SIO203C: PDE Notes B

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Lecture 9

Fourier series

9.1 Least squares approximation

Consider the periodic function $\text{sqr}(x)$ defined by $\text{sqr}(x) = \text{sgn} [\sin(x)]$. AKA the square wave. Can we find the “best” approximation of $\text{sqr}(x)$ in terms of $\sin x$,

$$\text{sqr}(x) \approx b \sin x, \quad (9.1)$$

by picking the parameter $b$? There is no unique answer to this question because any choice will involve compromises somewhere. But here is an answer suggested by Gauss: pick $b$ so that the mean square error

$$\varepsilon(b) = \int_{-\pi}^{\pi} [\text{sqr}(x) - b \sin x]^2 \frac{dx}{2\pi}, \quad (9.2)$$

is as small as possible.

We can figure out $b$ almost before evaluating any integrals:

$$\frac{d\varepsilon}{db} = -2 \int_{-\pi}^{\pi} \sin x [\text{sqr}(x) - b \sin x] \frac{dx}{2\pi}. \quad (9.3)$$

The optimal $b$ is determined by

$$\frac{d\varepsilon}{db} = 0, \quad \Rightarrow \quad b = \frac{\int_{-\pi}^{\pi} \sin x \text{sqr}(x) \, dx}{\int_{-\pi}^{\pi} \sin^2 x \, dx} = \frac{4}{\pi}. \quad (9.4)$$

With (9.2) we can’t expect to get a very good approximation. For instance, $4/\pi = 1.27$ and so our approximation has a 27% pointwise error at $x = 3$. 


Another measure of the error is the ratio
\[ \frac{\int_{-\pi}^{\pi} b^2 \sin^2(x) \, dx}{\int_{-\pi}^{\pi} \text{sqr}(x)^2 \, dx} = \frac{8}{\pi^2} = 0.81. \] (9.5)

The simple approximation contains 81% of the “energy” in \( \text{sqr}(x) \).

Now suppose we use more functions to approximate our target function. The approximation can only improve as we use more basis functions.

Fourier’s suggestion is to use sinusoids. Let us consider the general case of approximating a “target function” \( f(x) \) on the fundamental interval \( -\pi \leq x \leq \pi \) using a linear combination of sinusoids
\[ S_n(x) \equiv a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx. \] (9.6)

In other words, we hope to pick the coefficients in \( S_n \) so that \( f(x) \approx S_n(x) \).

We have \( 2n+1 \) adjustable parameters, and we define the mean square error as
\[ \varepsilon_n(a_k, b_k) \equiv \int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 \, dx \frac{1}{2\pi}, \] (9.7)
and we can proceed just as before by taking derivatives with respect to \( a_k \) and \( b_k \). We find
\[ \frac{d\varepsilon_n}{da_k} = -2 \int_{-\pi}^{\pi} \cos kx \, [f(x) - S_n(x)] \frac{dx}{2\pi} = 0, \]
\[ \frac{d\varepsilon_n}{db_k} = -2 \int_{-\pi}^{\pi} \cos kx \, [f(x) - S_n(x)] \frac{dx}{2\pi} = 0. \] (9.8)

These are \( 2n+1 \) equations for the \( 2n+1 \) unknowns \( a_k \) and \( b_k \).

The sinusoids have the extraordinary property that if \( k \) and \( m \) are positive integers
\[ \int_{-\pi}^{\pi} \cos kx \, \sin mx \, dx = 0, \] (9.9)
and
\[ \int_{-\pi}^{\pi} \cos kx \, \cos mx \, dx = \int_{-\pi}^{\pi} \sin kx \, \sin mx \, dx = \pi \delta_{km}, \] (9.10)
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where

\[ \delta_{km} = \begin{cases} 0 & \text{if } k \neq m, \\ 1 & \text{if } k = m. \end{cases} \]  

(9.11)

To prove the results above, you should recall that \( \cos kx \) and \( \sin kx \) can be written as linear combinations of \( e^{\pm ikx} \). So a product such as \( \cos kx \cos mx \) will contain combinations such as \( e^{\pm i(k \pm m)x} \), and

\[ \int_{-\pi}^{\pi} e^{\pm i(k \pm m)x} \, dx = 0, \quad \text{unless } k = m. \]  

(9.12)

Using the **orthogonality relations** in the previous paragraph, we determine \( a_k \) and \( b_k \) without having to solve linear equations. For instance, if \( k \geq 1 \):

\[ \int_{-\pi}^{\pi} S_n(x) \cos kx \, dx = \sum_{p=1}^{n} a_p \int_{-\pi}^{\pi} \cos kx \cos px \, dx, \]  

(9.13)

\[ = \pi \sum_{p=1}^{n} a_p \delta_{kp} = \pi a_k. \]  

(9.14)

We have used the handy rule

\[ \sum_{p=1}^{\infty} \text{[anything]}_p \delta_{kp} = \text{[anything]}_k. \]  

(9.15)

The optimal choice of the coefficients \( a_k \) and \( b_k \) in (9.6) is therefore:

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \]  

(9.16)

\[ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kx f(x) \, dx, \]  

(9.17)

\[ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin kx f(x) \, dx. \]  

(9.18)

Notice the irritating factors of \( 2 \).

If the target function \( f(x) \) is even, then all the \( b_k \)'s are zero i.e., we approximate an even function using only the cosines, so that our approximation is also even. **Mutatis mutandis** if \( f(x) \) is odd.

As an undergraduate you probably suffered through many enervating problems involving evaluation of the integrals in (9.17) through (9.18) using simple target functions such as \( \text{sqr}(x) \). The Fourier series is usually
introduced by writing down (9.6) and appealing to the orthogonality relations to deduce (9.17) through (9.18). The important connection to least-squares approximation is not sufficiently emphasized by this approach: the truncated Fourier series is the best least squares approximation to the target using the number of sinusoids in the truncation.

Another important property of the Fourier series is finality. Suppose we possess $S_{17}(x)$ and we want to improve the approximation to $S_{18}(x)$ by adding one more term to the series. Then we don’t need to recompute the earlier terms: the coefficients are given once and for all in (9.17) through (9.18).

### 9.2 Examples of Fourier series

This section is a medley of examples.

**Example:** Find the complete Fourier series of the square wave function $sqr(x)$. Applying the recipe above to $sqr(x)$, we begin by observing that because $sqr(x)$ is an odd function, all the $a_k$’s are zero. To evaluate $b_k$ notice that the integrand of (9.18) is even so that we need only integrate from 0 to $\pi$

$$b_k = \frac{2}{\pi} \int_0^\pi \sin kx \, dx = -\left[ \frac{2}{\pi k} \cos kx \right]_0^\pi = \left[1 - (-1)^k\right] \frac{2}{\pi k}.$$

$$\cos k\pi = (-1)^k$$
The even $b_k$'s are also zero — this is clear from the anti-symmetry of the integrand above about $x = \pi/2$. A sensitive awareness of symmetry is often a great help in evaluating Fourier coefficients. Thus we have

$$\text{sqr}(x) = \frac{4}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \cdots \right]. \quad (9.20)$$

The wiggly convergence of (9.20) is illustrated in figure 9.1 — more about the wiggles later.

Notice that if we evaluate (9.20) at $x = \pi/2$ we obtain the famous Gregory-Leibniz series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad (9.21)$$

**Example:** Consider the “sawtooth” function $\text{saw}(x)$ defined by

$$\text{saw}(x) = x, \quad \text{if } -\pi < x < \pi, \quad (9.22)$$

and $\text{saw}(-\pi) = \text{saw}(\pi) = 0$. Outside the fundamental interval, $\text{saw}(x) = \text{saw}(x + 2\pi)$. Represent $\text{saw}(x)$ as a Fourier series.

Because $\text{saw}(x)$ is odd, only sines appear in the Fourier series, and

$$b_k = \frac{2}{\pi} \int_0^\pi x \sin kx \, dx = 2 \frac{(-1)^{k+1}}{k}. \quad (9.23)$$

Thus the Fourier representation of $\text{saw}(x)$ is

$$\text{saw}(x) = 2 \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k} \sin kx, \quad (9.24)$$

$$= 2 \left[ \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right]. \quad (9.25)$$

Figure 9.2 shows that 20 terms in this series produces a wiggly version of $\text{saw}(x)$. ■

**Example:** Consider the function $\text{cusp}(x)$ defined by

$$\frac{d}{dx} \text{cusp}(x) = \text{saw}(x), \quad \int_{-\pi}^\pi \text{cusp}(x) \, dx = 0. \quad (9.26)$$

Obtain an explicit expression for $\text{cusp}(x)$, and represent $\text{cusp}(x)$ as a Fourier series.

On the fundamental interval the derivative of $\text{cusp}(x)$ is equal to $x$, so $\text{cusp}(x) = \frac{1}{2}x^2 + C$. The constant of integration $C$ is obtained from

$$0 = \int_{-\pi}^\pi \frac{1}{2}x^2 \, dx + 2\pi C, \quad \Rightarrow C = -\frac{\pi^2}{6}. \quad (9.27)$$
Figure 9.2: This shows the 20-term sum of the saw-series (9.25), and that of the cusp-series (9.28). sawcusp.eps

To quickly obtain the Fourier series of cusp we integrate (9.25):

$$\text{cusp}(x) = -2 \left[ \cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} + \cdots \right].$$  \hspace{1cm} (9.28)

The constant of integration is zero because $\text{cusp}(x)$ is defined to have zero DC content. Figure 9.2 shows that 20 terms in this series provides an excellent parabola in the interval $-\pi < x < \pi$. There are no wiggles.

There follows the matlab code that draws Figure 9.2:

```matlab
x = linspace(-4,4,200);
saw = zeros( 1,length(x) );
cusp = zeros( 1,length(x) );
for n=[1:20]
    saw = saw + 2*(-1)^(n +1)*sin(n*x)/n;
    cusp = cusp + 2 *(-1)^n*cos(n*x)/n/n;
end
figure(1)
```

1Uniform convergence is a sufficient condition for exchanging integration with summation. But the Fourier series of the discontinuous function $\text{saw}(x)$ is not uniformly convergent, so there is a red flag here. However JJ says this is not a problem: A Fourier series can always be integrated term by term to provide the integral of the target function.
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Figure 9.3: Three terms in (9.33) make a rough approximation to the dotted square. polarsquare.eps

```matlab
plot(x,saw,x,cusp)
hold on
plot([-4 4],[0 0],'k')
axis([-4,4 -4,4]), xlabel('x'), ylabel('saw(x) and cusp(x)')
```

Notice that the coefficients in the Fourier series for cusp(x) decrease like \( k^{-2} \), and this is much faster than the \( k^{-1} \) decrease in the saw-series (9.25). The relatively fast \( k^{-2} \) ensures that partial sums of the Fourier series of cusp(x) are not afflicted with ugly wiggles. We'll see as a general rule that functions with discontinuities have \( O(k^{-1}) \) Fourier coefficients. Integration produces a continuous function, with a more rapidly convergent Fourier series.

**Example:** Consider a square in the \((x,y)\)-plane defined by the four vertices \((1,0), (0,1), (-1,0)\) and \((0,-1)\). The square can be represented in polar coordinates as \( r = R(\theta) \). Find a Fourier series representation of \( R(\theta) \).

Since \( R(\theta) = R(-\theta) \) we only need the cosines. But we also have \( R(\theta) = R(\theta + \pi/2) \), and this symmetry implies that

\[
R(\theta) = a_0 + a_4 \cos 4\theta + a_8 \cos 8\theta + \cdots
\]

(9.29)

We leave out \( \cos \theta, \cos 2\theta, \cos 3\theta \) etc because these terms reverse sign if \( \theta \to \theta + \pi/2 \).

In the first quadrant of the \((x,y)\)-plane, the square is \( x + y = 1 \), or
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\( R(\theta) = (\cos \theta + \sin \theta)^{-1} \). The first term in the Fourier series is therefore

\[
a_0 = \frac{1}{2\pi} \int R(\theta) \, d\theta, \quad (9.30)
\]

\[
a_0 = \frac{2}{\pi} \int_{\pi/2}^{\pi} \frac{d\theta}{\cos \theta + \sin \theta}. \quad (9.31)
\]

We've used symmetry to reduce the integral to four times the integral over the side in the first quadrant. The mathematica command

\[\text{Integrate}[1/(\text{Sin}[x] + \text{Cos}[x]), \{x, 0, \text{Pi}/2\}]\]

tells us that

\[
a_0 = \frac{2\sqrt{2}}{\pi} \text{tanh}^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.7935. \quad (9.32)
\]

The higher terms in the series are

\[
a_{4k} = \frac{1}{\pi} \int \cos(4k\theta)s(\theta) \, d\theta = \frac{4}{\pi} \int_{\pi/2}^{\pi/2} \frac{\cos(4k\theta) \, d\theta}{\cos \theta + \sin \theta}. \quad (9.33)
\]

With mathematica, we find

\[
a_4 = \frac{4}{\pi} \left[\frac{4}{3} - \sqrt{2}\text{tanh}^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \approx 0.1106, \quad (9.34)
\]

\[
a_8 = -\frac{4}{\pi} \left[\frac{128}{105} - \sqrt{2}\text{tanh}^{-1}\left(\frac{1}{\sqrt{2}}\right)\right] \approx 0.0349. \quad (9.35)
\]

Mathematica gives a general formula in terms of hypergeometric series. This is more than we need to know about this example: Figure 9.3 shows that the first three terms in the series can be used to draw a pretty good square. The Fourier series in this example has moderately fast convergence — we’ll see later that \(a_{4k} = O(k^{-2})\) as \(k \to \infty\). This \(k^{-2}\) decrease is faster than \(a_k = O(k^{-1})\) in the Fourier series (9.20).

Other expansion intervals

Suppose we want to represent a function defined on an interval \(0 \leq x \leq a\). We can expand \(f(x)\) as a Fourier series with period \(a\):

\[
f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos \left(\frac{2\pi kx}{a}\right) + b_k \sin \left(\frac{2\pi kx}{a}\right). \quad (9.36)
\]

We need both the sines and the cosines to represent an arbitrary function \(f\) in this fashion.
But we can also extend the definition of \( f(x) \) to the interval \(-a \leq x \leq 0\) by \( f(-x) = -f(x) \), and so define an odd function with period 2\(a\). This function can then be expanded in sines alone. Or we might extend the definition of \( f(x) \) to the interval \(-a \leq x \leq 0\) by \( f(-x) = f(x) \) i.e., we define an even function of period 2\(a\), which can then be expanded in cosines alone.

For example, suppose \( a = \pi \) i.e., \( f(x) \) is given only for \( 0 < x < \pi \) (aka the half-range). Then we can expand \textit{nonuniquely} on this interval \( 0 < x < \pi \) using either a sine series

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx \, dx, \quad (9.37)
\]

or a cosine series

\[
f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx \, dx. \quad (9.38)
\]

In the first case we used an \textit{odd} extension: \( f(x) = -f(-x) \) and in the second case we use the \textit{even} extension \( f(x) = f(-x) \).

Of course we can extend the definition of \( f(x) \) to \(-\pi \leq x \leq 0\) in many different ways so the two choices above certainly don’t exhaust the possibilities of expanding \( f(x) \) on an interval \((0, a)\). However the rate of convergence of these various Fourier representations can be very different: often one series is much faster than another.

**Example:** Represent \( \cos x \) on \( 0 \leq x \leq \pi \) using only sine functions. Sketch the sum of the resulting Fourier series on the interval \([-\pi, 3\pi]\).

