG experiment

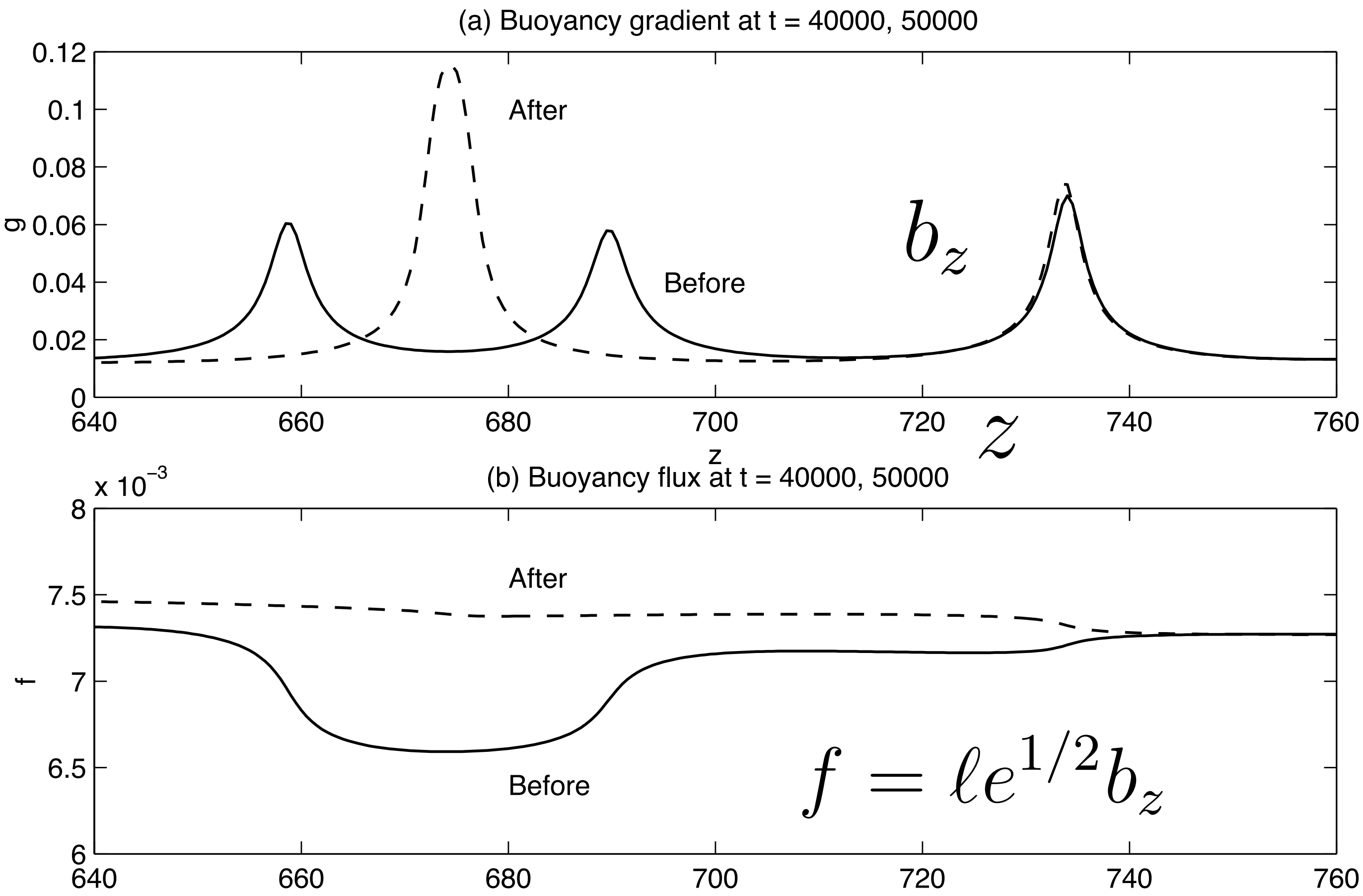


Buoyancy flux in the BLSY model



$$f = \ell e^{1/2} b_z$$

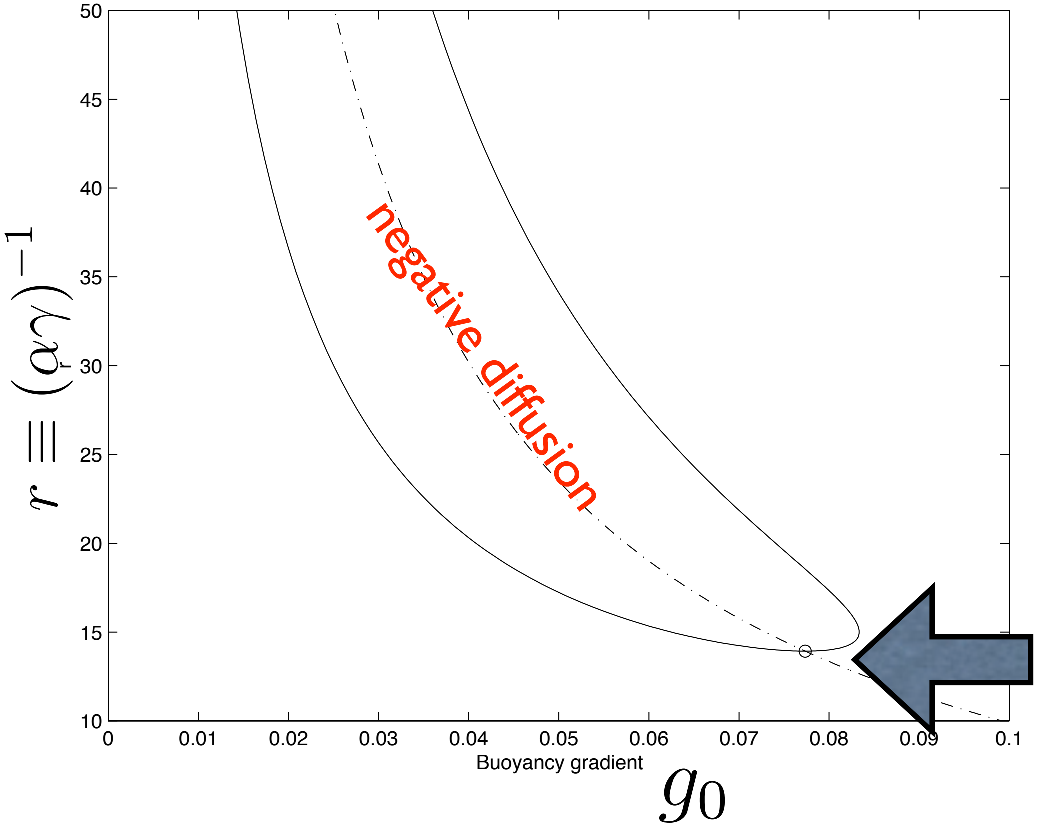
Mergers in the interior maintain constant flux



If the flux is slightly non-uniform then the steps slowly migrate, leading to layer merger.

This merger process keeps the interior flux close to constant (and we can calculate the constant).

General lessons?



