Sandström's theorem and the energetics of ocean mixing

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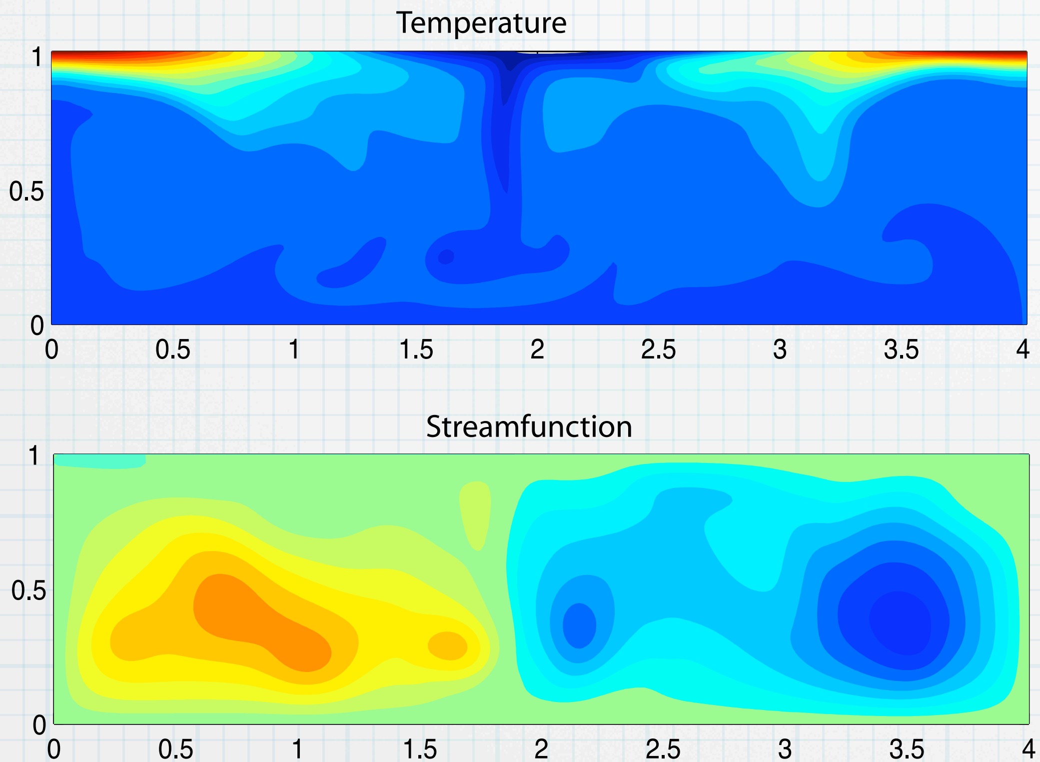
Sandström's (1908) "theorem": a closed steady circulation can only be maintained if the heat source is beneath the cold source (Defant 1965).



A 2-D counterexample: the box is heated non-uniformly at the top.

This is Rossby's horizontal convection (HC) problem.

The critical Rayleigh number of HC is zero --- the slightest ΔT imposed at the top surface causes motion.

TMFKAWS, June 05

Why bother with Sandström?

* There are many recent citations and endorsements of the 1908 "theorem": Munk & Wunsch (1998), Huang (1999), Emmanuel (2001), Wunsch & Ferrari (2004), etc.
* And also many counterexamples against the "theorem": Jeffreys (1925), Rossby (1965), Mullarney et al. (2004), Wang & Huang (2005), etc.
* What controls the THC? What drives (i.e., supplies energy to) the THC?

Sandström's (1908) "theorem": a closed steady circulation can only be maintained if the heat source is beneath the cold source (from Defant 1965).

Why isn't HC universally accepted as a counterexample to the "theorem"?

* HC isn't a vigorous flow.
* HC produces a thin thermocline --- you need wind, tides and breaking IGW's to explain deep ocean stratification.
* "strict interpretation of the theorem is difficult" (Houghton 1997).

The goal here: make the "theorem" into a theorem.