Because we’re using only sines, we’re constructing a function which is odd on the fundamental interval \(-\pi \leq x \leq \pi \) i.e., the target function in this problem is

\[
f(x) = \sqrt{x} \cos x. \quad (9.39)
\]

This function coincides with \( \cos x \) on \((0, \pi)\) and is an odd function with period \(2\pi\). The Fourier series representation is

\[
\sqrt{x} \cos x = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{m \sin 2mx}{4m^2 - 1}, \quad (9.40)
\]

\[
= \frac{8}{\pi} \left[ \frac{1}{3} \sin 2x + \frac{2}{15} \sin 4x + \frac{3}{35} \sin 6x + \cdots \right]. \quad (9.41)
\]

To obtain the series (9.40), we use the formula (9.37):

\textit{c.c.} = \textit{complex conjugate}
\[ b_n = \frac{2}{\pi} \int_0^{\pi} \cos x \sin nx \, dx, \]
\[ = \frac{2}{\pi} \int_0^{\pi} \frac{e^{ix} + e^{-ix}}{2} \left( e^{inx} - e^{-inx} \right) \, dx, \]
\[ = \frac{1}{2\pi i} \int_0^{\pi} \left( e^{(n+1)x} + e^{(n-1)x} - c.c. \right) \, dx, \]
\[ = \frac{1}{\pi} \int_0^{\pi} \sin(n+1)x + \sin(n-1)x \, dx, \]
\[ = \frac{1}{\pi} \left[ \frac{1 - \cos(n+1)\pi}{n+1} + \frac{1 - \cos(n-1)\pi}{n-1} \right]. \] (9.42)

As one can anticipate using a symmetry argument, all the odd \( b_n \)'s are zero. If \( n \) is even, we write \( n = 2m \) so that
\[ b_m = \frac{8m}{\pi} \left[ \frac{1}{4m^2 - 1} \right]. \]
Alternatively, we can do this integration with the mathematica command:

Assuming[Element[n, Integers],
Integrate[Cos[x]*Sin[n*x], {x, 0, Pi}]]

Mathematica dutifully responds with
\[ ((1 + (-1)^n) n)/(-1 + n^2) \]

\[ 9.3 \quad \text{Convergence of Fourier series} \]

Because the coefficients in the Fourier series
\[ \text{sqr}(x) = \frac{4}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \cdots \right]. \] (9.43)
decay rather slowly, like \( k^{-1} \), this series doesn’t converge very quickly. It’s churlish to complain about slow convergence because we are representing a discontinuous function as a linear superposition of smoothly varying sinusoids: it’s amazing that it works at all. Nonetheless, let’s examine the convergence in some detail. If we truncate the series with \( n \) terms, the maximum pointwise error is
\[ P_n = \max_{-\pi \leq x \leq \pi} |\text{sqr}(x) - S_n(x)|. \] (9.44)

We find \( P_1 = 0.27, P_3 = 0.2 \) and remarkably
\[ \lim_{n \to \infty} P_n = 0.18. \] (9.45)
In other words, the pointwise error never disappears. Figure 9.1 shows that the error squeezes into the close proximity of points such as \( x = 0 \) at which \( \sqrt{x} \) is discontinuous. This behavior means that \( S_n(x) \) does not converge uniformly to \( \sqrt{x} \). This problem with non-uniform convergence was discovered by Kelvin and first explained by Gibbs. We discuss Gibbs’ phenomenon in more detail below.

Gibbs’ phenomenon is not inconsistent with \( \lim_{n \to \infty} \epsilon_n = 0 \): the small region near \( x = 0 \) in which there is significant pointwise error makes an increasingly feeble contribution to \( \epsilon_n \) as \( n \) increases. If \( \epsilon_n \to 0 \) as \( n \to \infty \) we say that the Fourier series converges in the mean. In applications we are often concerned with convergence in the mean, rather than pointwise convergence. Consideration of convergence in the mean leads us to Parseval’s theorem in the next section.

Gibbs’ phenomenon

Truncating the Fourier series of \( \sqrt{x} \) in (9.20) we obtain the finite sum

\[
S_{2q+1}(x) = \frac{4}{\pi} \left[ \sin x + \frac{1}{3} \sin 3x + \cdots + \frac{1}{2q+1} \sin[(2q+1)x] \right].
\]

This sum be expressed exactly in terms of an integral:

\[
S_{2q+1}(x) = \frac{2}{\pi} \int_0^x \frac{\sin[2(q+1)t]}{\sin t} \, dt,
\]

(see problem 9.14). Carefully looking at the right panel in figure 9.1 we are inspired to simplify the integral (9.47) in the neighbourhood of \( x = 0 \). We can replace the \( \sin t \) in the denominator by \( t \) but we can’t touch the \( \sin[2(q+1)t] \) in the numerator because \( q \) is large. Thus we find that if \( x \) is close to zero

\[
S_{2q+1}(x) \approx \frac{2}{\pi} \int_0^{2(q+1)x} \frac{\sin u}{u} \, du.
\]

(9.48)

The approximation (9.48) is compared with the sum (9.46) in figure 9.4. Notice that (9.48) is a “similarity solution” in the sense that on the RHS

\[
\text{Recall that the mean square error is:} \\
\epsilon_n = \int_{-\pi}^{\pi} (f - S_n)^2 \, dx/2\pi
\]
Figure 9.4: Comparison of the approximation \(9.48\) with the partial sum \(S_{19}(x)\). Notice the 18% overshoot. gibbs.eps

\(q\) and \(x\) appear only in the combination \((q + 1)x\). Thus the truncated Fourier series, \(S_n\), differs from the target function \(sqr(x)\) in a neighbourhood of width \(\Delta x \sim 1/n\) centered on a point of discontinuity.

It follows from (9.48) that

\[
\lim_{q \to \infty} \lim_{x \to 0} S_{2q+1}(x) = 0,
\]

and

\[
\lim_{x \to 0} \lim_{q \to \infty} S_{2q+1}(x) = 1.
\]

The two limits are not exchangeable. We can also see that the maxima of \(S_{2q+1}(x)\) are at \(2(q + 1)x = \pi, 2\pi\) and so on. At the first maximum

\[
S_{2q+1} \left( \frac{\pi}{2(q + 1)} \right) \approx 2 \int_0^{\pi} \frac{\sin u}{u} \, du
\]

\[
= 2 \left( 1 - \frac{\pi^2}{3 \times 3!} + \frac{\pi^4}{5 \times 5!} - \frac{\pi^6}{7 \times 7!} + \cdots \right),
\]

\[
= 1.18.
\]

This is the source of the irreducible 18% pointwise error at the discontinuities.
9.4 Parseval’s equation

The accuracy of \( f(x) \approx S_n(x) \) in (9.6) increases, or at least can’t get worse, as we add more terms. And we can pass to the limit \( n \to \infty \) and obtain an infinite Fourier series.

Now that we’ve got the coefficients in (9.6) we can show (see problem 9.11) that provided the target function has a finite square integral,

\[
\int_{-\pi}^{\pi} f^2(x) \, \frac{dx}{2\pi} < \infty ,
\]

then the minimum mean square error is

\[
\min_{\forall a_k, b_k} \varepsilon(a_k, b_k) = \int_{-\pi}^{\pi} f^2(x) \, \frac{dx}{2\pi} - a_0^2 - \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2) .
\]

Since the mean square error is positive we must have

\[
\int_{-\pi}^{\pi} f^2(x) \, \frac{dx}{2\pi} \geq a_0^2 + \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2) .
\]

This is Bessel’s inequality. The sum on the right is non-decreasing and is bounded above by the integral of the squared target function on the left. This means the sum converges as \( n \to \infty \), and that the terms in the sum approach zero as \( n \to \infty \).

As \( n \) increases the finite Fourier expansion can be imagined as constantly reducing \( \varepsilon \) by piling on more and more orthogonal sinusoids. If in the limit \( n \to \infty \) the error \( \varepsilon_n(a_k, b_k) \) vanishes then Bessel’s inequality becomes Parseval’s equality:

\[
\int_{-\pi}^{\pi} f^2(x) \, \frac{dx}{2\pi} = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) .
\]

We often need to know quadratic integrals of functions (e.g., the energy of a field). Parseval’s equation is an important computational tool because it enables us to calculate these integrals using the Fourier coefficients.

In fact, mathematicians\(^3\) have shown that the sinusoidal basis set \( \{ \cos kx, \sin kx \} \) is complete on the interval \( -\pi \leq x \leq \pi \). This means

\(^3\)This is a special case of the Fischer-Riesz Theorem.
that \( \lim_{n \to \infty} \varepsilon_n = 0 \) for any \( f(x) \) satisfying (9.52). In other words, with enough terms the Fourier series captures all the energy in the target function. Because of Gibbs phenomenon this convergence in the mean not imply that the Fourier series converges pointwise to \( f(x) \).

The square wave again

We illustrate Parseval’s equation by noting that

\[
\int_{-\pi}^{\pi} \text{sqr}^2(x) \, dx = 2\pi. \tag{9.56}
\]

Thus, according to Parseval’s equation (9.55),

\[
\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots \tag{9.57}
\]

The examples in the first part of this lecture can now be used to generate other wonderful series.

**Example:** Sum the series

\[
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \tag{9.58}
\]

Recalling that

\[
\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots = \frac{1}{2} \text{saw}(x), \tag{9.59}
\]

we have

\[
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{4\pi} \int_{-\pi}^{\pi} \text{saw}^2(x) \, dx = \frac{\pi^2}{6}. \tag{9.60}
\]

9.5 The complex form of the Fourier series

Using Euler’s formula:

\[
e^{i\theta} = \cos \theta + i \sin \theta \tag{9.61}
\]

the real Fourier series in (9.6) and (9.18) is equivalent to

\[
f(x) = \sum_{m=-\infty}^{\infty} f_m e^{imx}, \quad \text{where} \quad f_m = \int_{-\pi}^{\pi} f(x) e^{-imx} \frac{dx}{2\pi}. \tag{9.62}
\]
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The key to remembering this result (or working it out very quickly) is the orthogonality property of the complex sinusoids:
\[
\int_{-\pi}^{\pi} e^{i(p-q)x} \frac{dx}{2\pi} = \delta_{pq}.
\]
(9.63)

The complex form of Parseval’s theorem is
\[
\int_{-\pi}^{\pi} f^2(x) \frac{dx}{2\pi} = \sum_{m=-\infty}^{\infty} f_m f^*_m.
\]
(9.64)

There are many reasons for preferring the complex form e.g., no more irritating factors of 2. And in the future lectures we can very quickly obtain the Fourier Integral Theorem starting from (9.62).

It is easy to show that if \( f(x) \) is a real function then the Fourier coefficients must satisfy the reality condition
\[
f_m = f_{-m}^*.
\]
(9.65)

This is is an important check on your algebra.

The complex form and the real form are completely equivalent. In particular you should show the relation between the coefficients is
\[
a_m = f_m + f_{-m} \quad \text{and} \quad b_m = i(f_m + f_{-m}).
\]
(9.66)

**Example:** expand \( f(x) = \exp(\alpha x) \) on the fundamental interval \((-\pi, \pi)\) in a complex Fourier series.

The complex Fourier coefficients are given by
\[
f_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\alpha-im)x} \frac{dx}{2\pi},
\]
\[
= \frac{1}{2\pi} \left[ e^{i(\alpha-im)x} \right]_{-\pi}^{\pi},
\]
\[
= \frac{1}{2\pi} \frac{\alpha + im}{\alpha^2 + m^2} \left( e^{i(\alpha-im)\pi} - e^{-(\alpha-im)\pi} \right).
\]
(9.67)

Notice that the reality condition is satisfied.

9.6 The Dirac comb

Throwing caution to the winds, we now calculate the Fourier series of the \( \delta \)-function. Because of the sifting property
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x)e^{-ikx} \frac{dx}{2\pi} = \frac{1}{2\pi},
\]
(9.68)
the result is
\[ \delta_c(x) = \frac{1}{2\pi} \sum_{k=\infty}^{\infty} e^{ikx}. \] (9.69)

Notice that on the infinite line, as opposed to the fundamental interval \((-\pi, \pi)\), the series on the right hand side of (9.69) is actually the Dirac comb i.e., a sequence of \(\delta\)-functions at \(x = 2m\pi\) where \(m = \cdots -1, 0, 1, \cdots\). This is why I have used the subscript \(c\) in \(\delta_c(x)\). So, to summarize, we have the basic result
\[ \delta_c(x) \equiv \sum_{m=\infty}^{\infty} \delta(x - 2\pi m) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx}. \] (9.70)

If we multiply (9.70) by some arbitrary function \(f(\xi - x)\) and integrate with respect to \(x\) over the interval \((-\pi, \pi)\) then we reproduce the Fourier series expansion of \(f(\xi)\). Here is the formal calculation:
\[
\begin{align*}
  f(\xi) &= \int_{-\pi}^{\pi} f(\xi - x) \delta_c(x) \, dx, \\
  &= \int_{-\pi}^{\pi} f(\xi - x) \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ikx} \, dx, \\
  &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} f(\xi - x) e^{ikx} \, dx, \\
  &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ik\xi} \int_{\xi-\pi}^{\xi+\pi} f(y) e^{-iky} \, dy, \\
  &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f_k e^{ik\xi}. \tag{9.71}
\end{align*}
\]

So we have a consistent set of results — the formal manipulations above are an expression of the Fourier series representation of a periodic function.

Even after these reassuring remarks about Fourier series, you might still be feeling nervous about the formal construction in (9.70). Here is another way of understanding (9.70). If we stop the sum, keeping only
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2n + 1 terms, we have a function:

\[ D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{k=n} e^{ikx}, \]  
(9.72)

\[ = \frac{\sin \left( \left( n + \frac{1}{2} \right) x \right)}{2\pi \sin(x/2)}, \]  
(9.73)

which is called the Dirichlet kernel. You’re asked to prove the result above in problem 9.15.

At \( x = 0 \) we have \( D_n(0) = (2n + 1)/2\pi \) and at \( x = \pi/(n + 1/2) \), \( D_n = 0 \). Both of these results should remind you of a \( \delta \)-sequence: the peak grows linearly with \( n \) and the peak width decreases as \( 1/n \). Moreover, it is obvious from (9.72) that

\[ \int_{-\pi}^{\pi} D_n(x) \, dx = 1, \]  
(9.74)

so the integral of \( D_n(x) \) is equal to one, independent of \( n \). The graph of \( D_n(x) \) in Figure 9.5 tells the story: as \( n \to \infty \) the Dirichlet kernel

Figure 9.5: The Dirichlet kernel with \( n = 10 \). DirichletKernel.eps
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becomes concentrated around $x = 0$. Thus if $f(x)$ is any reasonably smooth function

$$\int_{-\pi}^{\pi} f(x - y)D_n(y) \, dy \rightarrow f(x), \quad \text{as } n \rightarrow \infty.$$  \hspace{1cm} (9.75)

Notice that (9.75) does not converge at a fixed point: if we pick, say $x = 1$, and let $n \rightarrow \infty$ then $D_n(1)$ oscillates back and forth between the limits $\pm 1/2\pi \sin(1/2)$. The wavelength of the oscillations decreases as $n^{-1}$, and thus the sifting property of the $\delta$-function relies on the destructive interference between the alternately positive and negative waves.

Convergence of Fourier series for piecewise smooth functions

I feel obliged to prove (9.75) for the “reasonably smooth” functions appearing in applications. Let’s assume $f(x)$ has at least one derivative in sub-intervals of $(-\pi, \pi)$. At the ends of the sub-intervals, $f(x)$ has a simple jump discontinuities. Thus at every point $x \in (-\pi, \pi)$, $f(x)$ has both a left limit $f(x^-)$ and a right limit $f(x^+)$, and moreover the left and right derivatives also exist at every point $x$. The function $\mathrm{sqr}(x)$ is a simple example.

Now consider

$$f_n(x) \equiv \int_{-\pi}^{\pi} D_n(y)f(x - y) \, dy,$$

$$= \frac{f(x^+) + f(x^-)}{2}$$

$$+ \left[ \int_{-\pi}^{0} \sin \left( (n + \frac{1}{2})y \right) \frac{f(x - y) - f(x^-)}{2\pi \sin(y/2)} \, dy \right]$$

$$+ \left[ \int_{0}^{\pi} \sin \left( (n + \frac{1}{2})y \right) \frac{f(x - y) - f(x^+)}{2\pi \sin(y/2)} \, dy \right].$$ \hspace{1cm} (9.76)

Because $f$ has left and right derivatives, the ratios

$$\frac{f(x - y) - f(x^\pm)}{2\pi \sin(y/2)}$$

approach finite limits as $y \rightarrow 0$. Thus, invoking Riemann-Lebesgue, the integrals above are zero as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} f_n(x) = \frac{f(x^+) + f(x^-)}{2}.$$ \hspace{1cm} (9.77)

\footnote{The result (9.77) is true for a much wider class of functions that satisfy Dirichlet conditions.}

SIO203C, W.R. Young, March 21, 2011
9.7 References for Fourier series

Chapter 14 of JJ is a good reference. The slim book

\textbf{DymMc} \textit{Fourier Series and Integrals} by H. Dym & H.P. McKean

is an accessible account of the rigorous theory. For much information
on the rate of convergence of Fourier series see chapter 2 of

\textbf{Boyd} \textit{Chebyshev and Fourier Spectral Methods} by J.P. Boyd.