At the bottom of the banana the system can be reduced via an amplitude expansion. The critical point is:

$$r_\star = 7 + 4\sqrt{3}\,, \qquad g_{0\star} = \frac{1}{\sqrt{3}} - \frac{1}{2}$$

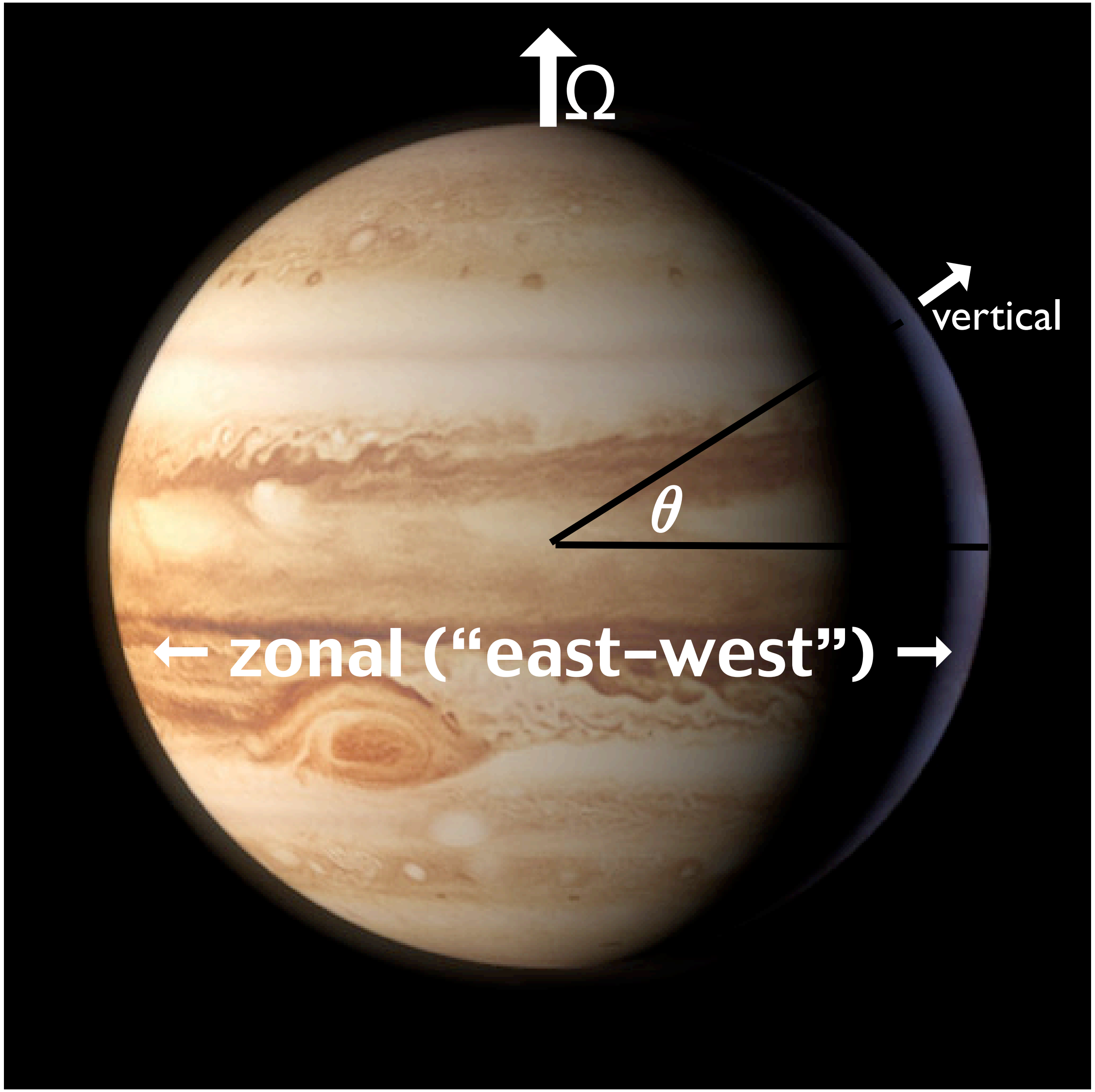
The expansion begins like this:

$$r = r_\star + \epsilon^2 r_2 \qquad g = g_{0\star} + \epsilon\left[g_1 + A(z,t)\right] + O(\epsilon^2)$$

and leads to the Cahn-Hilliard equation:

$$A_t = \left[\pm A - A_{zz} \pm A^2 + A^3\right]_{zz}$$

The CH equation - derived by CH from thermodynamic arguments - is the simplest complete description of negative diffusion (and negative viscosity).....



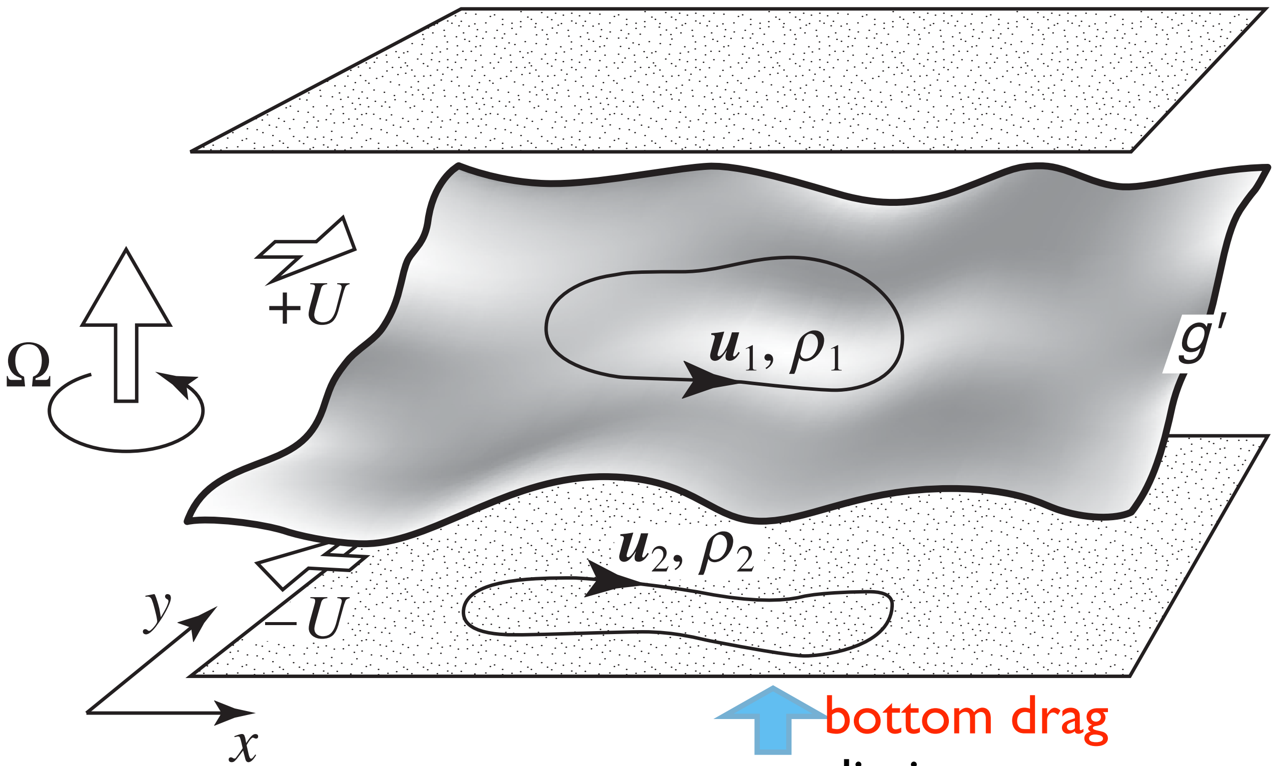
Negative viscosity: the prime example is zonal flows in the ocean and atmosphere.

This is a fluid "stratified" by vertical vorticity

In a thin spherical layer only the local-vertical component of Ω is important (the β effect).

$$2\Omega_{\mathrm{vert}} = 2\Omega \times \sin\theta \approx \underbrace{2\Omega\sin\theta_0}_{f_0} + \underbrace{2\Omega R^{-1}\cos\theta_0 \times R(\theta-\theta_0)}_{\beta\times y}$$

Stratified turbulence in a rapidly rotating β-plane fluid



Available Potential Energy in the tilted interface is released by baroclinic instability.

$$\lambda = \frac{\sqrt{2g'H}}{f_0} \sim 10 \text{ to } 40 \text{ km}$$

ocean

bottom drag dissipates energy

Geostrophic balance: $(u_1, v_1) = \left(-\frac{\partial \psi_1}{\partial y}, \frac{\partial \psi_1}{\partial x}\right)$ $\psi_1 \propto$ pressure in layer 1

The quasigeostrophic approximation

$$q_{1t} + Uq_{1x} + \left(\beta + \lambda^{-2}U\right)\psi_{1x} + J(\psi_1, q_1) = -\nu\nabla^8 q_1 \,,$$

$$q_{2t} - Uq_{2x} + \left(\beta - \lambda^{-2}U\right)\psi_{2x} + J(\psi_2, q_2) = -\kappa\nabla^2\psi_2 - \nu\nabla^8 q_2 \,.$$

Potential Vorticity: $q_n = \psi_{nxx} + \psi_{nyy} + \frac{1}{2}\lambda^{-2}(-)^n(\psi_1 - \psi_2)$

