This exam is open notes, but no computers, calculators, iphones etc.
Throughout $0 < \epsilon \ll 1$. There are five problems: turn the page.

**Problem 1**
Construct a uniformly valid, leading order, “composite solution” of the boundary value problem
\[ \epsilon y'' + xy' + xy^2 = 0, \]
posed on $0 < x < 1$ with boundary conditions $y(0) = 0$ and $y(1) = -1$

**Problem 2**
Find the leading-order dependence on $\epsilon$ of
\[ F(\epsilon) = \int_0^\infty \frac{du}{\sqrt{\epsilon^2 + u^2 + u^4}}. \]
The following indefinite integrals might be useful
\[ \int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left[ x + \sqrt{a^2 + x^2} \right], \quad \text{and} \quad \int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|. \]

**Problem 3**
Use multiple scale theory to find an approximate solution of the initial value problem
\[ u_{tt} + u = 2 \left[ \cos(\epsilon t) + \epsilon u^2 \right], \quad \text{with ICs} \quad u(0) = 0, \quad u_t(0) = 1. \]

**Problem 4**
When $\epsilon = 0$ the eigenproblem
\[ y'' + \epsilon \left( \frac{1}{2} y^2 \right)' + \lambda y = 0, \quad y(0) = y(\pi) = 0, \]
has the solution $\lambda = 1$ and $y = a \sin x$. Use perturbation theory to investigate the dependence of the eigenvalue $\lambda$ on $a$ and $\epsilon$. To check your calculation show that if $a = 1/2$ and $\epsilon = 1/5$ then $\lambda \approx 1201/1200$.
Some trig identities:
\[ \sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x), \quad 2 \sin x \cos x = \sin 2x. \]
Problem 5

Figure 1 shows the solution to one of the four initial value problems:

\[
\begin{align*}
\epsilon^2 y''_1 - e^{-x} y_1 &= 0, \quad y_1(0) = 0, \quad y'_1(0) = 1, \\
\epsilon^2 y''_2 - e^{x} y_2 &= 0, \quad y_2(0) = 0, \quad y'_2(0) = 1, \\
\epsilon^2 y''_3 + e^{-x} y_3 &= 0, \quad y_3(0) = 0, \quad y'_3(0) = 1, \\
\epsilon^2 y''_4 + e^{x} y_4 &= 0, \quad y_4(0) = 0, \quad y'_4(0) = 1.
\end{align*}
\]

(a) Which \(y_n\) is shown in figure 1?  (b) Use the WKB approximation to estimate the \(\epsilon\) used in figure 1. Some interesting numbers are

\[
\begin{align*}
e^1 &\approx \frac{19}{7}, \quad e^{1.15} \approx \pi, \quad e^{1.25} \approx \frac{7}{2}, \\
e^{1.5} &\approx \frac{9}{2}, \quad e^{1.75} \approx \frac{23}{4}, \quad e^2 \approx \frac{36}{5}.
\end{align*}
\]