9.8 Problems

\textbf{Problem 9.1.} Find a least squares approximation

\[ \text{sqr}(x) \approx \alpha x \]  \hspace{1cm} (9.78)

on the interval \((-\pi, \pi)\) by minimizing

\[ \varepsilon(\alpha) = \int_{-\pi}^{\pi} \left[ \text{sqr}(x) - \alpha x \right]^2 \frac{dx}{2\pi}. \]  \hspace{1cm} (9.79)

Show that the minimum error is achieved by \(\alpha = 3/(2\pi)\). Indicate the
difficulties which arise if you attempt to improve this approximation with

\[ \text{sqr}(x) \approx \alpha x - \beta x^3. \]  \hspace{1cm} (9.80)

Why this approach is inferior to that of Fourier in (9.6)?

\textbf{Problem 9.2.} Represent the function

\[ f(x) = x(\pi - x) \]

on the interval \(0 \leq x \leq \pi\) as: (i) A sine series; (ii) a cosine series. (iii) Sum
the two series at \(x = \pi/2\), keeping only the first three non-zero terms in
each. Which series is more rapidly convergent to \(f(\pi/2) = \pi^2/4\)?

\textbf{Problem 9.3.} Suppose \(f(x)\) is defined on the half range \(0 \leq x \leq \pi\)
and we extend the definition of \(f(x)\) to the entire range \(-\pi \leq x \leq \pi\)
by taking \(f(x) = 0\) if \(-\pi \leq x < 0\). Find the analog of the half-range
expansions in (9.37) and (9.38). Write out the three different half-range
expansions of the function \(f(x) = 1\).
Problem 9.4. Represent the function

\[ f(x) = \cos qx \]

on the interval \(-\ell \leq x \leq \ell\) as a Fourier series.

Problem 9.5. Find the Fourier coefficients

\[ |\sin x| = \sum_{k=0}^{\infty} a_k \cos kx. \]

Problem 9.6. Define the \(2\pi\)-periodic function \(\text{ramp}(x)\) by

\[ \frac{d}{dx}\text{ramp}(x) = -\text{sqr}(x), \quad \text{and} \quad \int_{-\pi}^{\pi} \text{ramp}(x) \, dx = 0. \] (9.81)

(i) Find an explicit expression for \(\text{ramp}(x)\) on the fundamental interval.
(ii) and also a Fourier series representation. (iii) Use MATLAB to compare ten-term partial sums of the Fourier series for \(\text{saw}(x)\) and \(\text{ramp}(x)\) (see Figure 9.6).

Problem 9.7. Sum the Fourier series

\[ f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}. \]
Problem 9.8. Figure 9.7 shows two functions of $x$ on the fundamental interval $-\pi < x < \pi$. Here are four possible Fourier series representations of these two functions:

$$f_1(x) = a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots$$
$$f_2(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \cdots$$
$$f_3(x) = a_1 \cos x + a_3 \cos 3x + a_5 \cos 5x + \cdots + b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \cdots$$
$$f_4(x) = a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \cdots + b_2 \sin 2x + b_4 \sin 4x + b_6 \sin 6x + \cdots$$

(i) Which representation might apply to the function shown in the top panel of the figure? (ii) Which representation might apply to the function shown in the bottom panel of the figure? Lucky guesses don’t count: explain your reasoning.

Problem 9.9. Develop the least-squares approximation from scratch using the complex sinusoids as a basis. That is, instead of (9.6), start with

$$f(x) \approx \sum_{k=-n}^{n} f_k e^{ikx}.$$  

(9.82)
Show that minimizing
\[
\varepsilon_n(f_k) \equiv \left( \int_{-\pi}^{\pi} \left[ f(x) - \sum_{k=-n}^{n} f_k e^{i k x} \right]^2 \frac{dx}{2\pi} \right)^{1/2}
\] (9.83)
by varying \(f_k\) leads to (9.62). Show further that using these optimal coefficients the minimal error is given by
\[
\min_{f_k} \varepsilon_n(f_k) = \int_{-\pi}^{\pi} f^2(x) \frac{dx}{2\pi} - \sum_{k=-n}^{n} f_k f_k^* .
\] (9.84)
Assume that for suitably smooth functions, \(f(x)\), the complex sinusoids are complete in the sense that \(\varepsilon_n \to 0\) as \(n \to \infty\) and deduce the complex form of Parseval’s theorem in (9.64).

**Problem 9.10.** Sum the series
\[
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots
\]

**Problem 9.11.** Prove (9.53). Hint: prove that
\[
\int_{-\pi}^{\pi} f(x) S_n(x) \frac{dx}{2\pi} = \int_{-\pi}^{\pi} S_n'(x) \frac{dx}{2\pi}
\]
and
\[
\int_{-\pi}^{\pi} S_n^2(x) \frac{dx}{2\pi} = a_0^2 + \frac{1}{2} \sum_{k=1}^{n} (a_k^2 + b_k^2) .
\]

**Problem 9.12.** Consider the Fourier coefficients \(a_{4k}\) defined by (9.29). Find the sum of the series
\[
a_0^2 + \frac{1}{2} \left( a_4^2 + a_8^2 + \cdots \right) .
\]

**Problem 9.13.** Suppose that \(a(t)\) and \(b(t)\) are \(2\pi\) periodic functions with Fourier series representations
\[
a(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt} , \quad b(t) = \sum_{m=-\infty}^{\infty} b_m e^{imt} .
\]
Find the Fourier series of the convolution
\[
a \circ b(t) \equiv \int_{0}^{2\pi} a(t')b(t - t') \, dt' .
\]
Problem 9.14. Fill in the steps between (9.46) and (9.47). Hint
\[ S_{2q+1}(x) = \frac{2}{\pi} \int_0^x dt \sum_{k=0}^q \left[ e^{(2k+1)it} + e^{-(2k+1)it} \right]. \]

Problem 9.15. Prove (9.73).

Problem 9.16. (i) Show that on the fundamental interval \(-\pi \leq x \leq \pi\) the “box-car” function
\[ b(x; \ell) \equiv \begin{cases} 1, & \text{if } |x| < \ell; \\ 0, & \text{if } |x| > \ell; \end{cases} \]
has the Fourier series representation
\[ b(x; \ell) = \sum_{n=\leftarrow}^{\infty} \frac{\sin n\ell}{n\pi} e^{in\pi}. \]
(ii) Compute the convolution
\[ \text{sqr}(x) \equiv \int_{-\pi}^{\pi} \text{sqr}(x') b(x-x'; \ell) \, dx'. \]
(iii) Use the results of problem 9.13 to obtain the Fourier series of the smoothed square wave \( \text{sqr}(x) \). Is there Gibbs phenomenon?

Problem 9.17. Suppose \( f(x) \) is defined on the half-range \( 0 < x \leq \pi \). (i) How must we define \( f(-x) \) if all of the cosine terms in the Fourier series are to vanish? (ii) How must we define \( f(-x) \) if all of the sine terms in the Fourier series are to vanish? (iii) How must we define \( f(-x) \) if all of the even harmonics in the Fourier series are to vanish? (iv) How must we define \( f(-x) \) if all of the odd harmonics in the Fourier series are to vanish?

Problem 9.18. Use Parseval’s theorem and the Fourier series with coefficients in (9.67) to deduce that
\[ \pi \coth \pi \alpha = \sum_{m=-\infty}^{\infty} \frac{\alpha}{\alpha^2 + m^2}. \] (9.85)

Solution. An intermediate result is
\[ |f_m|^2 = \frac{1}{4\pi^2} \frac{(e^{\pi\alpha} - e^{-\pi\alpha})^2}{\alpha^2 + m^2}. \] (9.86)
Lecture 10

More Fourier series

10.1 Integration of Fourier Series

We generate new Fourier series from old by the linear operations of differentiation and integration. We start by considering integration. It is essential to realize that the integral of a periodic function is not necessarily periodic. For example, consider the target function \( f(x) = 1 \). This function is periodic with period \( 2\pi \), and its Fourier series is splendidly simple:

\[
f(x) = 1 + \sum_{n=1}^{\infty} (0 \times \sin kx + 0 \times \cos kx).
\] (10.1)

But the integral of \( f(x) \) is \( F(x) = x \), which is not periodic. Let’s define a “sawtooth” function

\[
saw(x) = x, \quad \text{if } -\pi < x < \pi,
\] (10.2)

and \( saw(-\pi) = saw(\pi) = 0 \). We define \( saw(x) \) outside the fundamental interval by \( saw(x) = saw(x + 2\pi) \). Using our recipe, the Fourier representation of \( saw(x) \) is

\[
saw(x) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin kx.
\] (10.3)

This is the Fourier series of the integral of the target function \( f(x) = 1 \). The left panel of Figure 9.6 shows the Fourier representation of \( saw(x) \) using ten terms in the series (10.3).
Now consider a general target function \( f(x) \) with a Fourier representation:

\[
f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i k x}.
\]  

(10.4)

We define

\[
F(x) \equiv \int_{0}^{x} (f(x') - f_0) \, dx',
\]

(10.5)

\[
= \sum_{n=-\infty}^{\infty} F_n e^{i k x}.
\]

(10.6)

By definition, \( F(x) \) is the integral of a periodic function with no “DC content” and therefore \( F(x) = F(x + 2\pi) \). How are the coefficients \( F_n \) related to \( f_n \)?

The \( k = 0 \) coefficient is

\[
F_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{x} (f(x') - f_0) \, dx' \, dx.
\]

(10.7)

To obtain \( F_k \) with \( k \neq 0 \) we use integration by parts

\[
F_k = \int_{-\pi}^{\pi} F(x) \frac{d}{dx} \frac{e^{-ikx}}{-ik} \, dx,
\]

(10.8)

\[
= \frac{1}{ik} \int_{-\pi}^{\pi} e^{-ikx} (f(x) - f_0) \, dx,
\]

(10.9)

\[
= \frac{f_k}{ik}.
\]

(10.10)

Notice that the terms falling outside the integral when we integrate by parts above are zero because \( F(-\pi) = F(\pi) \).

Thus we have

\[
F(x) = F_0 + \sum_{n=-\infty}^{\infty} \frac{f_k}{ik} e^{i k x}.
\]

(10.11)

Using the definition of \( F(x) \), this is equivalent to

\[
\int_{0}^{x} f(x') \, dx' = f_0 x + F_0 + \sum_{n=-\infty}^{\infty} \frac{f_k}{ik} e^{i k x}.
\]

(10.12)
In (10.12) we have the integral of the periodic target function \( f(x) \): this integral is not periodic unless \( f_0 = 0 \). But we can define yet another function

\[
g(x) \equiv f_0 \text{saw}(x) + F_0 + \sum_{k=\infty}^{\infty} \frac{f_k}{1k} e^{ikx}.
\]

(10.13)

The function \( g(x) \) is periodic, \( g(x) = g(x+2\pi) \), and is equal to \( \int_0^\infty f(x') \, dx' \) on the fundamental interval. There is no real difficulty here: one just has to be careful to distinguish between (10.12) and (10.13).

### 10.2 Differentiation of a Fourier series

### 10.3 Fourier series without evaluating integrals

Every complex power series you know can be used to obtain a Fourier series at no extra charge. For example, consider

\[
\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots
\]

(10.14)

If we evaluate this with polar coordinates, \( z = r e^{i\theta} \), then

\[
\exp \left( r e^{i\theta} \right) = 1 + r e^{i\theta} + \frac{r^2}{2} e^{2i\theta} + \frac{r^3}{6} e^{3i\theta} + \cdots
\]

(10.15)

Taking the real part we have

\[
e^r \cos \theta \cos(r \sin \theta) = 1 + r \cos \theta + \frac{r^2}{2} \cos 2\theta + \frac{r^3}{6} \cos 3\theta + \cdots
\]

(10.16)

The imaginary part of (10.15) produces another monster. You’d find it challenging to obtain these Fourier series using the integration formulas in (9.17) through (9.18).

### 10.4 The rate of convergence of a Fourier series

We have encountered Fourier series with very different rates of convergence. The slowest series is that of \( \delta_c(x) \) in (9.69). In fact, that series
Lecture 10. More Fourier series

doesn’t really converge at all: the oscillations in the Dirichlet kernel become faster, but not smaller, with increasing $N$. The series for $\text{sqr}(x)$ in (9.20) is a little better, since there $f_k = O(n^{-1})$ as $k \to \infty$. This $O(k^{-1})$ behavior is typical of sectionally smooth functions (i.e., functions that are infinitely differentiable in finite intervals, with discontinuous jumps between the intervals). Another example of a sectionally smooth function with $f_k = O(k^{-1})$ is $\text{saw}(x)$.

The Fourier coefficients of $\text{ramp}(x)$ in (??) decrease as $O(n^{-2})$, so the $\text{ramp}(x)$ series converges much faster than the $\text{sqr}(x)$ and $\text{saw}(x)$ series. This is because $\text{ramp}(x)$ is less singular than $\text{sqr}(x)$ and $\text{saw}(x)$ i.e., a jump in a derivative is not as bad as a jump in the function itself. And if integrate $\text{ramp}(x)$ we obtain a function with $O(n^{-3})$ Fourier convergence. This new function has a jump only in its second derivative. The message is that the rate of convergence of a Fourier series is determined by the strength of the singularities in the target function.

Now let’s go to the other extreme and consider very rapidly convergent Fourier series, such as

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x .$$

(10.17)

Another example of a rapidly convergent Fourier series is

$$\frac{1 - r^2}{1 + r^2 - 2r \cos x} = 1 + 2r \cos x + 2r^2 \cos 2x + 2r^3 \cos 3x + \cdots$$

(10.18)

If $|r| < 1$ then the coefficients decrease as $r^k = e^{k \ln r}$, which is faster than any power of $k$. The series

$$e^{y \cos x} = I_0(y) + 2 \sum_{k=1}^{\infty} I_k(y) \cos nx$$

(10.19)

in (??) is yet another example of exponentially fast convergence. These series converge quickly because the target function is infinitely differentiable.

We can prove that if $f(x)$ and its first $p - 1$ derivatives is continuous and differentiable in the closed interval $-\pi \leq x \leq \pi$, and the $p$'th derivative exists apart from jump discontinuities at some points, then the Fourier coefficients are $O(n^{-p-1})$ as $n \to \infty$. Functions such as $\text{sqr}(x)$, with jump discontinuities, correspond to $p = 0$. Very smooth functions such as (10.17), (10.18) and (10.19) correspond to $p = \infty$. 

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These results are obtained by evaluating the Fourier coefficient

\[ f_n = \int_{-\pi}^{\pi} f(x)e^{inx} \, dx , \quad (10.20) \]

using integration by parts \(^1\). Suppose we can break the fundamental interval up into sub-intervals so that \( f(x) \) is smooth (i.e., infinitely differentiable) in each subinterval. Non-smooth behavior, such as a jump in some derivative, occurs only at the ends of the interval. Then the contribution of the sub-interval \((a, b)\) to \( f_n \) is

\[
I_n \equiv \int_{a}^{b} f(x)e^{inx} \, dx ,
\]

\[
= \frac{1}{in} \int_{a}^{b} f(x)\frac{de^{inx}}{dx} \, dx ,
\]

\[
= \frac{1}{in} \left[ f(x)e^{inx} \right]_{a}^{b} - \frac{1}{in} \int_{a}^{b} f'(x)e^{inx} \, dx . \quad (10.21)
\]

Since \( f(x) \) is smooth, we can apply integration by parts to \( J_n \) to obtain

\[
I_n = \frac{1}{in} \left[ f(x)e^{inx} \right]_{a}^{b} - \frac{1}{n^2} \left[ f'(x)e^{inx} \right]_{a}^{b} + \frac{1}{n^2} \int_{a}^{b} f''(x)e^{inx} \, dx . \quad (10.22)
\]

Obviously we can keep going and develop a series in powers of \( n^{-1} \). Thus we can express \( I_n \) in terms of the values of \( f \) and its derivatives at the end-points.