From the above, we obtain the energy equation:

$$U^2\lambda^{-2}\langle\psi_x\tau\rangle \approx \kappa\langle|\boldsymbol{\nabla}\psi_2|^2\rangle \quad \text{where} \quad \langle\rangle = \text{average over } x,\ y \text{ and } t$$

heat flux

bottom drag

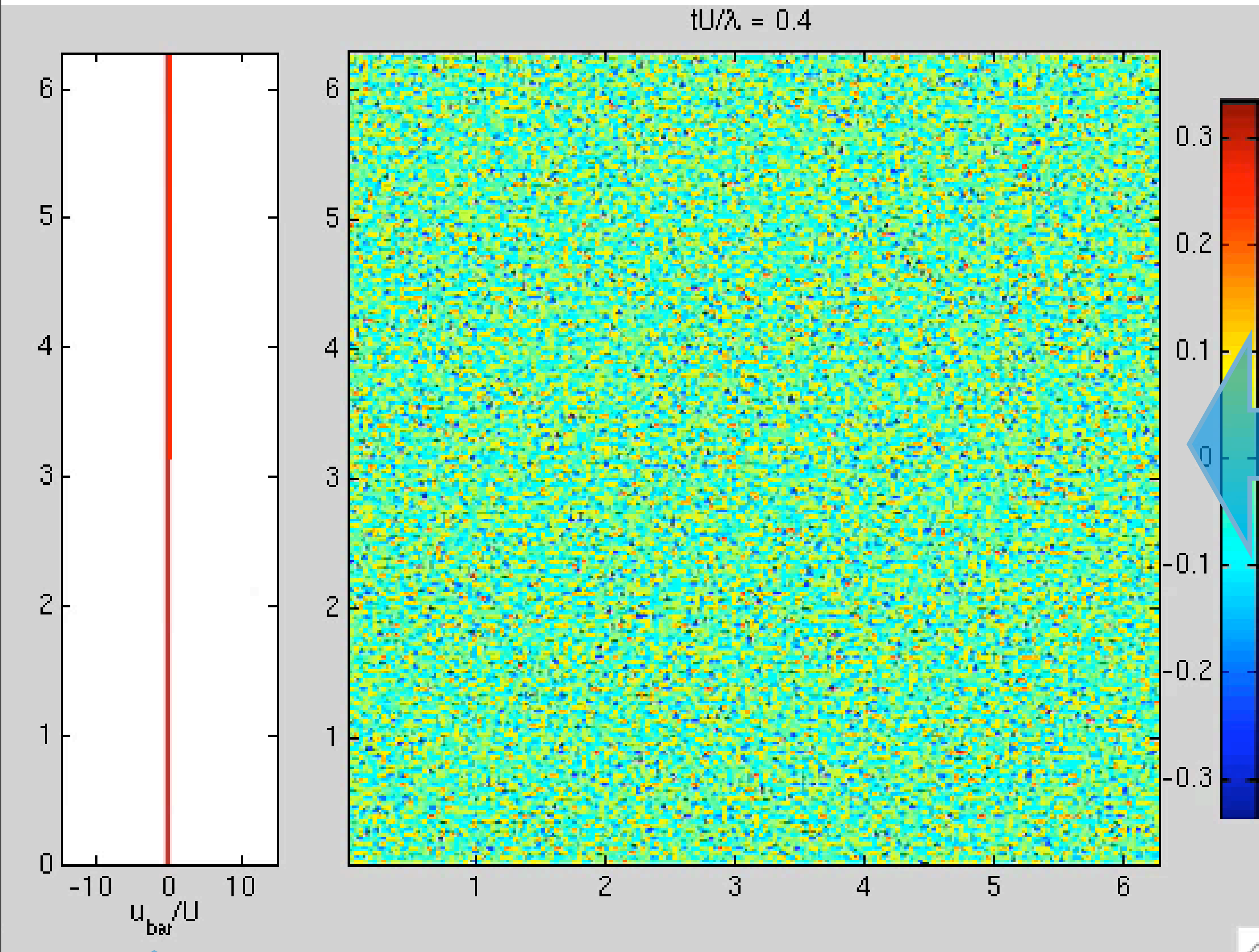
The release of APE by baroclinic eddies is proportional to the heat flux, and also to the supply the mechanical energy required to balance bottom friction.

Above: $\psi \equiv \frac{1}{2}(\psi_1 + \psi_2)\,, \qquad \tau = \frac{1}{2}(\psi_1 - \psi_2)$

The barotropic mode

The baroclinic mode (“heat”).

Spin-up: growth of linearly unstable disturbances



Upper layer PV

Barotropic zonally-averaged zonal flow: a negative-viscosity instability.

$$\bar{u}(y,t) \equiv L^{-1} \oint u(x,y,t)\,\mathrm{d}x$$

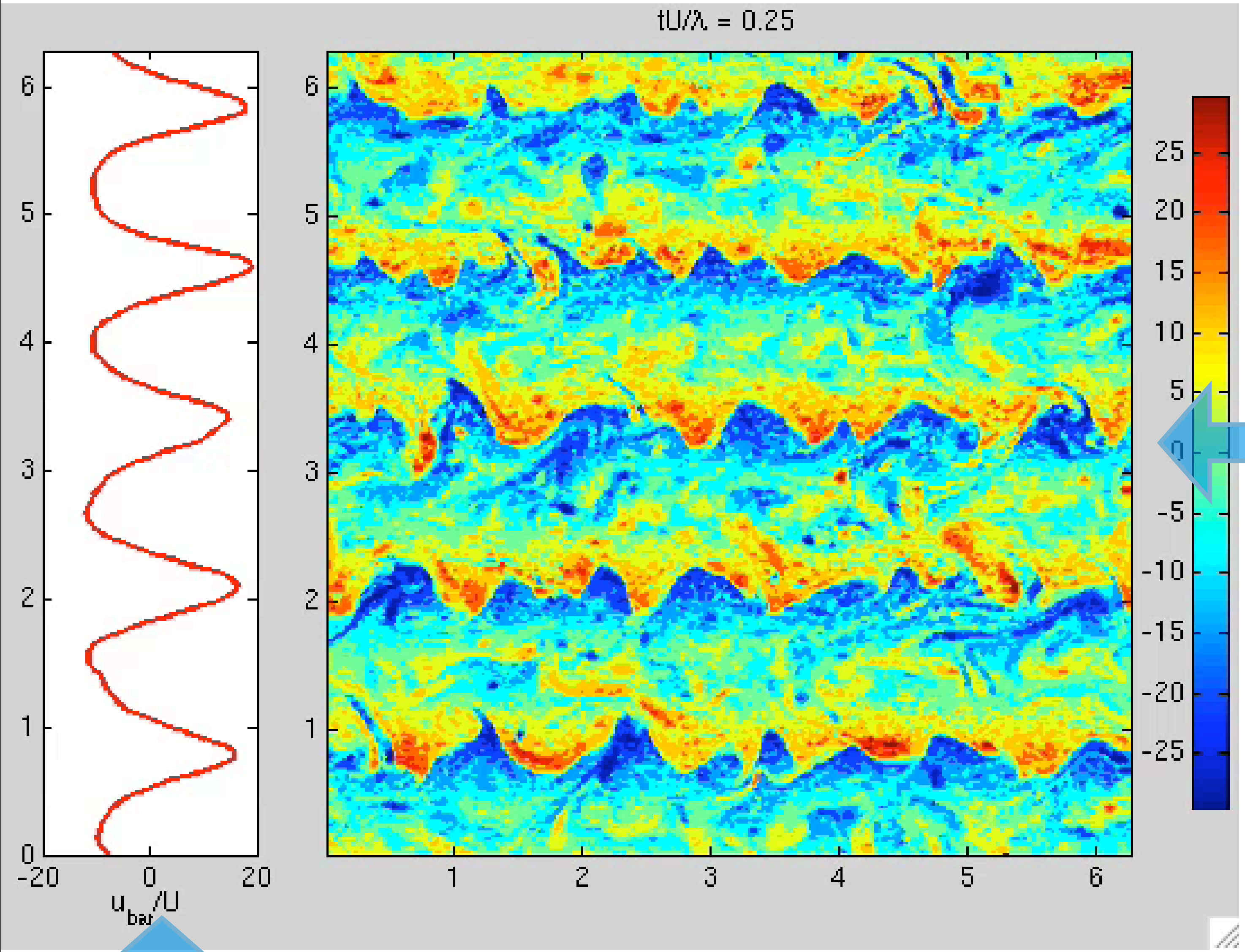
L=50π×λ

The most unstable linear disturbance is:

$$\psi_n \propto \exp\left[\mathrm{i}k_x(x-ct)\right]$$

$$k_y = 0 \quad \Rightarrow \quad J(\psi, \nabla^2\psi) = 0$$

Exponential growth is stopped by a secondary instability.