An enclosed Boussinesq fluid --- like the ocean.

$$\frac{D\boldsymbol{u}}{Dt} + \hat{\boldsymbol{z}} \times f\boldsymbol{u} + \boldsymbol{\nabla} p = g\alpha T \hat{\boldsymbol{z}} + \nu \nabla^2 \boldsymbol{u}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T$$

Note: linear EOS

Notation

$$\overline{\bullet} = \text{area and time average},$$
$$\langle \bullet \rangle = \text{volume and time average}$$

For example:

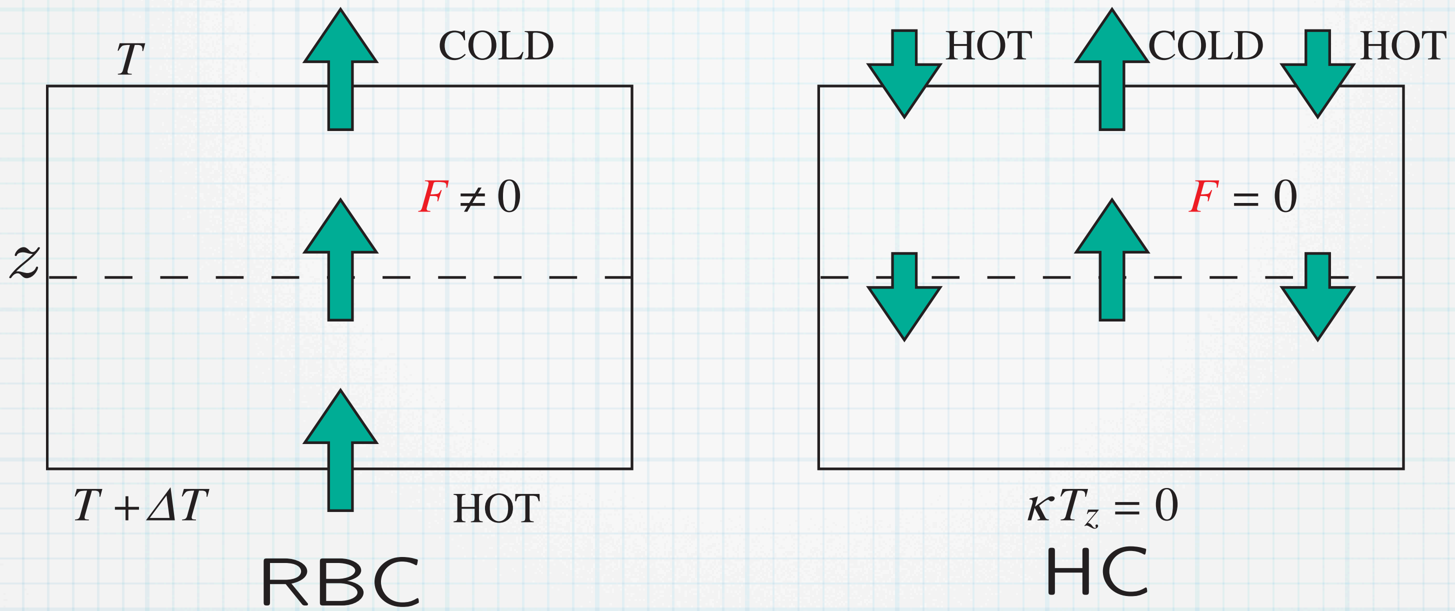
$$\overline{wT} = A^{-1} t_\infty^{-1} \int_0^{t_\infty} \iint wT \, \mathrm{d}x\mathrm{d}y \, \mathrm{d}t$$

$$KE = \langle \tfrac{1}{2} |\boldsymbol{u}|^2 \rangle = V^{-1} t_\infty^{-1} \int_0^{t_\infty} \iiint \tfrac{1}{2} |\boldsymbol{u}|^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z \, \mathrm{d}t$$

where $V = H \times A$.

We assume that all these averages are stationary.

The flux constraint



Average over x, y and t. :

$$\frac{DT}{Dt} = \kappa \nabla^2 T \qquad \Rightarrow \qquad \overline{wT} - \kappa \bar{T}_z = F \quad (\forall z)$$

F = the vertical heat flux

The momentum equation has not been used.

To obtain the **the mechanical energy budget**, take:

$$\int \boldsymbol{u} \cdot \left[\frac{D\boldsymbol{u}}{Dt} + \hat{\boldsymbol{z}} \times f\boldsymbol{u} + \boldsymbol{\nabla} p = g\alpha T \hat{\boldsymbol{z}} + \nu \nabla^2 \boldsymbol{u} \right] \mathrm{d}V$$

After <>, viscous dissipation = PE release:

$$\underbrace{\nu \int |\boldsymbol{\nabla u}|^2 \, \mathrm{d}V}_{\varepsilon} = g\alpha \langle wT \rangle$$

Now recall the zero-flux constraint:

$$\overline{wT} - \kappa \bar{T}_z = 0 \qquad \Rightarrow \qquad \langle wT \rangle = \frac{\kappa \Delta \bar{T}}{H}$$

Eliminate <wT>:

$$\boxed{\underbrace{\nu \langle |\boldsymbol{\nabla u}|^2 \rangle}_{\varepsilon} = \kappa \frac{\alpha g \Delta \bar{T}}{H}}$$

(A rehabilitation of Sandström's theorem.)