Suppose, for example, we have a function such as those on the right of \((10.17), (10.18)\) and \((10.19)\). These examples are smooth throughout the fundamental interval. In this case we take \( a = -\pi \) and \( b = \pi \) and use the result above. Since \( f(x) \) and all its derivatives have no jumps, even at \( x = \pm \pi \), all the end-point terms vanish. Thus in this case \( f_n \) decreases faster than any power of \( n \) e.g., perhaps something like \( e^{-n} \), or \( e^{-\sqrt{n}} \). In this case integration-by-parts does not provide the asymptotic rate of decay of the Fourier coefficients — we must deploy a more potent method such as steepest descent.

\(^1\)Section 6.3 of BO is a good reference.
10.5 Response of a black box to periodic input

10.6 Problems

**Problem 10.1.** Integrate the Fourier series of ramp\((x)\) term-by-term and deduce the Fourier series representation of the following functions

\[
f(x) \equiv \frac{\pi^2}{8} x - \frac{\pi}{8} x |x|, \quad \text{and} \quad g(x) \equiv \frac{\pi}{8} \left[ \frac{\pi^3}{12} - \frac{\pi}{2} x^2 - \frac{1}{3} |x|^3 \right]
\]
defined on the interval \(-\pi < x < \pi\). (Make sure that you explain where the “absolute values” \(||\) come from.) Discuss the rate of convergence of these Fourier series and the relationship between rate of convergence and the degree of singularity of the function being represented. As a byproduct of this exercise, you will find that

\[
\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots
\]

\[
\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots
\]

**Problem 10.2.** Sum the Fourier series

\[
J(x) = 1 + \sin x + \frac{1}{2!} \sin 2x + \frac{1}{3!} \sin 3x + \frac{1}{4!} \sin 4x + \cdots
\]

**Problem 10.3.** (i) The series

\[-\ln(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots,\]

converges if \(|z| < 1\). The series also converges on the unit circle, except at \(z = 1\). Set \(z = re^{i\theta}\), and separate real and imaginary parts. Deduce that

\[
r \cos \theta + \frac{r^2}{2} \cos 2\theta + \frac{r^3}{3} \cos 3\theta + \cdots = -\frac{1}{2} \ln \left[ 1 - 2r \cos \theta + r^2 \right];
\]

\[
r \sin \theta + \frac{r^2}{2} \sin 2\theta + \frac{r^3}{3} \sin 3\theta + \cdots = \tan^{-1} \left[ \frac{r \sin \theta}{1 - r \cos \theta} \right].
\]

(ii) Which branch of \(\tan^{-1}\) should be used in the formula above? (iii) Take \(r = 1\) (the worst case) and use MATLAB to compare truncated sums with the analytic expressions on the right.
Problem 10.4. The function

\[ f(z) = \frac{1}{z + a}, \quad \text{with } a \text{ real and greater than one}, \]

is analytic in the region \(|z| < a\). Evaluate this function on the unit circle and so obtain the Fourier series

\[ \frac{a + \cos \theta}{1 + a^2 + 2a \cos \theta} = a^{-1} \sum_{n=0}^{\infty} a^{-n} \cos n\theta, \]

\[ \frac{\sin \theta}{1 + a^2 + 2a \cos \theta} = -a^{-1} \sum_{n=1}^{\infty} a^{-n} \sin n\theta. \]

(10.23)

Fiddle about with the results above and deduce the Fourier series in (10.18).

Problem 10.5. In (9.33) we encountered the integral

\[ a_{4k} = \frac{4}{\pi} \int_{0}^{\pi/2} \frac{\cos(4k\theta) \, d\theta}{\cos \theta + \sin \theta}. \]

(10.24)

(i) Find the large-\(k\) asymptotic expansion of this integral and verify that \(a_{4k} = O(k^{-2})\). (ii) Make a numerical comparison of your asymptotic formula with exact evaluation of \(a_4\) through \(a_{16}\) using mathematica.

Problem 10.6. Consider a damped oscillator forced by periodic impulses:

\[ \ddot{\theta} + \epsilon \dot{\theta} + \sigma^2 \theta = F \delta_c(t/T). \]

(10.25)

Calculate the mean square displacement \(\overline{\theta^2}\) using two methods:

(i) Represent the forcing as a Fourier series, solve the differential equation and use Parseval’s theorem.

(ii) Explicit construction of a periodic solution.

Plot the mean square displacement as a function of the forcing frequency \(\omega = 2\pi/T\). In case (ii), let \(\theta_*\) and \(\dot{\theta}_*\) denote the unknown displacement and velocity immediately after the oscillator gets a kick at \(t = 0\). Then solve the differential equation as an initial value problem in the interval \(0 < t < 2\pi T\). Determine \(\theta_*\) and \(\dot{\theta}_*\) by requiring that the solution is periodic.
Lecture 11

Diffusion on a finite interval

11.1 The Fourier method

Suppose we need to solve the diffusion equation

$$u_t = \kappa u_{xx}, \quad (11.1)$$

on the interval $0 < x < \ell$ with boundary conditions

$$u(0, t) = u(\ell, t) = 0, \quad (11.2)$$

and some initial condition

$$u(x, 0) = f(x). \quad (11.3)$$

We suppose that $f(x)$ is square integrable so that there is no problem expanding in a Fourier series. With suitable nondimensionalization we can take $\ell \to \pi$ and $\kappa \to 1$.

Now we ignore the initial condition for a just a bit and find solutions of the PDE which also satisfy the boundary conditions. We can do this using separation of variables:

$$u(x, t) = \phi(x)\psi(t), \quad \Rightarrow \quad \frac{\psi'}{\psi} = \frac{\phi''}{\phi} = -\lambda^2. \quad (11.4)$$

We get a Sturm-Liouville eigenproblem for $\phi(x)$:

$$\phi'' + \lambda^2 \phi = 0, \quad \phi(0) = \phi(\pi) = 0. \quad (11.5)$$
Lecture 11. Diffusion on a finite interval

There is a complete set of eigenfunctions

\[ \phi_n(x) = \sin(nx), \quad \lambda = n. \]  \hfill (11.6)

Thus we have an infinite number of solutions (one for each \( n \)) of our diffusion problem:

\[ u(x, t) = e^{-n^2t} \sin nx. \]  \hfill (11.7)

You can check in seconds by substitution that this is a solution of the diffusion equation for every \( n \).

We are solving a linear problem and so we can use superposition to write down a more complicated solution as a half-range Fourier expansion

\[ u(x, t) = \sum_{n=1}^{\infty} f_n e^{-n^2t} \sin n x. \]  \hfill (11.8)

Our hope is by picking the coefficients \( f_n \) we can ensure that the solution above satisfies the initial condition \( u(x, 0) = f(x) \). This leads to

\[ f_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \]  \hfill (11.9)

Notice we are working on the half range with a sine series so we have implicitly used an odd extension of \( f(x) \) to \(-\pi < x < 0\).

As soon as \( t \) is a little bit greater than zero we can happily plug (11.8) into the PDE by blithely differentiating under the \( \sum \) -sign — because of the \( \exp(-n^2t) \) this series converges very quickly. Hence \( u(x, t) \) in (11.8) satisfies the PDE, the boundary conditions and the initial condition.

**Example:** \( f(x) = 1 \)

Using the Fourier series for \( \text{sqr}(x) \), the solution of (11.1) through (11.3) with \( f(x) = 1 \) is

\[ u(x, t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k+1)x)}{2k+1} e^{-(2k+1)^2t}. \]  \hfill (11.10)

This solution, using 20 terms in the series (11.10), is shown in figure 11.1. The original problem back in (11.1) through (11.3) was posed on the interval \( 0 \leq x \leq \pi \); in figure 11.1 I have plotted (11.10) on a larger
Figure 11.1: Solution of $u_t = u_{xx}$ with the initial condition $u(x, 0) = \text{sqr}(x)$. The initial square wave is represented approximately taking 20 terms in (11.10). The solution is shown at $t = 0.001, 0.01, 0.1$ and 0.5. Viewed on the half-range $0 < x < \pi$ this is the solution of a problem with inconsistent initial and boundary data. sqrdiff.png
interval so you can see how we’ve used symmetry to extend the half-range to the full range.

Because we can use only a finite number of terms in the series the initial condition exhibits Gibbs’ phenomenon. But these rapid oscillations disappear very quickly because of diffusion. The lesson is that a little bit of diffusion is an effective low-pass filter. aka a “Gaussian filter”.

Inconsistent boundary conditions and initial conditions

Suppose, as in the example above, that either \( f(0) \) or \( f(\pi) \) is nonzero. Then there is a ‘corner layer’ at \( (x, t) = (0, 0) \) or \( (x, t) = (\pi, t) \) as the initial condition adjusts to the boundary condition. (Locally we have something which looks just like the erf solution from an earlier lecture.)

How quickly do the Fourier coefficients in (11.9) decrease? We use integration by parts to estimate

\[
    f_n = -\frac{2}{\pi} \int_0^{\pi} f(x) \frac{d}{dx} \left( \frac{\cos nx}{n} \right) \, dx,
    \]

\[
    = \frac{2}{n\pi} [f(0) - (-1)^n f(\pi)] + \frac{2}{n\pi} \int_0^{\pi} f'(x) \cos nx \, dx. \tag{11.11}
\]

Invoking the Riemann-Lebesgue lemma the second term on the RHS is much less than the first as \( n \) becomes large. Thus \( b_n \sim n^{-1} \) as \( n \to \infty \).

The Fourier series of \( \text{sqr}(x) \) illustrates this slow convergence.

Suppose \( f(0) = f(\pi) = 0 \) but \( f''(0) \) or \( f'''(\pi) \) is nonzero. Notice that if we evaluate the PDE at \( x = 0 \) or \( \pi \), where \( u = u_t = 0 \), we conclude that \( u_{xx} = 0 \). Thus we still have an inconsistency between initial and boundary data at the corners of the domain. In this case repeated integration by parts shows that \( b_n \sim n^{-3} \). Because of this improved convergence there is no Gibbs phenomenon in our representation of \( f(x) \).

However if we calculate \( f'''(x) \) by twice differentiating the Fourier series we again encounter the problem. It is clear that this process can be continued to show that Gibbs phenomenon and nonuniform convergence are associated with the corner layers which are created by inconsistent boundary and initial data.

Example:

\( f(x) = x(\pi - x) \).
11.2 Inhomogeneous boundary conditions

Another way to solve (11.1) through (11.3) is to write down a Fourier series
\[ u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx, \quad (11.12) \]
and plug it into the PDE. We blithely “differentiate under the Σ sign” and quickly see that modal amplitudes \( u_n(t) \) satisfy simple ODES
\[ \frac{du_n}{dt} = -n^2 u_n . \quad (11.13) \]
We obtain \( u_n(0) \) by applying the initial condition so we recover the earlier solution in (11.8).

The procedure above seems natural provided \( u(x, t) \) satisfies homogenous boundary conditions. But suppose instead we have inhomogenous boundary conditions
\[ u(0, t) = a(t) , \quad u(\pi, t) = b(t) . \quad (11.14) \]
It might seem that the guess in (11.12) is wrong because this function always vanishes on the boundary. But recall that apart from a few isolated embarrassments at points of discontinuity we can expand any function in a Fourier series. So our guess in (11.12) must work! The problem is that we can’t take two \( x \) derivatives “under the Σ sign” because the resulting series doesn’t converge.

To get around this problem we recall that the Fourier coefficient is given by
\[ u_n(t) = \frac{2}{\pi} \int_0^\pi u(x, t) \sin nx \, dx . \quad (11.15) \]
We differentiate the above with respect to \( t \) and use \( u_t = u_{xx} \):
\[ \frac{du_n}{dt} = \frac{2}{\pi} \int_0^\pi u_{xx}(x, t) \sin nx \, dx , \quad (11.16) \]
and now we integrate by parts twice
\[ \frac{2}{\pi} \int_0^\pi u_{xx}(x, t) \sin nx \, dx = \frac{2}{\pi} \left[ \sin nx u_x(x, t) - n \cos nx u(x, t) \right]_0^\pi - n^2 u_n . \quad (11.17) \]
The terms which fall outside the integral simplify to
\[ \cos(n\pi) = ( -1 )^n \]
Lecture 11. Diffusion on a finite interval

\[ [\sin nx u_x(x, t) - n \cos nx u(x,t)]_0^\pi = na(t) - n(-1)^n b(t) \]  

(11.18)

Hence with inhomogeneous boundary conditions the modal amplitudes satisfy inhomogenous ODE’s:

\[ \frac{du_n}{dt} = \frac{2n}{\pi} [a - (-1)^n b] - n^2 u_n. \]  

(11.19)

This is OK, but not great because convergence is slow — for large \( n \) \( u_n(t) \sim n^{-1} \). In fact, this slow convergence is precisely why our solution manages to satisfy the inhomogeneous BC as we approach \( x = 0 \) and \( x = \pi \) from the interior of the interval.

**Improvement of convergence**

We can improve convergence by reformulating the PDE so that it has homogenous boundary conditions:

\[ u(x, t) = a(t) + [b(t) - a(t)] \frac{x}{\pi} + w(x,t). \]  

(11.20)

The function \( w(x,t) \) defined above vanishes at both \( x = 0 \) and \( x = \pi \). Throwing this into the PDE shows that

\[ w_t - w_{xx} = -\dot{a}(t) - \left[ \dot{b}(t) - \dot{a}(t) \right] \frac{x}{\pi}, \quad w(0,t) = w(\pi,t) = 0. \]  

(11.21)

Now we have an inhomogenous equation rather than inhomogenous boundary conditions. This is progress because if we use a Fourier series to represent \( w \):

\[ w(x,t) = \sum_{n=1}^\infty w_n(t) \sin nx, \quad w_n(t) = \frac{2}{\pi} \int_0^\pi w(x,t) \sin nx \, dx, \]  

(11.22)

then we expect \( w_n \sim n^{-3} \) — a factor \( n^{-2} \) faster than the Fourier expansion of \( u(x,t) \). This improved convergence is because \( w \) satisfies homogenous boundary conditions.

**11.3 Inhomogeneous equations**

We still have to determine \( w_n(t) \) in (11.21) and (11.22). We might as well consider the general case of an inhomogeneous diffusion equation

\[ w_t - w_{xx} = h(x,t), \]  

(11.23)
with the homogeneous boundary conditions

\[ w(0, t) = w(\pi, t) = 0 \] (11.24)

We Galerkin by projecting the PDE (11.23) onto sin nx. Galerking means multiply the PDE by sin nx and integrate:

\[ \int_0^\pi \sin nx \ [w_t - w_{xx}] \, dx = \int_0^\pi \sin nx \ h(x, t) \, dx. \] (11.25)

We get

\[ \dot{w}_n + n^2 w_n = h_n, \] (11.26)

where \( w_n(t) \) and \( h_n(t) \) are the coefficients of the Fourier sine-series of \( w(x, t) \) and \( h(x, t) \).

A crucial point is that in the Galerkin procedure we never take \( \partial^2_x \) under the \( \Sigma \)-sign.

OK, so now we have an ODE, (11.26), for \( w_n(t) \) — how fast does \( w_n \) decrease as \( n \to \infty \)? That is how rapidly does (11.22) converge? Since \( w(x, t) \) has homogeneous boundary conditions we expect \( w_n \sim n^{-3} \). One way to see this to argue that as \( n \to \infty \) the dominant balance in (11.26) is

\[ w_n \sim h_n / n^2. \] (11.27)

Thus if \( h_n \sim n^{-p} \) then \( w_n \sim n^{-p-2} \). Typically if the source is nonzero at \( x = 0 \) and \( \pi \) then \( p = 1 \) and \( w_n \sim n^{-3} \). This means we can differentiate the Fourier series twice and still maintain convergence.

**An example**

For the problem back in (11.21) and (11.22) we have the source term

\[ h(x, t) = -\dot{a}(t) - [\dot{b}(t) - \dot{a}(t)] \frac{x}{\pi}. \] (11.28)

After a festival of integration-by-parts I found

\[ \dot{w}_n + n^2 w_n = \frac{2}{n} o_n \dot{a} + \frac{1}{n} \dot{b}. \] (11.29)

(The above might be incorrect — must check.)
11.4 Problems

Problem 11.1. Solve (11.1) through (11.3) with \( f(x) = x(\ell - x) \).