Statistically steady turbulence

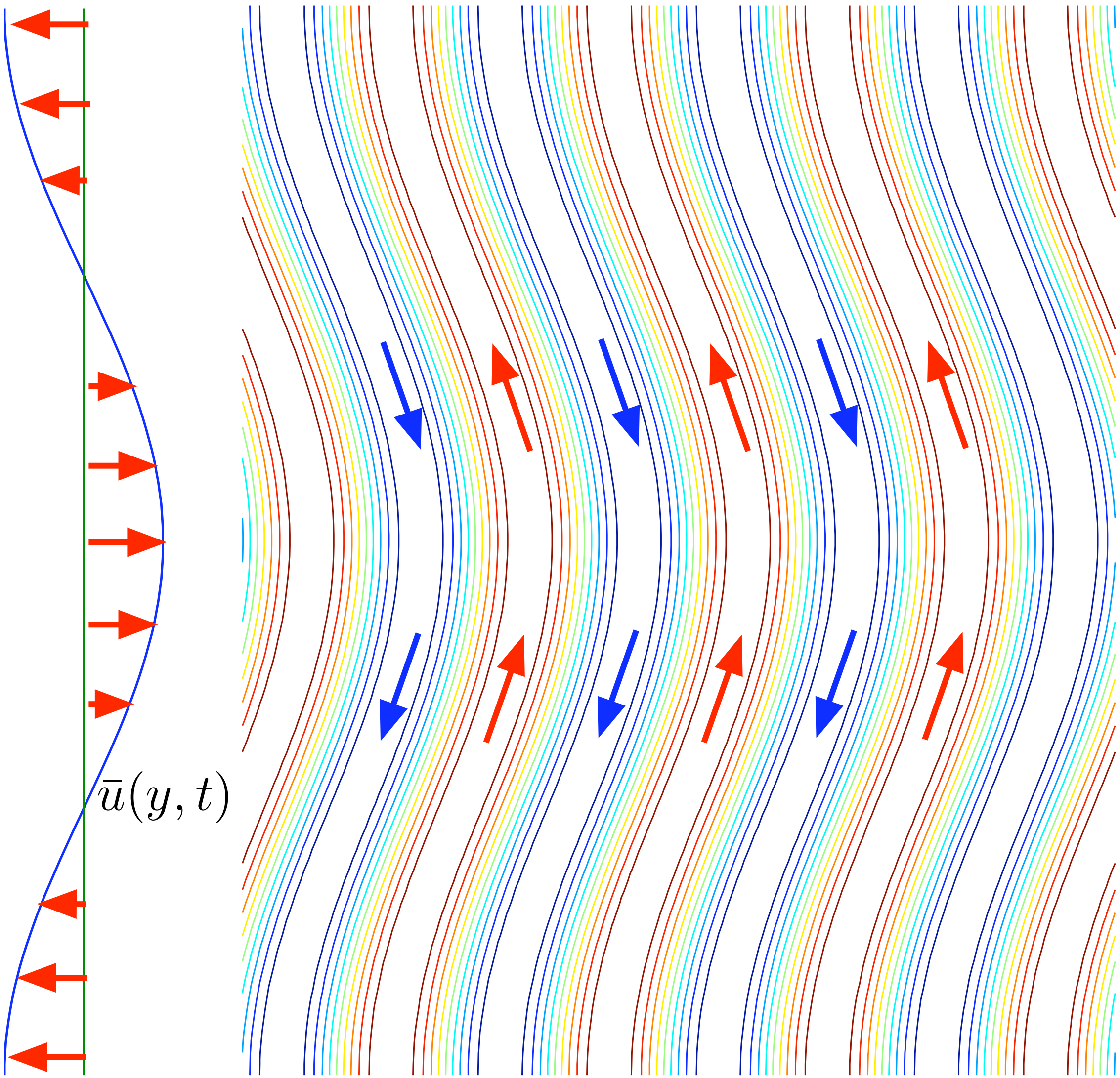
Upper layer PV

L=50π×λ

Barotropic zonal mean flow $\bar{u}(y,t) \sim 10U$

(Williams 1979, Panetta 1993)

Physical basis of negative viscosity



$\overline{v'u'} < 0$
$\bar{u}_y < 0$

$$\bar{u}_t + \left(\overline{u'v'}\right)_y = -\kappa\bar{u}$$

$\overline{v'u'} > 0$
$\bar{u}_y > 0$

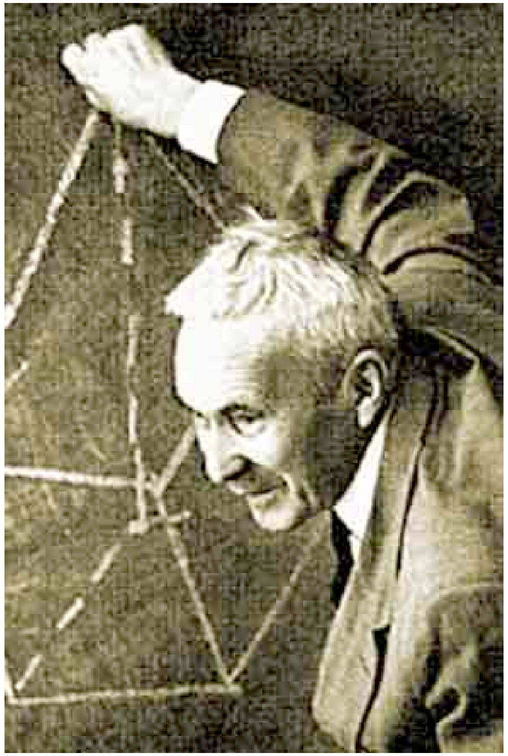
Weak zonal distortion of the elevator mode un-mixes momentum and so maintains the jets against bottom drag.

$$\psi' = a\cos\left[k\left(x - \bar{u}(y)t\right)\right] \Rightarrow \overline{u'v'} \sim +\tfrac{1}{2}(ka)^2 t\bar{u}_y \Rightarrow \bar{u}_t = -\tfrac{1}{2}(ka)^2 t\bar{u}_{yy} - \kappa\bar{u}$$

(Starr 1968, Sivashinsky 1985, Manfroi & Young 1999)

jump to CH

The Kolmogorov problem



An illustrative model of negative viscosity is the stability of:

$$\Psi = -\Psi_{\max} \cos kx\,, \qquad V = \Psi_x = k\Psi_{\max} \sin kx$$

The 2D Navier-Stokes equation is:

$$\nabla^2\psi_t + R_c(1+\epsilon^2)\sin x\left(\nabla^2\psi + \psi\right)_y + J(\psi, \nabla^2\psi) + \beta\psi_x = \nabla^4\psi - \kappa\nabla^2\psi$$

differential rotation
and bottom drag

This flow is unstable (in d=2) if:

$$R \equiv \frac{\Psi_{\max}}{\nu} > R_c = \sqrt{2}$$

(Meshalkin & Sinai 1961)

The Kolmogorov problem - continued

The weakly nonlinear expansion of:

$$\nabla^2\psi_t + R_c(1+\epsilon^2)\sin x\left(\nabla^2\psi+\psi\right)_y + J(\psi,\nabla^2\psi) + \beta\psi_x = \nabla^4\psi - \kappa\nabla^2\psi$$

starts like this:

$$\psi = A(\underbrace{\epsilon y, \epsilon^4 t}_{Y\ \&\ T}) + \epsilon\sin x R_c A_Y + O(\epsilon)^2$$

Again we obtain the Cahn-Hilliard equation:

$$U_T = -\mu U - \left[(2-\beta^2)U + 3U_{YY} + 2\beta U^2 - \tfrac{2}{3}U^3\right]_{YY}$$

Bottom drag ($-\mu U$); Negative viscosity ($(2-\beta^2)U$); Symmetry breaking ($2\beta U^2$)

Notation: $U \equiv -A_Y$

The Taylor-Prandtl debate: mix momentum or vorticity?