The top-to-bottom ΔT is: $\Delta \bar{T} = \bar{T}(0) - \bar{T}(-H)$

An estimate using:

$$\underbrace{\nu\langle|\boldsymbol{\nabla u}|^2\rangle}_{\varepsilon} = \kappa\frac{\alpha g \Delta \bar{T}}{H}$$

We can easily estimate the mechanical energy dissipation:

$$\varepsilon \approx 10^{-7} \times \frac{2\times 10^{-2}}{4000} = 2\times 10^{-12}\mathrm{W\ kg}^{-1}\,.$$

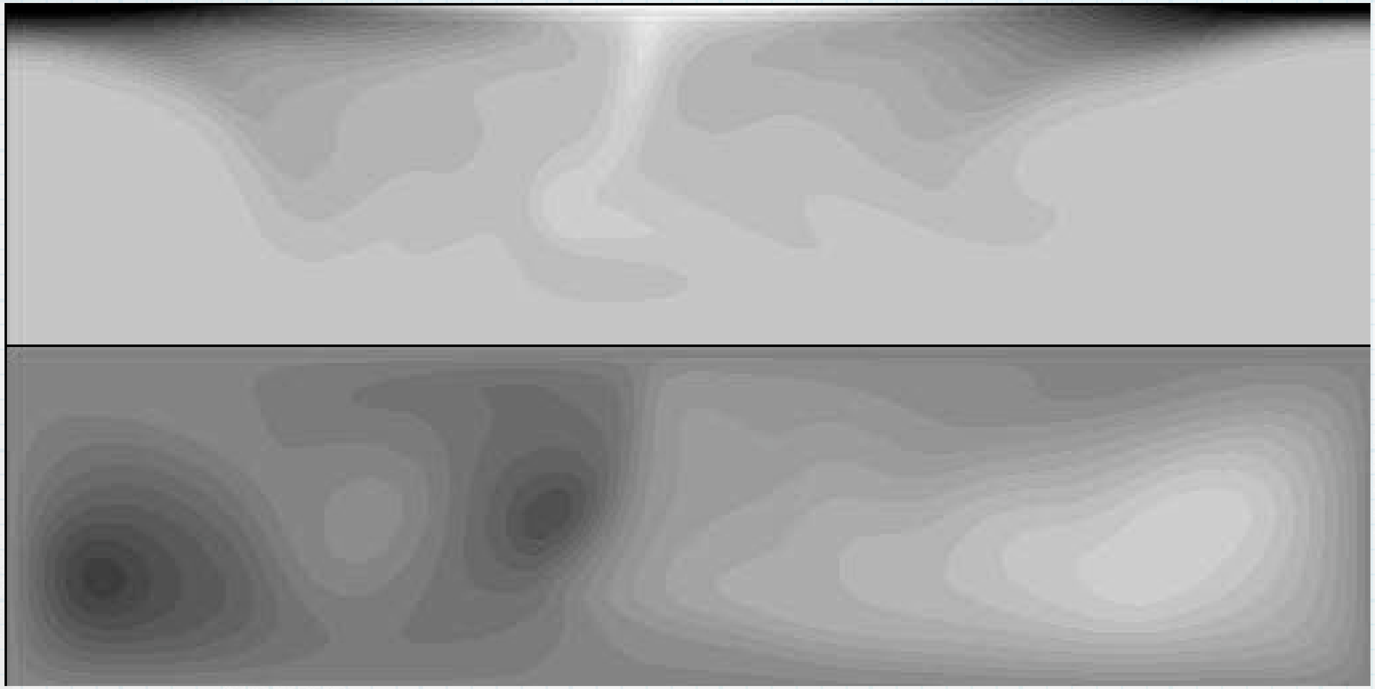
This is smaller by a factor of one thousand than observations.

Conclusion 1: the observed level of ocean turbulence requires other sources of KE (winds, tides and fish).

Conclusion 2: there is a kernel of truth in Sandström's theorem, or:

PY (2002) theorem: the mechanical energy supplied by non-uniformly heating only the top surface is directly proportional to the molecular diffusion of heat.