Problem 11.2. Consider a diffusion problem defined on the interval \( 0 \leq x \leq \ell \):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2}, \\
\begin{array}{c}
\left. u(x,t) \right|_{x=0} &= 0, \\
\left. \frac{\partial u}{\partial x} \right|_{x=\ell,t} &= 0,
\end{array}
\end{align*}
\tag{11.30}
\]

with initial condition \( u(x,t) = 1 \). (i) If you use separation of variables then it is easy to anticipate that you’ll find a Sturm-Liouville eigenproblem with sinusoidal solutions. Sketch the first two eigenfunctions before doing this algebra. Explain why you are motivated to nondimensionalize so that \( 0 \leq x \leq \pi/2 \) (a quarter-range expansion). (ii) With \( \ell \to \pi/2 \) and \( \kappa \to 1 \), work out the Sturm-Liouville algebra and find the eigenfunctions and eigenvalues just as we did in (11.4) through (11.7). (iii) With \( f(x) = 1 \) find the solution as a Fourier series and use MATLAB to visualize the answer.

Problem 11.3. Consider the inhomogeneous diffusion equation

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \frac{1}{\sqrt{x(\pi - x)}}, \\
\begin{array}{c}
\left. u(x,t) \right|_{x=0} &= u(x,t) = 0,
\end{array}
\end{align*}
\]

(i) Using a Fourier sine series

\[
u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin nx,
\]

and the Galerkin procedure, find the ODE’s satisfied by the modal amplitudes \( u_n(t) \). (ii) Your answer to (i) will involve the integral

\[
h_n \equiv \frac{2}{\pi} \int_{0}^{\pi} \frac{\sin nx}{\sqrt{x(\pi - x)}} \, dx,
\]

which you probably can’t evaluate off-the-cuff. (Try looking this one up?) However you should deduce that deduce that \( h_n \) is zero if \( n \) is even and

\[
h_n \equiv \frac{4}{\pi} \int_{0}^{\pi/2} \frac{\sin nx}{\sqrt{x(\pi - x)}} \, dx, \quad \text{if } n \text{ is odd.}
\]

Find the asymptotic expansion of \( h_n \) above as \( n \to \infty \). (Refer to chapter 6 of Bo if necessary.) (iii) I believe \( u_n \sim n^{-q} \) as \( n \to \infty \). Find \( q \).
Problem 11.4. A rod occupies $1 \leq x \leq 2$ and the thermal conductivity depends on $x$ so that diffusion equation is

$$u_t = (x^2 u_x)_x.$$ 

The boundary and initial conditions are

$$u(1, t) = u(2, t) = 0, \quad u(x, 0) = 1.$$ 

(i) The total amount of heat in the rod is

$$H(t) = \int_1^2 u(x, t) \, dx.$$ 

Show that $H(0) = 1$ and

$$\frac{dH}{dt} = 4u_x(2, t) - u_x(1, t).$$

Physically interpret the two terms on the right hand side above. What is the sign of the $u_x(2, t)$ and the sign $u_x(1, t)$? (ii) Before solving the PDE, show that roughly 61% of the heat escapes through $x = 2$. (There is a simple analytic expression for the fraction $0.61371 \cdots$ which you should find.) (iii) Use separation of variables to show that the eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{x}} \sin \left( \frac{n\pi \ln x}{\ln 2} \right).$$

Find the eigenvalue which is associated with the $n$'th eigenfunction. (iv) Use modal orthogonality to find the series expansion of the initial value problem.

Problem 11.5. (i) Solve

$$u_t = u_{xx}, \quad u(0, t) = 0, \quad u(\pi, t) = 0, \quad (11.31)$$

with the initial condition $u(x, 0) = \sin x$ using a Fourier sine series.

Now suppose, perversely, that we try solve the same problem using cosine expansion:

$$u(x, t) = \sum_{n=0}^{\infty} a_n(t) \cos nx.$$ 

Because $\{\cos nx\}$ is a complete set it must be possible to represent the solution like this. (ii) Use symmetry to deduce that the cosine series
above includes only terms with even $n$ i.e., we need only worry about $n = 2m$ with $m = 0, 1, 2 \cdots$ (iii) Use integration by parts to show that $a_n(t) \sim n^{-2}$ as $n \to \infty$ and deduce that
\[
0 = \sum_{m=0}^{\infty} a_{2m}(t).
\]
(iv) Show that
\[
\dot{a}_{2m} + (2m)^2 a_{2m} = -\frac{4c_{2m}}{\pi} \sum_{p=0}^{\infty} pa_{2p}
\]
where $c_0 = 1$ and if $m \geq 1$, $c_{2m} = 2$. (v) Deduce the solution of the system above by taking the solution of part (i) and expanding as a cosine series. Verify by substitution (numerical if necessary) that this works.
Lecture 12

The Fourier Transform

12.1 Definition of the Fourier transform

We begin by defining the Fourier transform of a function \( f(x) \)

\[
\mathcal{F}[f(x); x \to k] \equiv \int_{-\infty}^{\infty} e^{-ikx} f(x) \, dx.
\]  

(12.1)

\( \mathcal{F} \) is a linear operator. We start with \( f \), which is a function of \( x \), but the Fourier transform is a function of the transform variable, \( k \). For brevity we will often denote the transform by \( \hat{f}(k) \). The explicit notation in (12.1) is useful if \( f \) depends on more than one variable and we need to be precise about which variable we are transforming against.

Example: \( \mathcal{F} \left[ e^{-\alpha|x|}; x \to k \right] \)

As an example of a Fourier transform we begin by finding \( \mathcal{F} \left[ e^{-\alpha x} H(x) \right] \). To ensure convergence we assume that the real part of the constant \( \alpha \) is positive. Using the definition in (12.1) we have

\[
\mathcal{F} \left[ e^{-\alpha x} H(x); x \to k \right] = \frac{1}{\alpha + ik}.
\]  

(12.2)

Notice that in this example \( f(x) \) is real, but \( \hat{f}(k) \) is complex.

We can also quickly see that

\[
\mathcal{F} \left[ e^{\alpha x} H(-x); x \to k \right] = \frac{1}{\alpha - ik}.
\]  

(12.3)
Therefore, since
\[ e^{-\alpha|x|} = e^{\alpha x}H(-x) + e^{-\alpha x}H(x), \]  
we have
\[ \mathcal{F}[e^{-\alpha|x|}; x \rightarrow k] = \frac{1}{\alpha + ik} + \frac{1}{\alpha - ik} = \frac{2\alpha}{\alpha^2 + k^2}. \]  

**Example:** \( \mathcal{F}[\delta(x); x \rightarrow k] \)

A second example is a trivial formal calculation
\[ \mathcal{F}[\delta(x); x \rightarrow k] = 1, \]  
or more generally
\[ \mathcal{F}[\delta(x - \xi); x \rightarrow k] = \exp(-ik\xi). \]  

Noticing
\[ \delta(x) = \lim_{\alpha \to \infty} \frac{1}{2\alpha} e^{-\alpha|x|}, \]  
we give an alternative derivation of (12.6) by taking \( \alpha \to \infty \) in (12.5).

**A punctuation identity:** \( \mathcal{F}[f(x - a); x \rightarrow k] = e^{-ika}\tilde{f}(k) \)

The result announced in the subsection heading is obtained by shuffling symbols in the definition of the Fourier theorem. Other simple but important examples of “punctuation” identities are collected in the problems.

**Another punctuation identity: the reality condition**

A useful algebra check is provided by the reality condition:
\[ \text{if } f(x) \text{ is real, then } \tilde{f}(k) = \tilde{f}^*(-k). \]  
Notice that the examples in (12.2) through (12.8) pass this test (provided that \( \alpha \) is real).
More examples

To establish Fourier-literacy, we collect some important examples of Fourier transforms in the following examples.

Example: Consider the indicator function of the interval \(-a < x < a\). That is
\[
\text{ind}(x; a) = \begin{cases} 
1, & \text{if } |x| < a, \\
0, & \text{if } |x| > a.
\end{cases}
\] (12.10)

The Fourier transform of the indicator function \(\text{ind}(x; a)\) is
\[
\tilde{\text{ind}}(k; a) = \int_{-a}^{a} e^{-ikx} \, dx = \frac{2 \sin ka}{k}.
\] (12.11)

Notice as \(a \to \infty\) the right hand side above is, apart from normalization, a \(\delta\)-sequence. Thus we have
\[
\mathcal{F}[1; x \to k] = \lim_{a \to \infty} \mathcal{F}[\text{ind}(x; a); x \to k],
\] (12.12)

Other examples — the Gaussian, \(\text{sgn}(x)\).

12.2 Inverse transforms: the FIT

The Fourier Integral Theorem (FIT) states that if \(\hat{f}(k)\) is the Fourier transform of \(f(x)\) then
\[
f(x) = \mathcal{F}^{-1}[\hat{f}(k); k \to x],
\]
\[
= \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \frac{dk}{2\pi}.
\] (12.13)

Note carefully the difference in the signs of the arguments of the exponentials in (12.1) and (12.13). This difference in sign implies that \(\mathcal{F}^{-1} = \mathcal{F}^*\). With (12.1) and (12.13) we can go back and forth between the function and its transform.

We can combine (12.1) and (12.13) into a single equation
\[
f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left[ \int_{-\infty}^{x'} dx' e^{-ikx'} f(x') \right],
\]
\[
= \int_{-\infty}^{x'} f(x') \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')} \right] dx'.
\] (12.14)
Comparing the result above with the identity
\[ f(x) = \int_{-\infty}^{\infty} f(x') \delta(x - x') \, dx' \]  
(12.15)
we have the fundamental conclusion that
\[ \delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x - x')} \, dk. \]  
(12.16)
Equation (12.16) is a concise statement of the FIT.

**Example: inverse transform of \( \tilde{f}(k) = 1/(\alpha - ik) \), with \( \alpha_r > 0 \).**

The inverse transform
\[ f(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{\alpha - ik} \frac{dx}{2\pi} \]  
(12.17)
can be evaluated using the Cauchy residue theorem. Notice that the integrand has a pole at \( k = -i\alpha \) in the lower half plane. The contour of integration is closed with a big semi-circle of radius \( R \). If \( x > 0 \) we close in the upper-half and there are no poles within the contour. We use Jordan’s lemma to argue that the contribution from the semi-circular arc vanishes as \( R \to \infty \). Thus \( f(x) = 0 \) if \( x > 0 \). If \( x < 0 \) we close in the lower-half plane and enclose the pole at \( k = -i\alpha \). Again we use Jordan’s lemma to show there is no contribution from the semi-circle in the limit \( R \to \infty \). Thus the integral above is determined by the residue theorem as:
\[ f(x) = -2\pi i \times \text{residue} \left( \frac{e^{ikx}}{\alpha - ik} ; k = -i\alpha \right) \frac{1}{2\pi} = e^{\alpha x} H(-x). \]  
(12.18)
The minus sign in front is because when we close in the lower half-plane, we're going around the contour in the clockwise (negative) direction.

### 12.3 Transforms of derivatives

From the definition in (12.1) we can prove a great many straightforward results about Fourier transforms. Some of these are collected in the problems. The most important result is that
\[ \mathcal{F}\left[ \frac{df}{dx}(x) ; x \to k \right] = ik\mathcal{F}[f(x) ; x \to k]. \]  
(12.19)
In other words, if \( \hat{f}(k) \) is the Fourier transform of \( f(x) \), then \( ik\hat{f}(k) \) is the Fourier transform of \( df/dx \). Notice the signs here: in (12.1) we have defined the Fourier transform with \( \exp(-ikx) \) and then the operational rule is

\[
\frac{d}{dx} \rightarrow +ik.
\] (12.20)

Using the operational rule we can transform many differential equations at a glance.

**Solution of the diffusion equation again**

Consider once again

\[
u_t = \kappa u_{xx}, \quad u(x,0) = f(x).
\] (12.21)

The Fourier transform of the problem is

\[
\tilde{u}_t = -\kappa k^2 \tilde{u}, \quad \tilde{u}(k,0) = \hat{f}(k), \quad \Rightarrow \quad \tilde{u}(k,t) = \hat{f}(k)e^{-\kappa k^2 t}.
\] (12.22)

We used (12.20) twice to get the transformed equation. Now we invoke the FIT to write down the integral representation of the solution

\[
u(x,t) = \int_{-\infty}^{\infty} e^{ikx-\kappa k^2 t} \hat{f}(k) \frac{dk}{2\pi}.
\] (12.23)

There is one initial condition for which we can easily invert the transform in (12.23), namely

\[
f(x) = \delta(x), \quad \text{and therefore} \quad \hat{f}(k) = 1.
\] (12.24)

We know the answer in this case: it is the Gaussian similarity solution of the diffusion equation from lecture 7:

\[
g(x,t) = \frac{e^{-x^2/4\kappa t}}{\sqrt{4\pi \kappa t}}
\] (12.25)

Let’s verify this by doing the integral in (12.23) with \( \hat{f}(k) = 1 \). The key is to complete the square in the exponential:

\[
\kappa k^2 t - ikx = \kappa t \left( k - \frac{ix}{2\kappa t} \right)^2 - \kappa t \left( \frac{ix}{2\kappa t} \right)^2.
\] (12.26)
Thus, with \( \tilde{f}(k) = 1 \), (12.23), is
\[
g(x, t) = e^{-x^2 / 4\kappa t} \int_{-\infty}^{\infty} e^{-\kappa t(k - \frac{i}{\kappa}x)^2} \frac{dk}{2\pi}.
\] (12.27)

To evaluate the integral we translate the contour in the complex \( k \)-plane so that it runs along the line \( k = (i\kappa x / 2\kappa t) + \kappa' \), where \( \kappa' \) runs from \( -\infty \) to \( \infty \) i.e., the new contour is parallel to the real-\( k \) axis. Since the integrand has no singularities the integral is unchanged by this translation, and we find
\[
g(x, t) = e^{-x^2 / 4\kappa t} \int_{-\infty}^{\infty} e^{-\kappa t^2 \kappa'^2} \frac{dk}{2\pi} = e^{-x^2 / 4\kappa t} \sqrt{\frac{\pi}{\kappa t}} \frac{1}{2\pi}.
\] (12.28)

Now return to (12.23), and write it as
\[
u(x, t) = \mathcal{F}^{-1} \left[ \tilde{g}(k, t) \times \tilde{f}(k) \right],
\] (12.29)
where
\[
\tilde{g}(k, t) = e^{-\kappa t^2 k^2} = \mathcal{F} \left[ g(x, t); x \rightarrow k \right].
\] (12.30)

In (12.29) we desire the inverse transform of the product of two Fourier transforms. This leads us to the convolution theorem.

### 12.4 Convolution

We define the convolution of two functions by
\[
f \circ g \equiv \int_{-\infty}^{\infty} f(x - x') g(x') \, dx'.
\] (12.31)

This definition should remind you of the Green’s function solutions from earlier lectures. Convolution is a linear operation:
\[
f \circ (g + h) = f \circ g + f \circ h, \quad f \circ (cg) = cf \circ g,
\] (12.32)
where \( c \) is a constant and \( f, g \) and \( h \) are functions of \( x \). Convolution is commutative and associative:
\[
f \circ g = g \circ f, \quad (f \circ g) \circ h = f \circ (g \circ h).
\] (12.33)

Do you see why mathematicians call the \( \delta \)-function the “identity element of the convolution algebra”?

Convolutions are important because
\[
\mathcal{F} [f \circ g] = \tilde{f}(k) \tilde{g}(k), \quad \text{or equivalently} \quad f \circ g = \mathcal{F}^{-1} \left[ \tilde{f} \tilde{g} \right].
\] (12.34)

In other words, the inverse transform of a product of transforms is the convolution of the two functions in the spatial domain.
Diffusion yet again

Armed with the convolution theorem, let us pick-up the diffusion equation at (12.29). We have simply

$$u(x,t) = g \circ f.$$  \hspace{1cm} (12.35)

This is the general solution of the diffusion equation as a convolution of the Greens function $g(x,t)$ with the initial condition $f(x)$ (see Lecture 7).