(Sivashinsky 1985, Manfroi & Young 1999).

The Kolmogorov problem - continued

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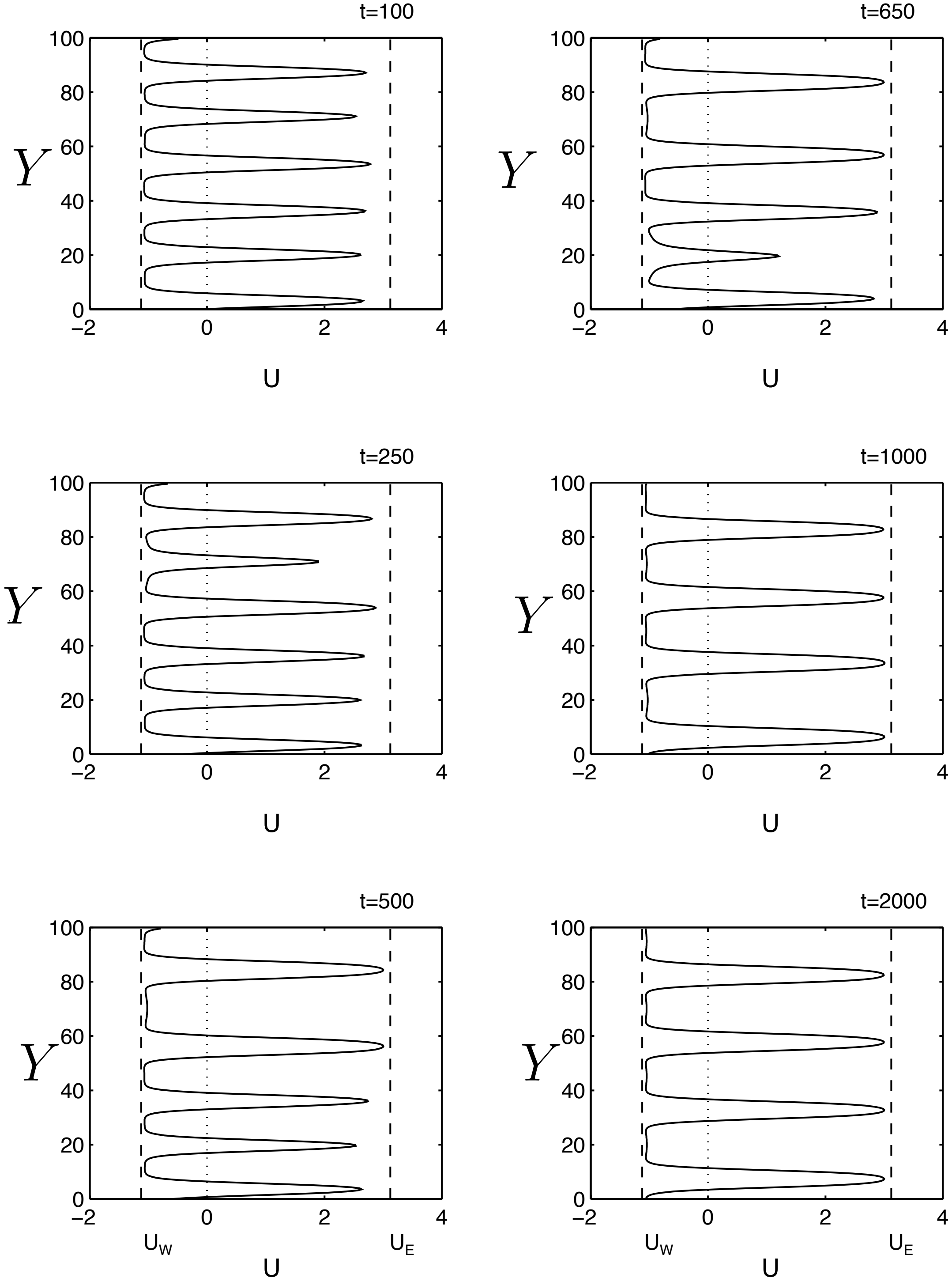
The Taylor-Prandtl debate: mix momentum or vorticity?

One point to Prandtl for the weakly nonlinear case above.

(Sivashinsky 1985, Manfroi & Young 1999).

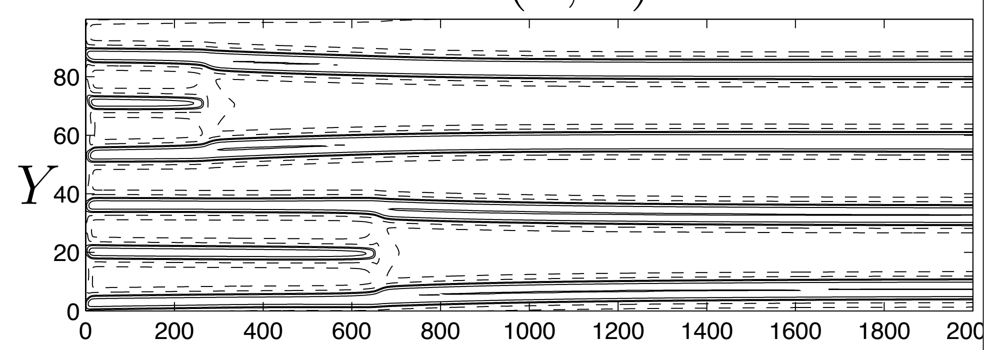
Solution of the CH equation, β=1



$$U_T = -\frac{1}{100}U - \left[U + 3U_{YY} + 2U^2 - \frac{2}{3}U^3\right]_{YY}$$

(1) Initial jet spacing according to linear instability.
(2) Jet spacing increases due to merger.
(3) Mergers are halted by bottom drag (not the Rhines scale).
(4) E-W asymmetry due to U^2.

$U(Y,T)$



Back to Baroclinic instability

The mechanical energy budget is: $U\lambda^{-2}\underbrace{\langle \psi_x \tau \rangle}_{=DU} \approx \underbrace{\kappa \langle |\boldsymbol{\nabla}\psi_2|^2 \rangle}_{=\varepsilon}$

