Sandström’s theorem and the energetics of ocean mixing

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Sandström’s (1908) “theorem”: a closed steady circulation can only be maintained if the heat source is beneath the cold source (Defant 1965).

A 2-D counterexample: the box is heated non-uniformly at the top.

This is Rossby’s horizontal convection (HC) problem.

The critical Rayleigh number of HC is zero --- the slightest ΔT imposed at the top surface causes motion.
Why bother with Sandström?

* There are many recent citations and endorsements of the 1908 "theorem": Munk & Wunsch (1998), Huang (1999), Emmanuel (2001), Wunsch & Ferrari (2004), etc.

* And also many counterexamples against the "theorem": Jeffreys (1925), Rossby (1965), Mullarney et al. (2004), Wang & Huang (2005), etc.

* What controls the THC? What drives (i.e., supplies energy to) the THC?

Sandström's (1908) "theorem": a closed steady circulation can only be maintained if the heat source is beneath the cold source (from Defant 1965).
Why isn’t HC universally accepted as a counterexample to the “theorem”? 

- HC isn’t a vigorous flow.
- HC produces a thin thermocline --- you need wind, tides and breaking IGW’s to explain deep ocean stratification.
- “strict interpretation of the theorem is difficult” (Houghton 1997).

The goal here: make the “theorem” into a theorem.
An enclosed Boussinesq fluid --- like the ocean.

\[ \frac{Du}{Dt} + \hat{z} \times f u + \nabla p = g \alpha T \hat{z} + \nu \nabla^2 u \]

\[ \nabla \cdot u = 0 \]

\[ \frac{DT}{Dt} = \kappa \nabla^2 T \]

Note: linear EOS
Notation

\[ \bar{\bullet} = \text{area and time average}, \]
\[ \langle \bullet \rangle = \text{volume and time average} \]

For example:

\[ \bar{wT} = A^{-1} t^{-1} \int_{0}^{t} \int \int wT \, dx \, dy \, dt \]

\[ KE = \langle \frac{1}{2} |u|^2 \rangle = V^{-1} t^{-1} \int_{0}^{t} \int \int \int \frac{1}{2} |u|^2 \, dx \, dy \, dz \, dt \]

where \( V = H \times A \).

We assume that all these averages are stationary.
The flux constraint

Average over x, y and t. :

\[
\frac{DT}{Dt} = \kappa \nabla^2 T \quad \Rightarrow \quad \overline{wT} - \kappa \overline{T_z} = F \quad (\forall z)
\]

\(F = \) the vertical heat flux

The momentum equation has not been used.
To obtain the **mechanical energy budget**, take:

$$\int u \cdot \left[ \frac{Du}{Dt} + \mathbf{\hat{z}} \times f u + \nabla p = g\alpha T \mathbf{\hat{z}} + \nu \nabla^2 u \right] \, dV$$

After $\langle \rangle$, viscous dissipation = PE release:

$$\nu \int |\nabla u|^2 \, dV = g\alpha \langle wT \rangle$$

Now recall the zero-flux constraint:

$$\overline{wT} - \kappa \overline{T_z} = 0 \quad \Rightarrow \quad \langle wT \rangle = \frac{\kappa \Delta \overline{T}}{H}$$

Eliminate $\langle wT \rangle$:

$$\nu \langle |\nabla u|^2 \rangle = \kappa \frac{\alpha g \Delta \overline{T}}{H}$$

(A rehabilitation of Sandström's theorem.)

The top-to-bottom $\Delta T$ is: $\Delta \overline{T} = \overline{T}(0) - \overline{T}(-H)$
An estimate using: $$\nu \langle | \nabla u |^2 \rangle = \kappa \frac{\alpha g \Delta \bar{T}}{H}$$

We can easily estimate the mechanical energy dissipation:

$$\varepsilon \approx 10^{-7} \times \frac{2 \times 10^{-2}}{4000} = 2 \times 10^{-12} \text{W kg}^{-1}.$$

This is smaller by a factor of one thousand than observations.

**Conclusion 1:** the observed level of ocean turbulence requires other sources of KE (winds, tides and fish).

**Conclusion 2:** there is a kernel of truth in Sandström’s theorem, or:

**PY (2002) theorem:** the mechanical energy supplied by non-uniformly heating only the top surface is directly proportional to the molecular diffusion of heat.
Back to the zero-flux constraint

\[ w T - \kappa \bar{T}_z = 0 \]

The zero-flux constraint follows only from the thermodynamic equation and the bottom BC --- no consideration of the momentum equation is necessary.

At every depth: Although the energy supply is small, the flow is not laminar.
The zero flux constraint almost* applies.

There is almost* no net conversion between PE and KE in the ocean.

*“almost” because of molecular diffusion, geothermal heating, nonlinear EOS and radiation

\[ \overline{wT} \approx 0 \quad \forall z \]
\[ \langle wT \rangle = \langle w\tilde{T} \rangle + \langle w'T' \rangle \approx 0 \]

large scales \( > 0 \)  
small scales \( < 0 \)

- Large-scale THC lowers the center of gravity.
- Small-scale mixing raises the center of gravity.
- These processes are almost* in perfect balance.
- Large scale overturning is linked to small scale mixing.

* "almost" because of molecular diffusion, geothermal heating, nonlinear EOS and radiation.

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Conclusions

* Sandström’s arguments are not completely wrong...

* The mechanical energy supplied by non-uniform surface heating is directly proportional to molecular diffusion.

* The net vertical buoyancy flux in the ocean is a small energy source (e.g., less than bioturbation).

* The zero-flux constraint links small-scale mixing and large-scale TH overturning.