12.5 Proof by intimidation of the FIT

Suppose we represent a compact function such as $f(x) = e^{-x^2}$ on the interval $-L/2 < x < L/2$ using the complex form of a Fourier series. We can represent the "clipped" version of $f(x)$ via a Fourier series as

$$f_L(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \exp{(in \, dk \, x)},$$  \hspace{1cm} (12.36)

and the "inverse" relation is

$$\hat{f}_n = \frac{1}{L} \int_{-L/2}^{L/2} \exp{(-in \, dk \, x)} f(x) \, dx,$$  \hspace{1cm} (12.37)

where

$$dk = \frac{2\pi}{L}. \hspace{1cm} (12.38)$$

The subscript $L$ on the left of (12.36) is to remind us that the series on the right represents $f(x)$ only on the interval $-L/2 < x < L/2$. We don't need to write $f_L(x)$ on the right of (12.37) because $f(x) = f_L(x)$ throughout the range of the integral. To be really cautious we should also decorate $\hat{f}_n$ with a subscript $L$ to remind us that these Fourier coefficients also change with $L$. Let's live dangerously by not decorating the $\hat{f}_n$'s.

Now let $L \to \infty$, so that $dk \to 0$. The sequence of wavenumbers

$$k_n \equiv ndk,$$  \hspace{1cm} (12.39)
then becomes very dense on the $k$-axis i.e., as $L \to \infty$, the difference between adjacent wavenumbers in the Fourier series (12.36) becomes smaller, $dk \to 0$. Now define $\hat{f}(k_n)$ via

$$\hat{f}(k_n) = \frac{L}{2\pi} \hat{f}_n,$$  

(12.40)

With this notation, (12.36) and (12.37) can be rewritten as

$$f_L(x) = \sum_k \hat{f}(k) e^{i k x} \frac{dk}{2\pi},$$  

(12.41)

and

$$\hat{f}(k) = \int_{-L/2}^{L/2} e^{-ikx} f(x) \, dx.$$  

(12.42)

So far there have been no assumptions: we have introduced some suggestive notation and rewritten the basic formula used to obtain the coefficients in a Fourier series. But now we take the limit $L \to \infty$, and make some plausible approximations.

If $L \to \infty$, then limits of integration in (12.42) become $\pm \infty$: this is evidently our definition of the Fourier transform back in (12.1). And in (12.41) the terms in the sum are obtained from a slowly changing function of $k_n = ndk$. Thus (12.41) is the definition of the Riemann integral of the function $\hat{f}(k) \exp(-ikx)$ with respect to $k$. In other words we claim that

$$\lim_{L \to \infty} f_L(x) = f(x),$$  

(12.43)

and

$$\lim_{dk \to 0} \sum_k \hat{f}(k) \exp(ikx) \frac{dk}{2\pi} = \int_{-\infty}^{\infty} \hat{f}(k) \exp(ikx) \, dk.$$  

(12.44)

Thus as $L \to \infty$, and $dk \to 0$, (12.41) and (12.42) are equivalent to the definition of the Fourier transform and its inverse given at the start of this lecture.

### 12.6 Problems

**Problem 12.1.** Prove the following results: (i) If $f(x)$ is a real function then $\hat{f}(-k)^* = \hat{f}(k)$. This is the reality condition. (ii) Show that

$$\mathcal{F} \{ \exp(iyx) f(x) \} = \hat{f}(k - y).$$
(iii) Suppose that \( \hat{f}(k) \) is the Fourier transform of \( f(x) \). Express the Fourier transform of \( f(a(x-b)) \), with \( a \) and \( b \) real constants, in terms of \( \hat{f}(k) \). (Hint: the correct answer involves \(|a|\) — make sure you understand how the absolute value of \( a \) arises.)

**Problem 12.2.** (i) Prove the operational rule

\[
\mathcal{F}[xf(x)] = i \frac{d}{dk} \hat{f}(k).
\]

(ii) The Airy function, \( Ai(x) \), is defined by

\[
Ai'' - xAi = 0, \quad \lim_{x \to \pm\infty} Ai(x) = 0,
\]

and

\[
\int_{-\infty}^{\infty} Ai(x) \, dx = 1.
\]

Use the FIT find an integral representation of \( Ai(x) \). Check your answer against the appendix in BO.

**Problem 12.3.** In the lecture we obtained the Fourier transform

\[
\frac{2\alpha}{\alpha^2 + k^2} = \int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-ikx} \, dx.
\]

Apply “punctuation identities” other tricks to the formula above, and so obtain the Fourier transforms of the following functions

\[
\begin{align*}
    f_1(x) &= |x| \exp(-\alpha|x|), \\
    f_2(x) &= x \exp(-\alpha|x|), \\
    f_3(x) &= \left[ 1 - e^{-\alpha|x|} \right] / |x|, \\
    f_4(x) &= \cos(yx) \exp(-\alpha|x|), \\
    f_5(x) &= \sin(yx) \exp(-\alpha|x|).
\end{align*}
\]

Your goal is to avoid the honest but tedious work of direct integration.

**Problem 12.4.** (i) Prove the convolution theorem \( (12.34) \). (ii) Prove Parseval's theorem

\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \, dk.
\]
Consider the “tent-function”

\[ f(x) = \begin{cases} 
1 - |x|, & |x| \leq 1; \\
0, & |x| > 1; 
\end{cases} \]

and its Fourier transform. Use Parseval’s theorem to show that

\[ \int_0^\infty \left( \frac{\sin y}{y} \right)^4 \, dy = \frac{\pi}{3}. \]

**Problem 12.5.** Solve the ODE:

\[ u_{xx} - u = \delta(x), \quad \lim_{x \to \pm \infty} u(x) = 0, \]

by Fourier transforming with respect to \( x \). Invert the transform and check your answer by substitution.

**Problem 12.6.** (i) Prove that

\[ I(a, b) \equiv \int_{-\infty}^{\infty} e^{-ax^2} \cos bx \, dx = \sqrt{\frac{\pi}{a}} e^{-b^2/4a}. \]

Hint: if you want to avoid contour integration, change variables to \( v \equiv \sqrt{a}x \) and let \( J(\eta) = \sqrt{a}I(\eta) \) where \( \eta \equiv b/\sqrt{a} \). Find and solve a simple first-order ODE satisfied by \( J(\eta) \). (ii) Calculate the Fourier transforms of \( f_1(x) \equiv \exp(-\alpha^2 x^2) \) and \( f_2(x) \equiv x \exp(-\alpha^2 x^2) \).

**Problem 12.7.** (i) Use the Fourier transform \( x \rightarrow k \) to obtain an integral representation of the solution of the dispersive wave equation

\[ u_t = \alpha u_{xxx}, \quad u(x, 0) = u_0(x), \quad \alpha \text{ a real positive constant}. \]

(ii) Calculate the inverse transform in the special case \( u_0(x) = \delta(x) \) i.e., find the Green’s function of this third-order dispersive wave equation. You can express the answer in terms of the Airy function defined by

\[ \text{Ai}(z) \equiv \frac{1}{\pi} \int_0^\infty \cos \left( \frac{v^3}{3} + zv \right) \, dv \]

(iii) In the problems at the end of an earlier lecture you were asked to solve this problem with the similarity method. Make sure your Fourier-transform answer agrees with your similarity solution. (iv) Write the general solution of the initial value problem in terms of the Green’s function.
Problem 12.8. Consider Laplace’s equation

\[ u_{xx} + u_{yy} = 0 \]

in a strip \(-\infty < x < \infty\) and \(0 < y < a\), with the condition that \(u \to 0\) as \(|x| \to \infty\). There are prescribed boundary values \(u(x, 0) = f(x)\) and \(u(x, a) = g(x)\). Express \(u(x, y)\) in terms of the Fourier transforms of \(f(x)\) and \(g(x)\). Suppose that

\[ f(x) = 0, \quad g(x) = \frac{x}{x^2 + a^2} - \frac{x}{x^2 + 9a^2}. \]

Find \(u(x, y)\) by inverting the Fourier transform \(\tilde{u}(k, y)\).

Problem 12.9. (i) The characteristic function of an interval \((a, b)\) is denoted by \(\chi(x; a, b)\). \(\chi = 1\) if \(x \in (a, b)\) and \(\chi = 0\) otherwise. Show that

\[ \mathcal{F}[\chi(x; -\ell, \ell)] = 2 \frac{\sin k\ell}{k}. \]

(ii) Use the Cauchy residue theorem to calculate the inverse transform and verify the FIT. Be sure to explain carefully how the discontinuous function \(\chi(x; -\ell, \ell)\) emerges from this calculation. (iii) Calculate

\[ \wedge(x) \equiv \chi(x, -1, 1) \circ \chi(x, -1, 1) \quad \text{and} \quad \tilde{\lambda}(k). \]

Problem 12.10. Find the Fourier transform of \(\text{sgn}(x)\).

Solution. Noticing that

\[ \text{sgn}(x) = \lim_{\alpha \to 0} \left[ e^{-\alpha x}H(x) - e^{\alpha x}H(-x) \right], \quad (12.45) \]

we argue from (12.2) and (12.3) that

\[ \mathcal{F}[\text{sgn}(x); x \to k] = \lim_{\alpha \to 0} \left[ \frac{1}{\alpha + i k} - \frac{1}{\alpha - i k} \right] \]

\[ = \lim_{\alpha \to 0} \frac{-2i k}{\alpha^2 + k^2}, \]

\[ = \frac{2}{ik}. \quad (12.46) \]

We now have a Fourier transform which is singular at \(k = 0\). The limiting process which led to this result indicates that we should view \(2i/k\)
Lecture 12. The Fourier Transform

as a generalized function or distribution. In other word, we interpret
integrals such as
\[
I \equiv \int_{-\infty}^{\infty} \frac{2}{ik} \tilde{u}(k) \, dk
\]
(12.47)
by retreating to the penultimate line in (12.46):
\[
I = \lim_{\alpha \to 0} \int_{-\infty}^{\infty} \frac{-2ik}{\alpha^2 + k^2} \tilde{u}(k) \, dk,
\]
\[
= PV \int_{-\infty}^{\infty} \frac{2}{ik} \tilde{u}(k) \, dk,
\]
(12.48)
where \( PV \) stands for “Principal Value”.

Problem 12.11. Using a Fourier transform, solve the integral equation
\[
e^{-x^2/2} = \int_{-\infty}^{\infty} e^{-|x-u|} f(u) \, du
\]

Problem 12.12. Solve the elastic wave equation
\[
\psi_{tt} + \psi_{xxxx} = 0, \quad \psi(x,0) = \delta(x), \quad \psi_t(x,0) = 0.
\]
using the Fourier transform. To invert the transform you'll need to "complete the square". To check your algebra, show that
\[
\psi(0,t) = \frac{1}{2\sqrt{2\pi t}}.
\]

Problem 12.13. Solve the Schrödinger equation
\[
i\psi_t = \psi_{xx}, \quad \psi(x,0) = e^{-m^2x^2/2}.
\]

Problem 12.14. The boundary value problem
\[
Q'' - \gamma^2 Q = \gamma, \quad Q(0) = 0, \quad Q(\pm \infty) = 0,
\]
occurs in equatorial oceanography (the Yoshida jet). \(i\) Calculate a few terms in the expansion of \(Q(\gamma)\) around \(\gamma = 0\) and \(\gamma = \pm \infty\). This should convince you that \(Q'(0)\) is an unknown constant which must be determined so that \(Q(\pm \infty) = 0\). \(ii\) Fourier transform the ODE and solve the transformed equation in terms of the modified Bessel function \(K_{1/4}\). Use the resulting integral representation of \(Q(\gamma)\) to show that
\[
Q'(0) = -\sqrt{\pi} \Gamma(3/4)/\Gamma(1/4).
\]
Figure 12.1: Solution of problem [12.13] with $m = 1/4$; the solid curves show $\Re \psi$ and the dashed curves show $\Im \psi$. schroedingerSoln.eps
Lecture 13

The wave equation \( u_{tt} = c^2 u_{xx} \)

13.1 Strings

The transverse deformation of a stretched string is described by its displacement, \( \eta(x, t) \), from the \( x \)-axis – see figure 13.1. The equation of motion for \( \eta(x, t) \) is Newton’s law applied to a small segment of string that lies between \( x \) and \( x + \delta x \). If \( \rho \) is the mass per unit length of the string then the mass of this segment is \( \rho \delta x \) and its acceleration in the vertical direction is \( \eta_{tt}(x, t) \). The product of mass times acceleration in the vertical direction is \( \delta x \rho \eta_{tt} \) and this must be equal to the sum of the forces acting in the vertical direction. One of these is certainly gravity, \( -\delta x \rho g \). But there is also a vertical force because the tension \( T \) in the string acts in different directions at the left and right ends of the segment. At the left end of the segment, \( x \), the vertical component of the tension is

\[
F_L = -T \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right](x, t),
\]

(13.1)

while at the right end, \( x + \delta x \), the vertical component of the tension is

\[
F_R = T \left[ \frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right] (x + \delta x, t).
\]

(13.2)

Newton’s law is then

\[
\delta x \rho \eta_{tt} = F_R + F_L - \delta x \rho g.
\]

(13.3)
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

Letting $\delta x \to 0$:  

$$
\eta_{tt} - c^2 \frac{\eta_x}{\sqrt{1 + \eta_x^2}} = -g, \quad c^2 \equiv \frac{T}{\rho}.
$$

(13.4)

c is the wave speed. This is a nonlinear equation and the first thing we do in an attempt to understand the structure of its solutions is to linearize it. This means that we assume $\eta_x \ll 1$, i.e. the angle that the wire makes with the $x$-axis is very small. With this approximation (13.4) reduces to  

$$
\eta_{tt} - c^2 \eta_{xx} = -g.
$$

(13.5)

Because (13.5) is linear the RHS can be removed by writing  

$$
\eta(x, t) = \left( \frac{\rho g}{2T} \right) x(x - L) + u(x, t).
$$

(13.6)

The first term on the RHS of (13.6) is the static sag produced by gravity. We have assumed that the string is fixed at $x = 0$ and $x = L$ so that there is no sag at these points. The nonstatic displacement, $u(x, t)$, now satisfies the homogeneous equation  

$$
u_{tt} - c^2 u_{xx} = 0.
$$

(13.7)

13.2 Energy conservation

Now we discuss the energetics of a vibrating string. We also generalize (13.7) including forcing and dissipation:  

$$
u_{tt} + \epsilon u_t - c^2 u_{xx} = f.
$$

(13.8)

Here $f(x, t)$ is some externally specified forcing function. To obtain the energy equation, multiply (13.8) by $\rho u_t$ and rearrange it as  

$$
\mathcal{E}_t + J_x = -\epsilon \rho u_t^2 + \rho f u_t,
$$

(13.9)

where the energy density $\mathcal{E}$ and the energy flux $J$ are  

$$
\mathcal{E} \equiv \frac{1}{2} \rho \left( u_t^2 + c^2 u_x^2 \right), \quad J \equiv -T u_t u_x.
$$

(13.10)
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

Figure 13.1: A string is a one-dimensional continuum i.e., a curve $z = \eta(x, t)$. The force of tension, $T$, is always parallel to the string.
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

To get the total energy

$$E \equiv \frac{1}{2} \rho \int_0^L u_t^2 + c^2 u_x^2 \, dx,$$

we integrate (13.10) over the length of the string, from $x = 0$ to $x = L$.

If the string is clamped at these end-points then $J$ is zero and

$$E_t = \rho \int_0^L f u_t - \epsilon u_t^2 \, dx.$$  \hspace{1cm} (13.12)

With no forcing and no dissipation the energy is conserved because the right hand side of (13.12) is zero.