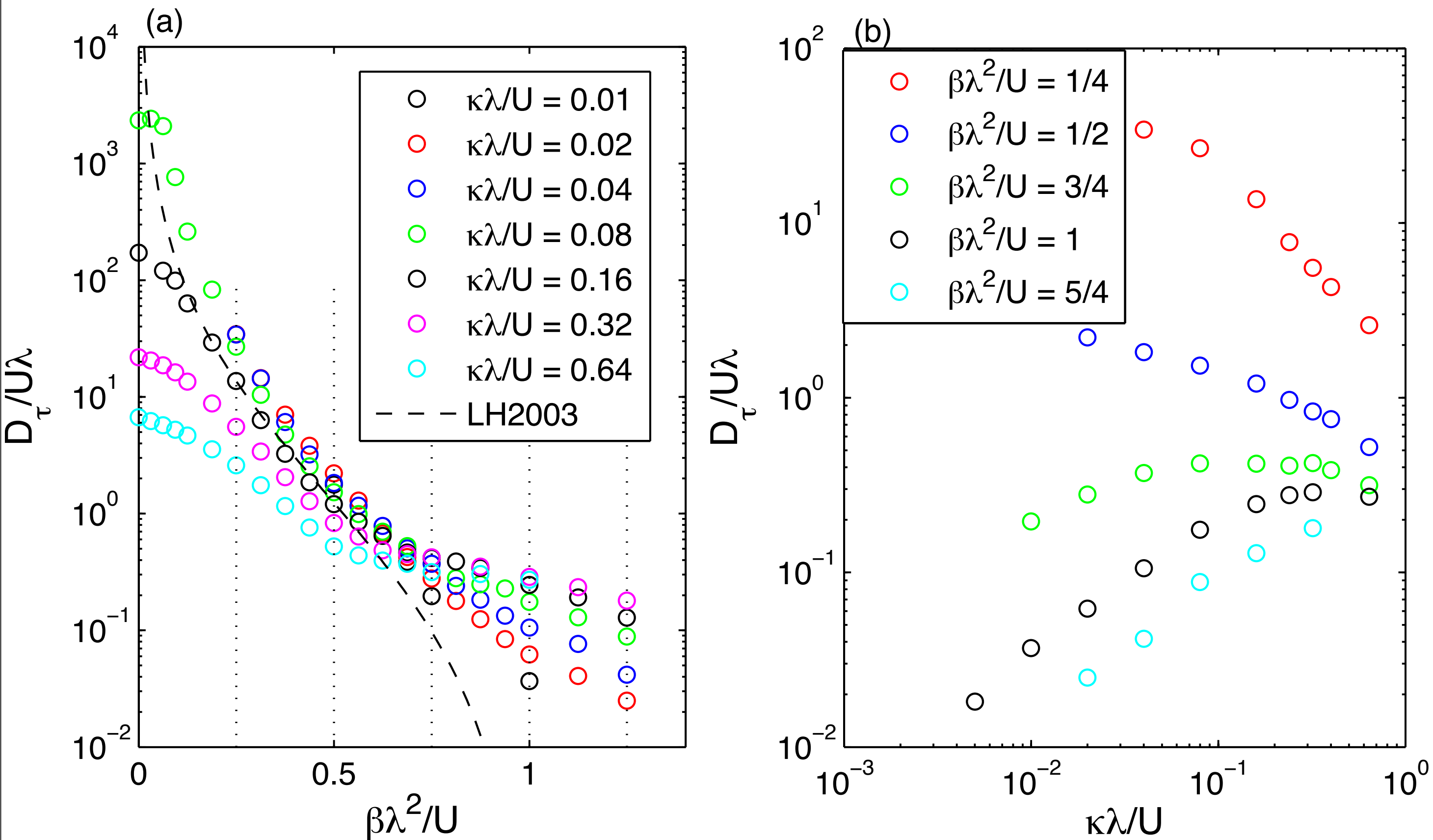
Always important

Dimensional analysis (HH 1980): $D = U\lambda \times D_* \left(\frac{\lambda}{L}, \frac{\kappa\lambda}{U}, \frac{\beta\lambda^2}{U}, \frac{\nu}{UL^7} \right)$

Small and unimportant

The leading contender (LH 2003): $D_\tau^{LH03} = U\lambda \times 1.75\beta_*^{-2}(1-\beta_*)^{5/2}$

Note: no dependence on bottom drag and no zonal jets.....



100 simulations
$L = 50\pi \times \lambda$

A satisfactory theory for D must include bottom drag.....

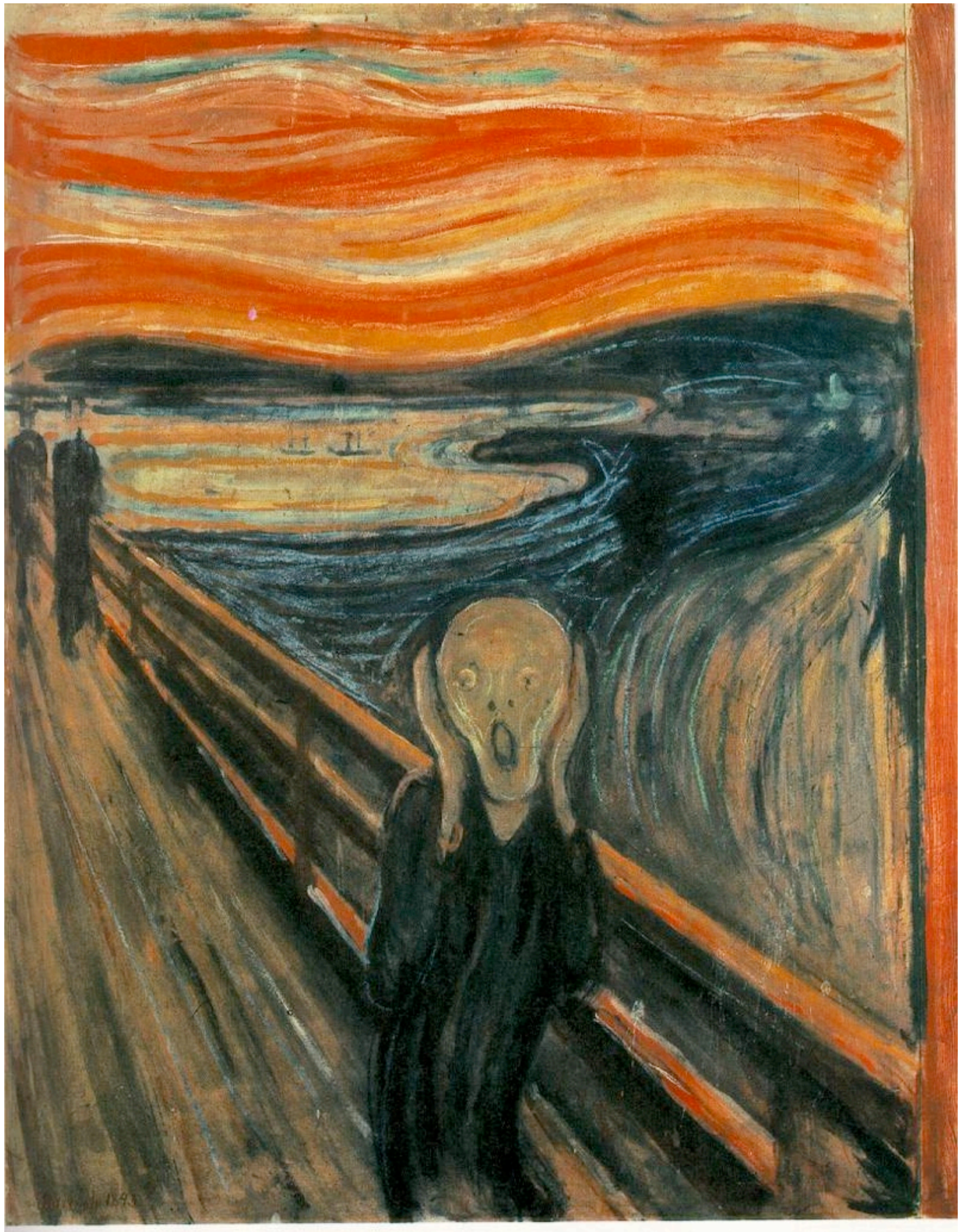
$$D = U\lambda \times D_* \left(\frac{\lambda}{L}, \frac{\kappa\lambda}{U}, \frac{\beta\lambda^2}{U}, \frac{\nu}{UL^7} \right)$$

e.g., $D_\tau^{LH03} = U\lambda \times 1.75\beta_*^{-2}(1-\beta_*)^{5/2}$

Conclusion

- Is turbulence stable? Not always.
- For experiments like PWG there is an almost-convincing theory (BLSY).
- There is no equivalent theory for baroclinic turbulence problem. why?
- Rhines scaling for the jets doesn't work either what!

Conclusion



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- Rhines scaling for the jets doesn't work either what!