Back to the zero-flux constraint



At every depth:

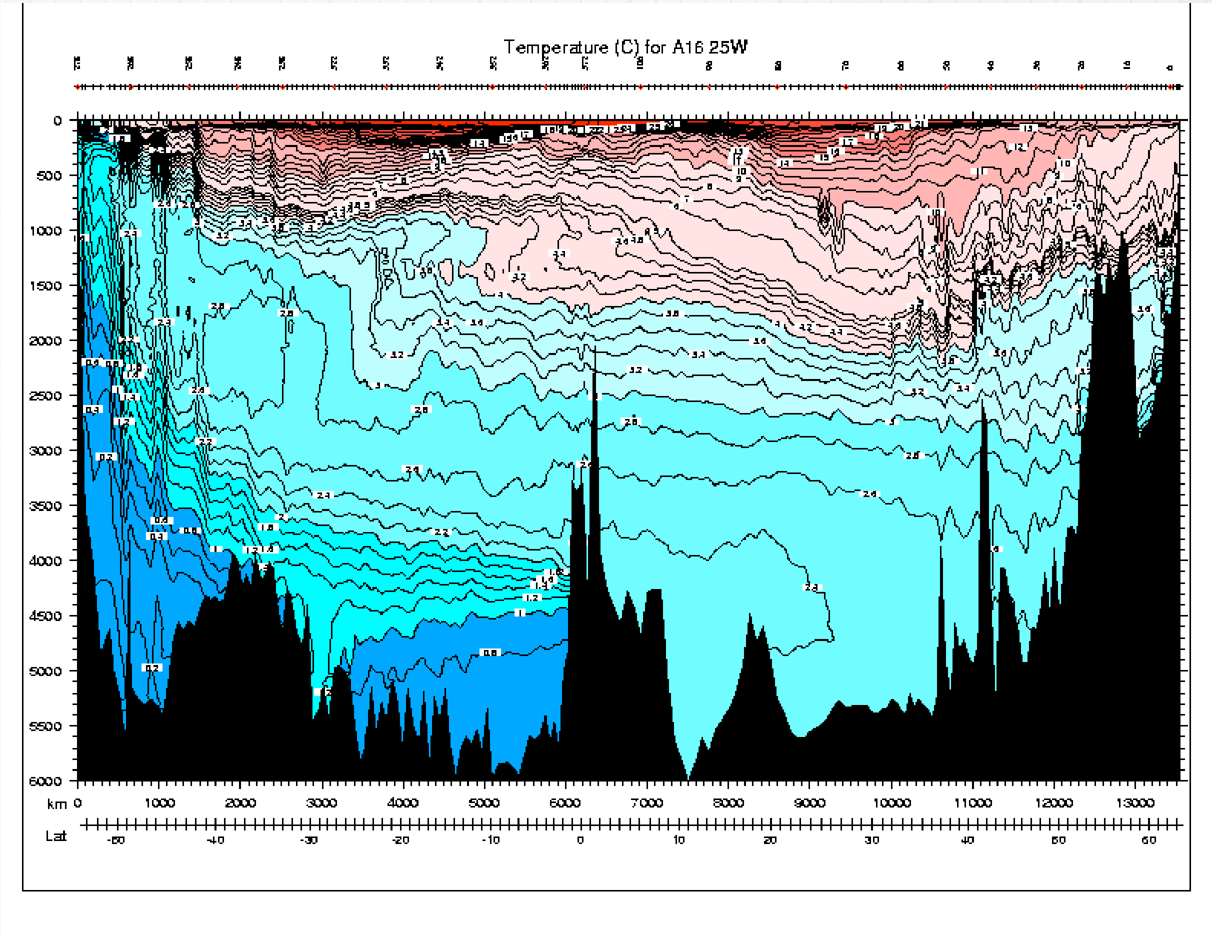
$$\overline{wT} - \kappa \bar{T}_z = 0$$

Although the energy supply is small, the flow is **not laminar**.

The zero-flux constraint follows only from the thermodynamic equation and the bottom BC --- no consideration of the momentum equation is necessary.

THE OCEAN....

* The zero flux constraint almost* applies.
* There is almost* no net conversion between PE and KE in the ocean.