### 13.3 D’Alembert’s solution

We begin by noting that there are two very simple solutions of (13.7) viz.

$$u(x, t) = R(x - ct) \quad \text{and} \quad u(x, t) = L(x + ct).$$ \hspace{1cm} (13.13)

Here $R$ and $L$ are arbitrary functions. You can check this by substituting into (13.7). $R(x - ct)$ is a right moving wave and $L(x + ct)$ is a left moving wave. Of course the equation is linear so that we can use superposition to obtain a solution that is a sum of a left and right moving disturbance

$$u(x, t) = R(x - ct) + L(x + ct).$$ \hspace{1cm} (13.14)

Using these solutions we can solve the simplest initial value problem. Suppose there is an infinite string and at $t = 0$

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = w_0(x).$$ \hspace{1cm} (13.15)

Can we find $R$ and $L$ in (13.14) so that at $t = 0$ we satisfy (13.15)? It turns out that this is straightforward. We obviously need

$$R + L = u_0 \quad \text{and} \quad c(L' - R') = w_0,$$ \hspace{1cm} (13.16)

where the prime denotes a derivative. The system in (13.16) is easy to solve:

$$R(x) = \frac{1}{2} u_0(x) - \frac{1}{2c} \int_{x_0}^{x} w_0(x') \, dx'$$ \hspace{0.5cm} \text{and} \hspace{0.5cm} $$L(x) = \frac{1}{2} u_0(x) + \frac{1}{2c} \int_{x_0}^{x} w_0(x') \, dx'.$$ \hspace{1cm} (13.17)
Lecture 13. The wave equation \( u_{tt} = c^2 u_{xx} \)

There is an arbitrary constant of integration, \( x_0 \), but we don’t have to worry too much about this because if we form the sum \( R + L \) then \( x_0 \) cancels. Finally we have

\[
u(x, t) = \frac{1}{2} \left[ u_0(x - ct) + u_0(x + ct) + \frac{1}{c} \int_{x - ct}^{x + ct} w_0(x') \, dx' \right]. \tag{13.18}\]

The formula in (13.18) is d’Alembert’s solution of the wave equation.

**Example: propagation of a discontinuity**

As an example of d’Alembert’s formula, consider the initial condition

\[
u_0 = \text{sgn}(x), \quad w_0(x) = 0. \tag{13.19}\]

The solution of the wave equation with this initial condition is

\[
u(x, t) = \frac{1}{2} \text{sgn}(x - ct) + \frac{1}{2} \text{sgn}(x + ct), \tag{13.20}\]

(see figure ??).

### 13.4 The forced string: Duhamel again

How do we solve the wave equation

\[
u_{tt} - c^2 \nu_{xx} = f(x, t), \tag{13.21}\]

driven by an arbitrary distributed force? Without loss of generality we assume that the string is initially motionless.

The relevant Green’s function is

\[
g_{tt} - c^2 g_{xx} = \delta(x) \delta(t), \quad g(x, 0) = 0. \tag{13.22}\]

Once we possess \( g(x, t) \) the solution of (13.21) is

\[
u(x, t) = \int_{-\infty}^{\infty} dx' \int_0^\infty dt' g(x - x', t - t') f(x', t'). \tag{13.23}\]

This is easy to check by subsitution.
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

Figure 13.2: The domain of dependence of the point $(x,t)$. Inside the shaded region $B$, $g(x,t,x',t') = 1/(2c)$, and outside the shaded region $g(x,t,x',t') = 0$. JoFig2.eps

Now to obtain the Green’s function we integrate from $t = 0^-$ to $t = 0^+$ and we see that (13.22) is equivalent to the initial value problem

$$g_{tt} - c^2 g_{xx} = 0, \quad g_t(x,0^+) = \delta(x).$$

(13.24)

The solution is apparent from D’Alembert’s formula:

$$g(x,t) = \frac{1}{2c} \left[H(x + ct) - H(x - ct)\right],$$

$$g_t(x,t) = \frac{1}{2} \left[\delta(x - ct) + \delta(x + ct)\right].$$

(13.25)

Associated with the spacetime point $(x,t)$ there is “backwards light-cone” — see figure 13.2, $B(x,t)$. The Green’s function $g(x-x',t-t')$

$$g(x-x',t-t') = \begin{cases} 1/2c, & \text{if } (x',t') \in B(x,t), \\ 0, & \text{otherwise}. \end{cases}$$

(13.26)

Thus the integral in (13.23) is equivalent to

$$u(x,t) = \frac{1}{2c} \iint_B f(x',t') \, dx' \, dt',$$

$$= \frac{1}{2c} \int_{t-c(t-t')}^{t} f(x',t') \, dx' \, dt'.$$

(13.27)
Thus the response at \((x,t)\) is proportional to the integral of the forcing over the backwards lightcone. Mathematicians also refer to the backwards lightcone as the domain of dependence of \((x,t)\). For similar reasons the forward facing region is the domain of influence of \((x,t)\).

A moving source

Consider the forced string
\[
 u_{tt} - c^2 u_{xx} = q(t) \delta(x - at), \quad u(x,0) = u_t(x,0) = 0. \quad (13.28)
\]
We suppose that \(0 < a < c\) i.e. the source is moving slower than the wave speed \(c\).
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

13.5 The method of images and reflections

What happens when a left moving pulse of width $\ell$

$$u(x,t) \approx f(x + ct) \quad \text{when} \quad x + ct \gg \ell \quad (13.29)$$
on a semi-infinite string $x > 0$, hits the fixed end, $x = 0$, where $u(0,t) = 0$? We can construct the solution of this problem using the method of images

$$u(x,t) = f(x + ct) - f(-x + ct). \quad (13.30)$$

It is easy to verify by substitution that this is a solution of the equation and also by inspection it satisfies the boundary condition. At large times the image pulse, that starts in the “imaginary” extension of the domain (i.e. $x < 0$), has moved into the real domain and this means that the pulse changes sign on reflection at a fixed end. A picture is worth a thousand words here – see figure 13.3

13.6 Problems

Problem 13.1. Find the energy conservation equation for the Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + \sigma_0^2 u = 0.$$ 

Problem 13.2. Find the energy conservation equation for the nonlinear wave equation

$$u_{tt} - c^2 \left[ \frac{u_x}{\sqrt{1 + u_x^2}} \right]_x = 0.$$ 

Your answer should reduce to the linear conservation law if $u_x \ll 1$.

Problem 13.3. Find the general solution of the PDES

$$u_{xy} = 0, \quad u_{xx} + u_{yy} = 0, \quad u_{xx} - 3u_{xy} + 2u_{yy} = 0,$$
in terms of “arbitrary functions”. (If you can’t guess, try substituting $u(x, y) = U(x + \lambda y)$ and determining $\lambda$.)

Problem 13.4. Find the general solution of the PDE

$$u_{xx} + 2u_{xy} + u_{yy} = 0,$$
in terms of “arbitrary functions”.

SIO203C, W.R. Young, March 21, 2011
Problem 13.5. Find the general solution of the PDE
\[ xu_{tt} - (x^{-1}u_x)_x = 0. \]
in terms of “arbitrary functions”. (Try transforming this PDE into the wave equation by changing variables.)

Problem 13.6. Show that if
\[ u = U(x \pm ct), \]
then \( J = \pm cE. \)

Problem 13.7. Solve the wave equation with the initial condition
\[ u_0(x) = 0, \quad w_0(x) = \text{sgn}(x). \]
Sketch \( u \) as a function of \( x \) at fixed times.

Problem 13.8. (i) Solve and visualize the wave equation \( u_{tt} = c^2 u_{xx} \)
with the initial condition
\[ u_0(x) = e^{-\alpha^2 x^2}, \quad w_0(x) = 0. \]
(ii) Repeat using
\[ u_0(x) = 0, \quad w_0(x) = e^{-\alpha^2 x^2}. \]

Problem 13.9. In the discussion following (13.19) we solved the wave equation using D’Alembert’s formula. However following the dimensional reasoning we used in our discussion of the diffusion equation one might try the similarity solution
\[ u(x,t) = U(\eta), \quad \eta \equiv \frac{x}{ct}. \]  
(13.31)

(i) Show that the solution in (13.20) does have the similarity form in (13.31).  (ii) Find the most general solution of the wave equation having the similarity form in (13.31).

Problem 13.10. Consider the forced wave equation
\[ u_{tt} - u_{xx} = -H(2t - x^2). \]
The string is undisturbed at \( t = 0 \). (i) On an \((x,t)\) diagram sketch the region in which the forcing on the RHS is nonzero. Also indicate the region in which \( u(x,t) \) is nonzero. (ii) Solve the equation. Check your algebra by showing
\[ u(0,t) = \frac{1}{3} + t - \frac{1}{3}(1 + 2t)^{3/2}. \]
Lecture 13. The wave equation $u_{tt} = c^2 u_{xx}$

Problem 13.11. What happens when a left moving pulse of width $\ell$

$$u(x, t) \approx f(x + ct) \quad \text{when} \quad x + ct \gg \ell$$

on a semi-infinite string $x > 0$, hits the free end, $x = 0$, where $u_x(0, t) = 0$?
14.1 Discussion of “forever problems”

If the initial condition is compact then the solution of

\[ u_{tt} - c^2 u_{xx} = 0 \]  \hspace{1cm} (14.1)

is also compact — the disturbance travels with finite speed \( c \). But when we deal with forced problems,

\[ u_{tt} - c^2 u_{xx} = f, \]  \hspace{1cm} (14.2)

where the force \( f(x,t) \) has been acting “forever” the waves have had plenty time to reach \( x = \pm \infty \). This is true even if \( f(x,t) \) is compact in space. In a case like this we don’t expect \( u(\pm \infty, t) = 0 \).

The most important example of this situation is a periodic-in-time, spatially compact force, such as \( f = \delta(x) \cos \omega t \). The force was switched on in the distant past and has been radiating waves with frequency \( \omega \) ever since. If we time-average the energy conservation equation

\[ \mathcal{E}_t + J_x = q u_t f, \]  \hspace{1cm} (14.3)

we obtain

\[ \bar{J}_x = q \bar{u}_t \bar{f}. \]  \hspace{1cm} (14.4)

The overbar denotes the time-average over a period \( \tau = 2\pi/\omega \):

\[ \bar{\theta}(x) = \frac{1}{\tau} \int_{t-\tau}^t \theta(x, t) \, dt. \]  \hspace{1cm} (14.5)
The average of $E_t$ is zero because the energy density at a fixed point is a periodic function of time. This is a special case of the general rule

$$\langle \text{any stationary function of time} \rangle_t = 0.$$ (14.6)

As $x \to \pm \infty$, the force $f(x) \to 0$, and the time-averaged energy equation (14.4) reduces to

$$\check{J}_x = 0 \quad \Rightarrow \quad \check{J}(\pm \infty) = \text{sgn}(x) J_* ,$$ (14.7)

where the constant asymptotic energy flux $J_*$ is positive i.e., energy is strictly radiating away from the source in both directions. This physical condition — the Sommerfeld condition — must be used to resolve mathematical ambiguities in the solution (see next section). The right hand side of (14.4) is the rate of working of the force, and the total rate of working of the source (power) is obtained by integrating (14.4) from $x = -\infty$ to $x = +\infty$:

$$2 J_* \equiv \varrho \int_{-\infty}^{\infty} u t f \, dx .$$ (14.8)

$J_*$ is the most important single number characterizing the source $f(x, t)$. Let’s calculate $J_*$.

### 14.2 Radiation from a compact source

A typical radiation problem assumes that

$$f(x, t) = e^{-i\omega t} F(x) + e^{i\omega t} F^*(x) .$$ (14.9)

Then we look for solutions of the forced wave equation (14.2) of the form

$$u(x, t) = e^{-i\omega t} U(x) + e^{i\omega t} U^*(x) .$$ (14.10)

We find that

$$U'' + \kappa^2 U = -c^{-2} F , \quad \kappa^2 = \omega^2 / c^2 .$$ (14.11)

Although $U(x)$ is not growing as $x \to \pm \infty$ it is not decaying either — we expect

$$U(x) \sim e^{\pm i\kappa x} , \quad \text{as} \quad x \to \pm \infty .$$ (14.12)

---

1 From the $x \to -x$ symmetry, we anticipate that $\check{J}(+\infty) = \check{J}(-\infty)$
This asymptotic behaviour ensures that energy is propagating away from the source. For example, suppose \( x > 0 \). Then in the far-field (a long way from the source) the condition in (14.12) ensures that the solution (14.10) becomes
\[
\begin{align*}
    u(x, t) &\sim A e^{ikx - i\kappa t} + A^* e^{-ikx + i\omega t}, \\
    &\quad \text{as } x \to +\infty.
\end{align*}
\] (14.13)

The radiation condition (14.12) ensures that the large-positive-\( x \) solution is purely a right-going wave.

We can solve (14.11), with the radiation condition (14.12), by finding the Green’s function
\[
    G'' + \kappa^2 G = \delta(x), \quad \Rightarrow \quad G(x) = \frac{e^{ik|x|}}{2i\kappa}. \quad (14.14)
\]
The solution above satisfies the patching conditions at \( x = 0 \): \( G(x) \) is continuous and to balance the \( \delta(x) \), the jump in \( G_x(x) \) is equal to unity.

Using the Green’s function, the solution of the radiation problem in (14.11) is
\[
    U(x) = -c^{-2} \int_{-\infty}^{\infty} F(x') \frac{e^{i\kappa|x-x'|}}{2i\kappa} \, dx', \quad (14.15)
\]
\[
    \quad \quad \quad -c^{-2} \int_{-\infty}^{\infty} F(x') \frac{e^{-i\kappa x'}}{2i\kappa} \, dx' e^{i\kappa x}, \quad \text{as } x \to +\infty. \quad (14.16)
\]
The argument above identifies the asymptotic constant \( A \) in (14.13) in terms of the Fourier transform of the source function \( F(x) \).

Now that we know the solution in the far-field, we can obtain the large-\( x \) energy flux as
\[
    J(x = +\infty, t) = -Tu_t u_x|_{x=\infty} = 2T\kappa AA^* + \text{oscillatory stuff}. \quad (14.17)
\]
This is only the right-going energy \( J^* \) — there is an equal amount going left. Thus the total rate of working is
\[
    2J^* = 4T \frac{\omega^2}{c} AA^*. \quad (14.18)
\]
14.3 Radiation damping of an oscillator

Consider a semi-infinite string \((x > 0)\) coupled to an oscillator at \(x = 0\) (see figure 14.1). At \(t = 0\) the string is at rest and the oscillator is kicked into motion. What happens? We expect that the oscillator will emit waves and lose energy: this is the simplest model of radiation damping. Now let’s work out the details.

The displacement of the oscillator away from its equilibrium position is \(u(0,t)\). The equation of motion of the oscillator, which is also the \(x = 0\) boundary condition for the string, is

\[
m\ddot{u}(0,t) + ku(0,t) = Tu_x(0,t).
\]

The right-hand side is the force of tension tugging on the oscillator.

As a sanity check we now verify that the total mechanical system (string + spring) conserves energy. The energy equation for the oscillator is

\[
\frac{d}{dt} \frac{1}{2} \left[ m\dot{u}^2 + ku^2 \right] = Tu_x u_t |_{x=0}.
\]

On the other hand, the wave equation

\[
\varrho u_{tt} - Tu_{xx} = 0
\]

on the half-line \(x > 0\) has the energy equation:

\[
\frac{d}{dt} \int_0^\infty \frac{1}{2} (\varrho u_t^2 + Tu_x^2) \, dx = -Tu_t u_x |_{x=0}.
\]
Adding (14.19) and (14.22) shows that the total energy is conserved: what the oscillator loses, the string gains.

Now we solve the pde (14.21) with the initial condition \( u(x,0) = u_t(x,0) = 0 \) and the boundary condition in (14.19). As an initial condition for the oscillator we suppose that

\[
\begin{align*}
    u(0,0) &= 0, & u_t(0,0) &= 1.
\end{align*}
\]

Thus the string is motionless and the mass at \( x = 0 \) is kicked into motion.

The problem is not completely posed till we insist that there is no energy impinging on the oscillator from \( x = \infty \) i.e., the flow of energy is strictly towards \( x = \infty \). This means that the disturbance on the string is travelling only to the right:

\[
\begin{align*}
    u(x,t) &= U(x - ct) \Rightarrow u_x = -c^{-1}u_t. & (14.24)
\end{align*}
\]

Thus the oscillator equation in (14.19) becomes

\[
\begin{align*}
    c^2 = \frac{T}{\rho}
    m\ddot{u}(0,t) + Tc^{-1}u_t(0,t) + ku(0,t) &= 0. & (14.25)
\end{align*}
\]

We have managed to get a simple ODE whose solution gives the motion of the end of the string. The key here is the argument in (14.24) that the disturbances on the string go only towards the right.