Taylor's identity:

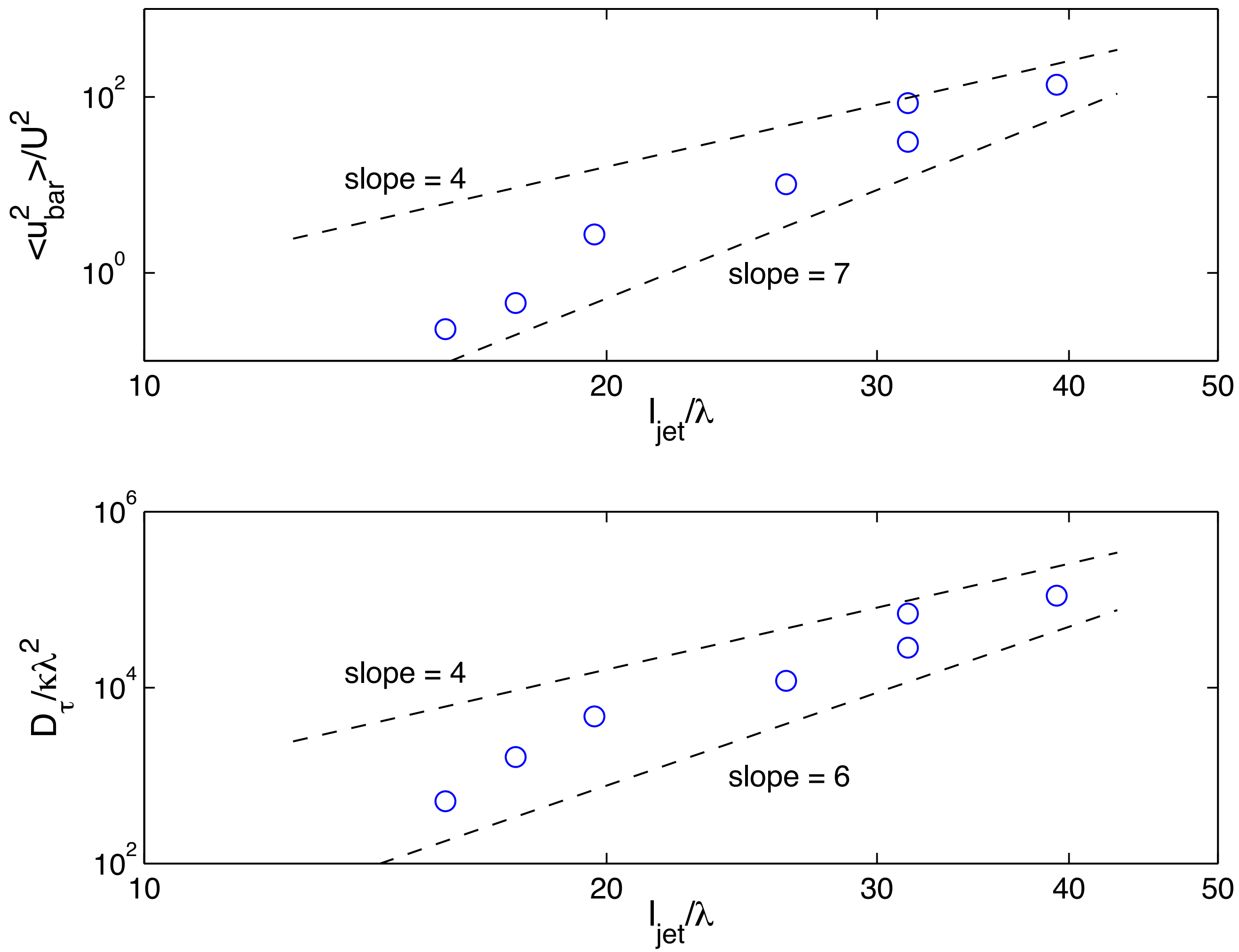
$$\overline{v_1'q_1'} + \overline{v_2'q_2'} = \left(\overline{u_1'v_1'} + \overline{u_2'v_2'}\right)_y \quad \Rightarrow \quad \int \overline{v_1'q_1'} + \overline{v_2'q_2'}\,\mathrm{d}y = 0$$

is incompatible with PV diffusion:

$$\overline{v_1'q_1'} = -D_1\bar{q}_{1y} \qquad \overline{v_2'q_2'} = -D_2\bar{q}_{2y}\,, \qquad D_n = \sqrt{e}\ell\,?$$

Rhines scaling?

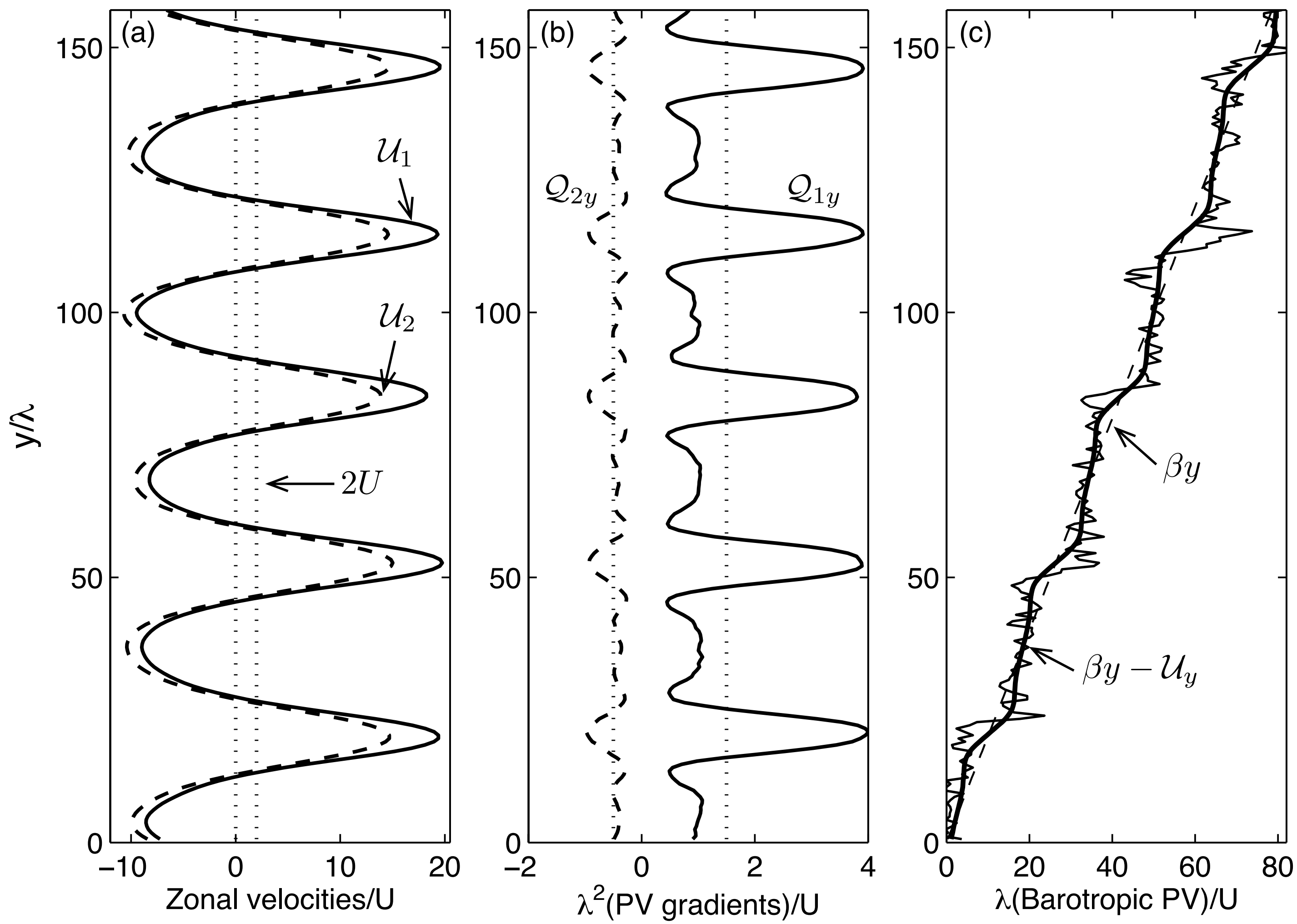
$$u_J \sim \beta \ell_J^2 \quad \Rightarrow \quad \frac{\langle \bar{u}^2 \rangle}{U^2} \sim \beta_*^2 \left(\frac{\ell_J}{\lambda} \right)^4$$



