Recitation on Thursday April 9th

Reading assignment

Read lecture 4 Why integrals? I won’t have time to cover this material in class.

Problem 1.5 — more algebraic perturbation theory

Use perturbation theory to find two-term approximations ($\epsilon \to 0$) to all roots of:

\[(f) \quad \epsilon x^3 + x^2 + (2 + \epsilon)x + 1 = 0, \]
\[(g) \quad \epsilon x^3 + x^2 + (2 - \epsilon)x + 1 = 0. \]

Problem 2.3 — perturbation of a differential equation

\[(i) \quad \text{Solve the problem} \]
\[\ddot{x} + (1 + \epsilon e^{\alpha t})x = 0, \quad \text{with IC} \quad x(0, \epsilon, \alpha) = 1, \quad \dot{x}(0, \epsilon, \alpha) = 0, \quad (1)\]
with the RPS
\[x(t, \epsilon, \alpha) = x_0(t, \alpha) + \epsilon x_1(t, \alpha) + \cdots \quad (2)\]
Calculate $x_0$ and $x_1$. \[(ii) \quad \text{Bearing in mind that } \alpha \text{ might be positive or negative, discuss the utility of the RPS when } t \text{ is large.} \]

Hint: To solve for $x_1(t)$ may help to recognize that
\[e^{\alpha t} \cos t = \Re e^{(\alpha + i)t}. \quad (3)\]

Problem 2.6 — the over-damped oscillator

Consider the non-dimensional oscillator problem
\[\ddot{x} + \beta \dot{x} + x = 0 \quad (4)\]
with the initial conditions
\[x(0) = 0, \quad \text{and} \quad \dot{x}(0) = 1. \quad (5)\]

\[(i) \quad \text{Supposing that } \beta > 2, \text{ solve the problem exactly.} \quad (ii) \quad \text{Show that if } \beta \gg 1 \text{ then the long-time behaviour of your exact solution is} \]
\[x \propto e^{-t/\beta}, \quad (6)\]
i.e., the displacement very slowly decays to zero. \[(iii) \quad \text{Motivated by this exact solution, “rescale” the problem (and the initial condition) by defining the slow time} \]
\[\tau \overset{\text{def}}{=} \frac{t}{\beta}, \quad (7)\]
and $X(\tau) = ?x(t)$. Show that with a suitable choice of $\tau$, the rescaled problem is
\[\epsilon \dot{X} + X + X = 0 \quad \text{with the IC:} \quad X(0) = 0, \quad X(\tau) = 1. \quad (8)\]
Make sure you give the definition of $X(\tau)$ and $\epsilon \ll 1$ in terms of the parameter $\beta \gg 1$ and the original variable $x(t)$. (iv) Try to solve the rescaled problem (8) using an RPS

$$X(\tau, \epsilon) = X_0(\tau) + \epsilon X_1(\tau) + \cdots$$

(9)

Discuss the miserable failure of this approach by analyzing the dependence of the exact solution from part (i) on $\beta$. That is, simplify the exact solution to deduce a useful $\beta \to \infty$ approximation, and explain why the RPS (9) cannot provide this useful approximation.

**Problem 3.9**

Find a leading-order $x \to \infty$ asymptotic approximation to

$$A(x; p, q) \overset{\text{def}}{=} \int_x^\infty e^{-pt^2} \, dt.$$ (10)

Above, $p$ and $q$ are both positive real numbers.

**Problem 4.11 — recognition of a well known special function**

Solve the half-plane ($y > 0$) boundary value problem

$$yu_{xx} + u_{yy} = 0$$ (11)

with $u(x, 0) = \cos qx$ and $\lim_{y \to \infty} u(x, y) = 0$.

**Due in class on Tuesday April 13th**

**Problem 1.9 — iteration and logarithms**

Read section 1.4 first. Consider $z(\epsilon)$ defined as the solution to

$$z^{1 \over \epsilon} = e^z.$$ (12)

(i) Use MATLAB to make a graphical analysis of this equation with $\epsilon = 1/5$ and $\epsilon = 1/10$. Convince yourself that as $\epsilon \to 0$ there is one root near $z = 1$, and second, large root that recedes to infinity as $\epsilon \to 0$. (ii) Use an iterative method to develop an $\epsilon \to 0$ approximation to the large solution. Calculate a few terms so that you understand the form of the expansion. (iii) Use MATLAB to compare the exact answer with approximations of various orders e.g., as in Figure 1.1. (iv) Find the dependance of the other root, near $z = 1$, on $\epsilon$ as $\epsilon \to 0$.

**Problem 2.1— regular perturbation of an ODE**

(i) Consider the projectile problem with linear drag:

$$\frac{d^2z}{dt^2} + \mu \frac{dz}{dt} = -g_0.$$ (13)
and the initial conditions $z(0) = 0$ and $dz/dt = u$. Find the solution with no drag, $\mu = 0$, and calculate the time aloft, $\tau$. (ii) Suppose that the drag is small — make this precise by non-
dimensionalizing the equation of motion and exhibiting the relevant small parameter $\epsilon$. (iii) Use a regular perturbation expansion to determine the first correction to $\tau$ associated with non-zero drag. (iv) Integrate the non-dimensional differential equation exactly and obtain a transcendental equation for $\tau(\epsilon)$. Asymptotically solve this transcendental equation approximately in the limits $\epsilon \to 0$ and $\epsilon \to \infty$. Make sure the $\epsilon \to 0$ solution agrees with the earlier RPS.

Problem 2.5 — constant flux of belligerent drunks

Let’s make a small change to the formulation of the belligerent-drunk example in section 2.2. Suppose that we model the bars using a Neumann boundary condition. This means that the flux of drunks, rather than the concentration, is prescribed at $x = 0$ and $\ell$: the boundary condition in (2.26) is changed to

$$\kappa u_x(0, t) = -F, \quad \text{and} \quad \kappa u_x(\ell, t) = F,$$

(14)

where $F$, with dimensions drunks per second, is the flux entering the domain from the bars. Try to repeat all calculations in section 2.2, including the analog of the $\beta \ll 1$ perturbation expansion. You’ll find that it is not straightforward and that a certain amount of ingenuity is required to understand the weakly interacting limit with fixed-flux boundary conditions. We should discuss this in the recitation session.