The solution of the damped oscillator equation (14.25) with the initial condition in (14.23) is

\[
\begin{align*}
    u(0,t) &= e^{-\gamma t} \frac{\sin \left( \sqrt{\omega^2 - \gamma^2} t \right)}{\sqrt{\omega^2 - \gamma^2}} H(t), & (14.26)
\end{align*}
\]

where

\[
\begin{align*}
    \omega &= \sqrt{\frac{k}{m}}, & \gamma &= \frac{T}{2cm}. & (14.27)
\end{align*}
\]

### 14.4 Problems

**Problem 14.1.** Find the Green’s function of the simple harmonic oscillator equation

\[
\ddot{\theta} + \sigma^2 \theta = f(t).
\]
Compare this calculation with that in (14.14) and carefully explain why the two Green’s function’s are not identical even though they satisfy the same ODE.

Problem 14.2. Consider radiation from a compact source on to an “elastically braced” string

\[ u_{tt} - c^2 u_{xx} + \sigma^2 u = F(x)e^{-i\omega t} + F^*(x)e^{i\omega t}. \]

This is the forced Klein-Gordon equation. Find an expression for the radiated energy in terms of \( F(x) \).

Problem 14.3. Work through all the calculations in this lecture again, but this time assume that the string is dissipative:

\[ u_{tt} + 2\nu u_t - c^2 u_{xx} = F(x)e^{-i\omega t} + F^*(x)e^{i\omega t}. \] (14.28)

Now we expect the disturbance to decay at \( x = \pm\infty \), and we should not have to make arguments about the direction of energy flux to resolve mathematical ambiguities. Show that this is the case, and that you recover the solution in the lecture by taking \( \nu \to 0 \).

Problem 14.4. Go back to (14.9), and let \( \omega \to \omega + iy \) where \( y \) is real, positive and very small. In this case we can consider that the forcing has switched on a long time in the past and has been very slowly growing exponentially as \( e^{yt} \). With small but nonzero \( y \) we can require that the solution will be small at \( x = \pm\infty \). Solve the problem with non-zero \( y \) and show that you recover the solution in the lecture in the \( y \to 0 \) limit.

Problem 14.5. Solve the initial value problem

\[ u_{tt} - c^2 u_{xx} = 0, \quad u(x, 0) = u_t(x, 0) = 0, \quad u(0, t) = \cos(\omega t). \]

The string is initially at rest and then the end at \( x = 0 \) is set into motion by shaking the end. Hint: draw the \( x-t \) diagram and carefully distinguish between characteristics originating at \( x = 0 \) and those at \( t = 0 \).

Problem 14.6. At a fixed time \( t \) sketch the displacement, \( u \) as a function of \( x \), produced by the boundary condition in (14.26).

Problem 14.7. Consider a mass-spring system attached to a string at \( x = 0 \); the mass is \( m \), the spring constant is \( K \) and the oscillator frequency is \( \sigma = \sqrt{m/K} \). A steady wave with frequency \( \omega \) is incident from the left \( (x = -\infty) \). The energy is partly reflected back to \( x = -\infty \) and
partly transmitted to $x = +\infty$. Find the reflection and transmission coefficients. Check your answer by showing that you recover the result in lectures if $K \to 0$. You should also find that there is no reflected wave if $\omega = \sigma$.

**Problem 14.8.** Consider an “elastically braced” string attached to an oscillator. The problem is the same as section 14.3 except that the PDE is

$$u_{tt} - c^2 u_{xx} + \sigma_0^2 u = 0.$$  

Find the damping rate of the oscillator.
Lecture 15

Nearest-neighbour coupling models

15.1 An example of the $\theta$-transform

Let us consider a infinite (in both directions) row of tanks. Each tank has volume $V$ m$^3$ and we label them $n = 0, \pm 1, \pm 2, \cdots$. There is a flux $Q$ m$^3$s$^{-1}$ going to the left and the same flux to the right; each tank has two outlets and two inlets. Then the equation for tracer conservation in this system is

$$\dot{C}_n = \alpha [C_{n-1} - 2C_n + C_{n+1}], \quad \text{with} \quad \alpha \equiv Q/V. \quad (15.1)$$

The initial condition for might be $C_n(0) = \delta_{n0}$; that is, tank 0 contains all of the dissolved chemical and the others are unpolluted.

Probabilists recognize this as a ‘continuous time random walk’. You can think of each tank as a site occupied by a number of random walkers; there is a constant probability per unit time, $\alpha$, of hopping to the right and the same probability of hopping to the left. This is also an ingredient in Turing’s model of morphogenesis in which cells exchange morphogens with their nearest neighbours.

Equation (1) is an infinite set of differential–difference equations. Solving it looks tough, but help arrives from an unexpected direction. Introduce the function

$$F(\kappa, t) \equiv \sum_{n=-\infty}^{\infty} C_n(t)e^{in\kappa}. \quad (15.2)$$

If $z \equiv e^{i\kappa}$ then

$$F(\kappa, t) \equiv \sum_{n=-\infty}^{\infty} C_n(t)z^n.$$

This is the "z-transform".

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Given the answer to our problem, \( C_n(t) \), we can obtain \( F(\kappa, t) \) by using the Fourier series in (15.2). But the inverse is also true: given \( F(\kappa, t) \) we can get the answer to our problem by evaluating the integral

\[
C_n(t) = \int_{-\pi}^{\pi} F(\kappa, t) e^{-ink} \frac{d\kappa}{2\pi}.
\]  

(15.3)

This is a very significant observation because it is easy to show from (15.1) that

\[
F_t = -2\alpha(1 - \cos \kappa)F, \quad \Rightarrow \quad F(\kappa, t) = F(\kappa, 0)e^{-2(1-\cos \kappa)\alpha t}.
\]  

(15.4)

Next, since we now possess \( F(\kappa, t) \), we use (15.3) to obtain an integral representation of the solution of (15.1):

\[
C_n(t) = \int_{-\pi}^{\pi} F(\kappa, 0)e^{-ink-2(1-\cos \kappa)\alpha t} \frac{d\kappa}{2\pi}.
\]  

(15.5)

Do not be too impressed with this yet — the integral in (15.5) is not a transparent representation of the solution. However having an integral is major progress because it is easier to numerically evaluate integrals than to timestep differential equations. And there are powerful analytic methods for approximately evaluating integrals.

Take a simple initial condition, namely \( C_n(0) = \delta_{n0} \), so that \( F(\kappa, 0) = 1 \). With this choice we can use symmetry arguments to reduce the solution in (15.4) to the form

\[
C_n(t) = e^{-2\alpha t} \int_{0}^{\pi} e^{2\alpha t \cos \kappa} \cos n\kappa \frac{d\kappa}{\pi}.
\]  

(15.6)
We try to evaluate (15.6) by looking up the integral in thick books like Gradshteyn & Ryzhik or Abramowitz & Stegun. In this problem our efforts are crowned with success because after some excavation we find the following result:

\[ I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos n\theta \, d\theta, \]  
(15.7)

(for example, see formula 9.6.19 in A&S) where \( I_n(z) \) is the modified Bessel function of integer order. So, now we get

\[ C_n(t) = e^{-2\alpha t} I_n(2\alpha t). \]  
(15.8)

This is progress; for instance, \matlab{} has the routine \texttt{besseli} which calculates \( I_n(z) \). Moreover, there are many results known about \( I_n \) so that we can make a pretty good qualitative visualization of the solution even without turning to numerical evaluation.

In figure ?? you can see that at large times the concentration is a slowly changing function of \( n \) and this suggests that we make a ‘continuum’ approximation by regarding \( n \) as a continuously changing variable, analogous to position, so that

\[ C(n + 1, t) = C(n, t) + C_n(n, t) + (1/2)C_{nn}(n, t) + \cdots \]  
(15.9)

Thus on the right hand side of (15.1)

\[ C_{n-1} - 2C_n + C_{n+1} \approx C_{nn}. \]  
(15.10)

At large times, when the solution is a slowly changing function of \( n \), the differential–difference equation (15.1) is approximated by the diffusion equation

\[ C_t = \alpha C_{nn}. \]  
(15.11)

It seems plausible that the initial condition which corresponds to releasing all of the chemical at \( n = 0 \) is probably

\[ C(0, t) = \delta(n). \]  
(15.12)

But we know the solution of this initial value problem is our friend the Gaussian

\[ C_n(t) \approx \frac{e^{-n^2/4\alpha t}}{2\sqrt{\pi \alpha t}}. \]  
(15.13)

Unlike our earlier exact result in (15.8) the approximation above is very transparent — it is very easy to visualize the solution without numerical evaluation. Partly, this is because the answer is in terms of elementary functions, but the main simpification is the discovery of the similarity form.
15.2 Mach bands

The physical basis for vision is that retinal cells “fire” when stimulated by light. A single cell, excised from the retina of a horseshoe crab, fires at a rate $R$ (spikes per second, traveling down the axon) which is proportional to the logarithm of the number of incident photons:

$$R = R_0 \frac{\ln I}{\ln I_0}. \quad (15.14)$$

Here $I_0$ is some standard illumination (photons per second) which elicits $R_0$ spikes per second. To avoid lots of logarithms we write

$$R = kL, \quad L \equiv \ln I. \quad (15.15)$$

Another experimental fact is lateral inhibition: the response of cell $n$ is reduced by stimulation of the neighbouring cells $n \pm 1$. Thus the model of a one-dimensional retina is

$$R_n = kI_n - \alpha [R_{n-1} + \alpha R_{n+1}]. \quad (15.16)$$

Now lets work out a few solutions.

Example 1: uniform illumination

Consider uniform illumination, $I_n = I$. Evidently

$$R = k \frac{L}{1 + 2\alpha}. \quad (15.17)$$

Since $\alpha > 0$ the response of a cell in the array is less than the response of an isolated cell.

Example 2: the Green’s function

Suppose we stimulate just the cell at $n = 0$:

$$G_n = k\delta_{n0} - \alpha [G_{n-1} + \alpha G_{n+1}]. \quad (15.18)$$
To solve this system we again consider the transform

$$G(\kappa) \equiv \sum_{n=-\infty}^{\infty} G_n e^{i n \kappa},$$

(15.19)

and we quickly find

$$G(\kappa) = \frac{k}{1 + 2\alpha \cos \kappa}.$$  

(15.20)

We could now recover $G_n$ by evaluating the integrals

$$G_n = \oint \frac{e^{-i n \kappa}}{1 + 2\alpha \cos \kappa} \frac{d\kappa}{2\pi}.$$  

(15.21)

This is a standard exercise in residue calculus. (Or see problem 15.5)

We summarize the result:

$$G = \frac{k}{\sqrt{1 - 4\alpha^2}} \sum_{n=-\infty}^{\infty} (-\beta)^{-|n|} e^{i n \kappa},$$  

(15.22)

where

$$\beta \equiv \sqrt{\frac{1}{4\alpha^2} - 1 + \frac{1}{2\alpha}}.$$  

(15.23)

We assume $0 < \alpha < 1/2$ — else it is lateral excitation rather than inhibition. (Why?)

The response at $n = 0$ is

$$G_0 = \frac{k}{\sqrt{1 - 4\alpha^2}}$$  

(15.24)

— this is always greater than the response of an isolated cell. The stimulated cell at $n = 0$ inhibits its neighbours at $n = \pm 1$. This in turn lowers the inhibition at $n = 0$, and also at $n = \pm 2$ and so on. In the trade this is known as disinhibition.

The response of the array is shown in the upper panel figure 15.2. Notice we can't have a negative response so the solution we have just found should be superposed on the response to a uniform illumination $L$. Thus the actual firing rate is

$$R_n = \frac{kL}{1 + 2\alpha} + \frac{kA}{\sqrt{1 - 4\alpha^2}} \left(\frac{1}{\beta}\right)^{|n|}.$$  

(15.25)
Figure 15.2: Top panel: the Green’s function with $\alpha = 1/4$. The response at $n = 0$ is about 15% greater than the response of an isolated cell. Lower panel: the response of the array if the seven central cells, $n = [-3, -2, \ldots, 2, 3]$ are uniformly excited. The cells at the edge ($n = \pm 4$ and $n = \pm 3$) have an enhanced response. This is how your visual system detects ‘edges’. MachBand.eps
Example 3: edges

If you look closely at the border between a uniformly light region and uniformly dark region you’ll notice that the edge is enhanced — there is a bright bar running along the border. This is a Mach band. The phenomenon was observed by impressionists who wondered if it was necessary to represent the band in their paintings. The physicist Mach said: “Of course not — just paint realistically and the observer’s nervous system will supply the band.” This advice was ignored by Degas, Monet and others who routinely overcompensate for the Mach band in their paintings.

15.3 Problems

**Problem 15.1.** Formulate a version of (15.1) whose continuum approximation is the advection equation

\[
C_t + C_n \approx 0.
\]  

(15.26)

Solve the discrete system and compare your answer with the relevant solution of (15.26).

**Problem 15.2.** Suppose that \( t \) is large in (15.6). Use Laplace’s method to asymptotically evaluate the integral and show that the answer agrees with the similarity solution in (15.13).

**Problem 15.3.** Consider an infinite set of coupled oscillators

\[
\dot{\theta}_n = \theta_{n-1} - 2\theta_n + \theta_{n+1}; \quad \theta_n(0) = \delta_{n0}, \quad \dot{\theta}_n(0) = 0.
\]

Find a solution of the system above in terms of the Bessel functions:

\[
J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos[z \sin \theta - n\theta] d\theta = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{iz \cos \theta} \cos n\theta d\theta,
\]

AS §9.1.21.

Give a simple expression for the decay of \( \theta_0 \) at large times. Use MATLAB to plot the displacements as a function of \( n \) at fixed times such as \( t = 1 \), \( t = 2 \) etcetera.

**Problem 15.4.** The impulse response of coupled oscillators in the previous problem is the solution of

\[
\ddot{G}_n = G_{n+1} - 2G_n + G_{n+1} + \delta_{n0} \delta(t).
\]
Use the Fourier series method to find a compact integral representation of the solution. Can the integrals be evaluated in terms of Bessel functions?

**Problem 15.5.** Fill in all the steps between (15.20) and (15.23).

**Solution.** There is interesting way of working out the coefficients in a Fourier series without evaluating integrals — this example is an opportunity to demonstrate that technique. We use the complex variable

\[ z = e^{i \kappa} = \cos \kappa + i \sin \kappa. \]

The idea is a Laurent series in \( z \) corresponds to a Fourier series in \( \kappa \). Let’s see how this works in this particular problem.

The function \( G(\kappa) \) in (15.20) can be written as

\[
G = \frac{k}{1 + 2\alpha \cos \kappa},
\]

\[
= \frac{k}{1 + \alpha(z + z^{-1})},
\]

\[
= \frac{k}{\alpha z^2 + \alpha^{-1}z + 1},
\]

\[
= \frac{k}{\alpha (z + \beta)(z + \beta^{-1})},
\]

\[
= \frac{k}{\alpha \beta - \beta^{-1}} \left( \frac{\beta}{z + \beta} - \frac{\beta^{-1}}{z + \beta^{-1}} \right).
\]

We now have to work out the Laurent series on the unit disk i.e \(|z| = 1\). We must be sensitively aware that \( \beta > 1 \). Thus

\[
\frac{\beta}{z + \beta} = \frac{1}{1 + (z/\beta)} = 1 - (z/\beta) + (z/\beta)^2 + \cdots
\]

while

\[
\beta^{-1} z + \beta^{-1} = \frac{1}{\beta z} \frac{1}{1 + (1/\beta z)} = \frac{1}{\beta z} \left[ 1 - (1/\beta z) + (1/\beta z)^2 + \cdots \right].
\]

Notice that both series converge because we use complex number whose modulus is less than unity (\( z/\beta \) in one case and \( 1/\beta z \) in the other). Thus we finally have

\[
G = \frac{k}{\alpha \beta - \beta^{-1}} \sum_{n=-\infty}^{\infty} \frac{(-z)^n}{\beta |n|}.
\]

\[
\frac{1}{\alpha \beta - \beta^{-1}} = \frac{1}{\sqrt{1 - 4\alpha^2}}.
\]