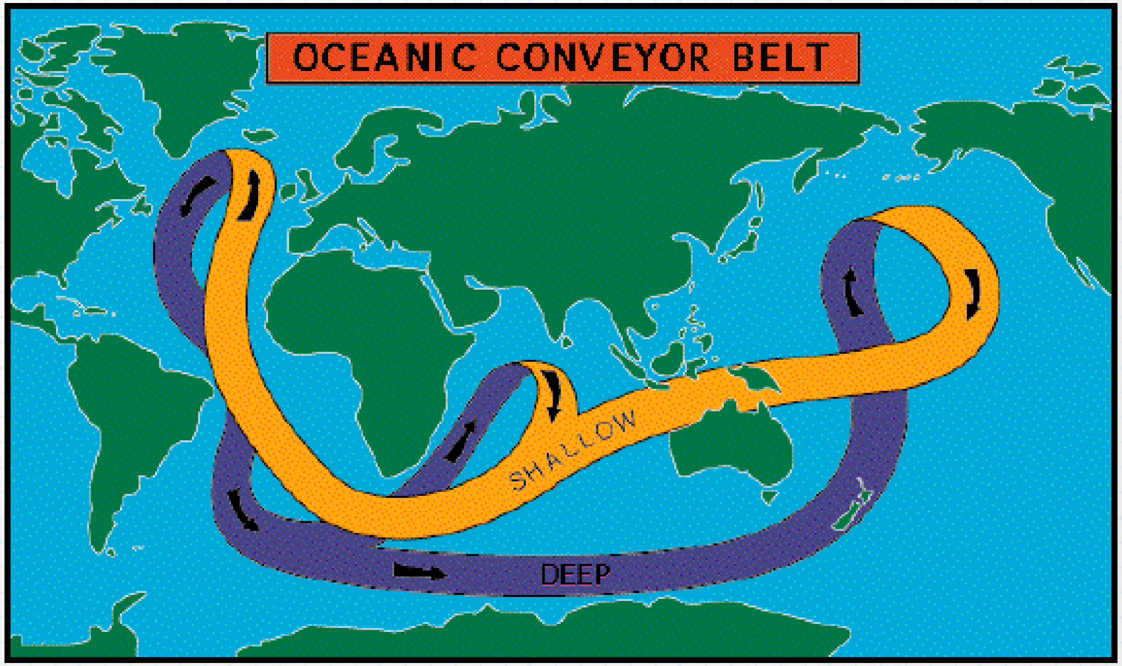
$$\overline{wT} \approx 0 \qquad \forall z$$

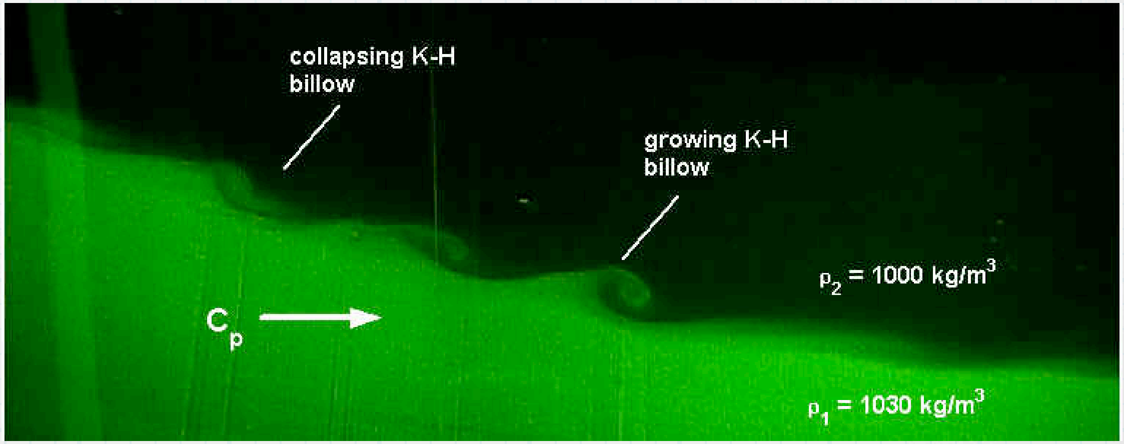
* "almost" because of molecular diffusion, geothermal heating, nonlinear EOS and radiation

$$\langle wT\rangle = \underbrace{\langle \bar{w}\bar{T}\rangle}_{\text{large scales}>0} + \underbrace{\langle w'T'\rangle}_{\text{small scales}<0} \approx 0$$

* Large-scale THC lowers the center of gravity.
* Small-scale mixing raises the center of gravity.
* These processes are almost* in perfect balance.
* Large scale overturning is linked to small scale mixing.

* "almost" because of molecular diffusion, geothermal heating, nonlinear EOS and radiation





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Conclusions

* Sandström's arguments are not completely wrong...
* The mechanical energy supplied by non-uniform surface heating is directly proportional to **molecular** diffusion.
* The net vertical buoyancy flux in the ocean is a small energy source (e.g., less than bioturbation).
* The zero-flux constraint links small-scale mixing and large-scale TH overturning.