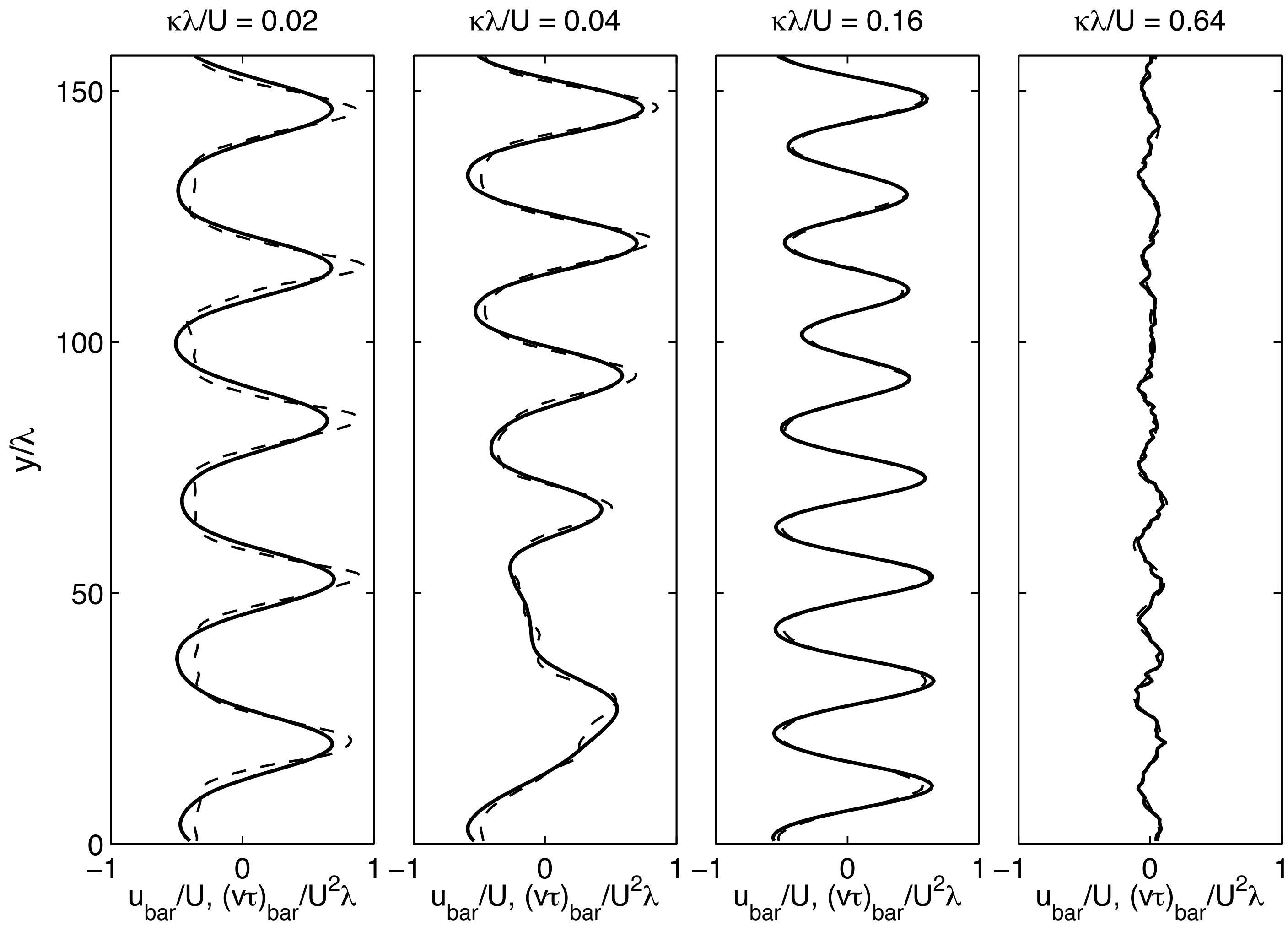
$\beta_* = 1/2$

Another consequence of Rhines scaling:

$$U^2 D/\lambda^2 \approx \kappa \langle |\nabla \psi_2|^2 \rangle \approx \kappa \langle \bar{u}^2 \rangle \quad \Rightarrow \quad \frac{D}{\kappa \lambda^2} \sim \beta_*^2 \left(\frac{\ell_J}{\lambda} \right)^4$$

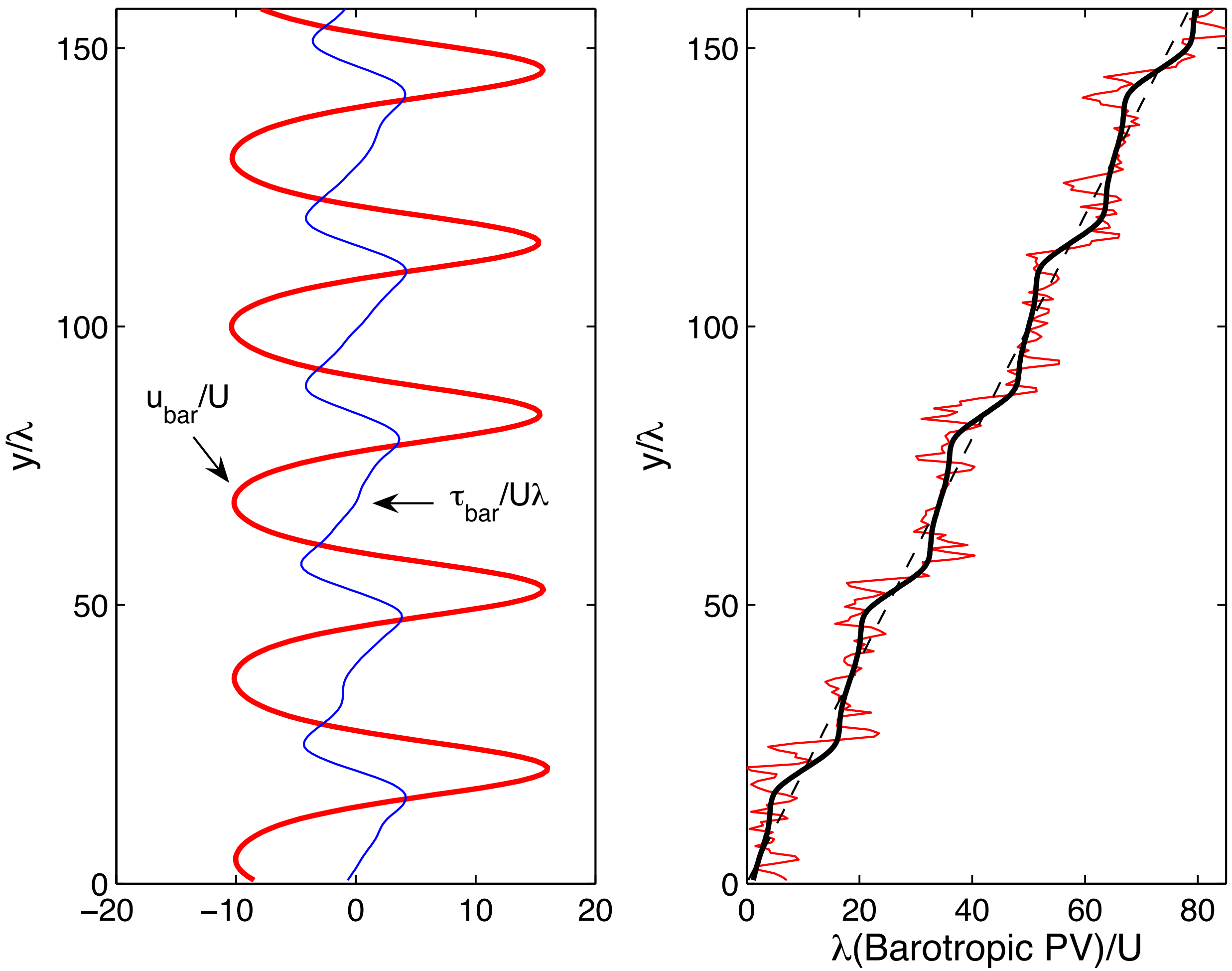


(a)
(b)
(c)
$\mathcal{U}_1$
$\mathcal{U}_2$
$2U$
$\mathcal{Q}_{2y}$
$\mathcal{Q}_{1y}$
βy
$\beta y - \mathcal{U}_y$
y/λ
Zonal velocities/U
λ^2(PV gradients)/U
λ(Barotropic PV)/U



κλ/U = 0.02
κλ/U = 0.04
κλ/U = 0.16
κλ/U = 0.64
y/λ
150
100
50
0
−1
0
1
u_bar/U, (vτ)_bar/U^2λ

A PV staircase formed by partially mixing the β-ramp



$$\frac{\beta\lambda^2}{U} = \frac{1}{2}$$

$$\frac{\kappa\lambda^2}{U} = \frac{1}{50}$$

$$\ell_J = 50\pi\lambda/5 = 10\pi\lambda$$